Lecture 17 — Tuesday, March 10

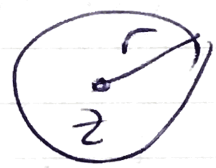
$\Omega \subseteq \mathbb{C}$ open domain

$u : \Omega \to [-\infty, \infty)$ is subharmonic if

① u is u.s.c ($u(z) \geq \overline{\lim}_{\xi \to z} \mu(\xi)$)

② u satisfies the submean inequality:

$$u(z) \leq \frac{1}{2\pi} \int u(z + re^{i\theta})\, d\theta$$



Aim today : To prove the following claim :

a subharmonic iff $\Delta u \geq 0$

$\Updownarrow$

Δu is a positive measure.

Obs : $u \equiv -\infty$ is NOT subharmonic

(This is just a convention)

$SH(\Omega) \subseteq L^1_{loc}(\Omega)$

Thm 4 : $u \in SH(\Omega)$. For any $v \in \mathcal{C}^2_0(\Omega)$ with compact support, $v \geq 0$, then

then $\displaystyle\int_\Omega u \,\Delta v \geq 0.$

<u>Remark</u>: In distribution theory,

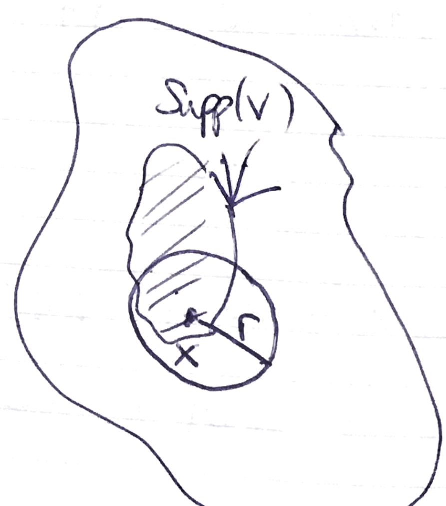
$u \in SH(\Omega) \Rightarrow \Delta u \geq 0.$

$$(\Delta u, v) \underset{def}{=} \int u \,\Delta v \geq 0.$$

<u>Proof</u>: Fix

$$0 < r < \operatorname{dist}(\operatorname{Supp}(v), \partial\Omega)$$

$z \in \operatorname{Supp}(v)$



For $z \in \operatorname{Supp}(v)$

$$u(z) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})\,d\theta$$

Multiply both sides by $v(z)$ to get, and then integrate

$$\int u(z)\,v(z)\,dLeb(z) \leq \int \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})\,v(z)\,d\theta \; dLeb(z)$$

$$\Rightarrow \int u(z)\left(\frac{1}{2\pi}\int_0^{2\pi} v(z-re^{i\theta})\,d\theta - v(z)\right) d\,Leb(z) \geq 0$$

(after change of variables: $z+re^{i\theta} \to z'$.

Taylor expansion + divide $\log\left(\frac{1}{r^2}\right) \rightsquigarrow$ get the result.

$$v(z+w) = v(z) + \frac{\partial v}{\partial x}x + \frac{\partial v}{\partial y}y + \frac{1}{2}\left(\frac{\partial^2 v}{\partial x^2}x^2 + 2\frac{\partial^2 v}{\partial x \partial y}xy + \frac{\partial^2 v}{\partial y^2}y^2\right) + o(u^3)$$

$w = x + iy$

$w = re^{i\theta}$

$\oint x = 0$, $\oint y = 0$; $\oint xy = 0$

$$\oint x^2 = \oint y^2 = \frac{r^2}{2}$$

$$\int u(z)\,\Delta v(z) + o(r) \geq 0.$$

Now let $r \to 0$ to get the result $\int u(z)\Delta V(z) \geq 0$

which exactly proves $\Delta u \geq 0$ ✓

Now, we will show the converse implication. First, we need to study regularization of subharmonic functions.

Thm 5: $u \in \mathcal{C}^2(\Omega)$. Then

$u \in SH(\Omega)$ if and only if $\Delta u \geq 0$.
Moreover, when $\Delta u \geq 0$, then for any $z \in \Omega$, the function

$$r \mapsto M(r,z) = \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})\, d\theta$$

is non-decreasing

Proof: We just need to prove:
$\Delta u \geq 0 \Rightarrow$ submean value inequality.

We shall prove
$M(r,z)$ is non-decreasing.

[Recall from previous page that we have already proved one of the implications:
$u \in SH(\Omega) \Rightarrow \forall v \in \mathcal{C}^2$
$\int u \, \Delta v \geq 0 \Rightarrow \int \Delta u \cdot v \geq 0$]

We use polar coordinates:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial^2 r^2} + \frac{1}{2}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$0 \leq \frac{1}{2\pi}\int_0^{2\pi} d\theta \cdot \left(\Delta u(z+re^{i\theta})\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta \left(\frac{\partial^2}{\partial r^2} u(z+re^{i\theta}) + \frac{1}{2}\frac{\partial}{\partial r} u(z+re^{i\theta})\right)$$

$$= M''(r,z) + \frac{1}{2} M'(r,z)$$

$$= \frac{1}{2}\left(rM'(r,z)\right)$$

Thus, the function $r \longrightarrow rM'(r,z)$ is non-decreasing.

As $r \to 0$, $rM'(r,z) \longrightarrow 0$

and thus $rM'(r) \geq 0$

$\Longrightarrow M$ is non-decresing!

Remark: One can find other proofs in the literature, e.g. in Demailly's book based on the representation of harmonic functions using Poisson kernels.

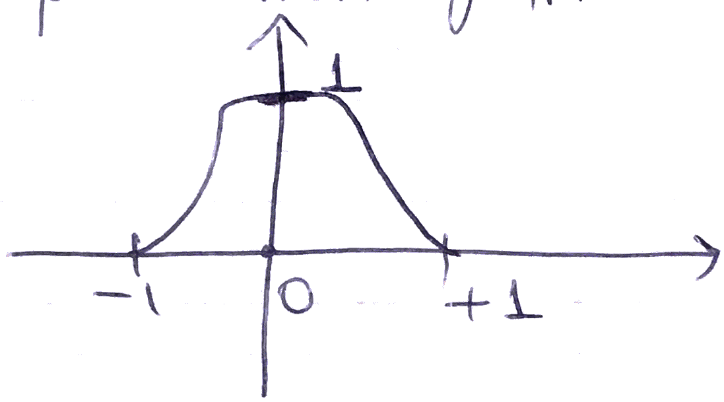
$\rightsquigarrow$ We need to develop regularization techniques for subharmonic functions

$u \in SH(\Omega) \rightsquigarrow u * \rho_\varepsilon \in \mathcal{C}^2 \cap SH(\Omega)$

and $u * \rho_\varepsilon \searrow u$ as $\varepsilon \to 0$.

Choice of a regularizing kernel

bump function $\rho : \mathbb{R} \to \mathbb{R}_+$



$\mathcal{C}^\infty$, and $\operatorname{supp}(\rho) \subseteq [-1, +1]$

and $\int \rho = +1$ and we also assume that $\rho = +1$ in a neighborhood of 0.

Smoothing kernel

$$\rho_\varepsilon(z) = \frac{1}{2\pi\varepsilon^2}\,\rho\left(\frac{|z|}{\varepsilon}\right)$$

Then, $p_\varepsilon \in \mathcal{C}^\infty$, $\int p_\varepsilon = +1$

and $\operatorname{supp}(p_\varepsilon) \subseteq \mathbb{D}(0,\varepsilon)$

<u>Thm 6</u>: Take $u \in SH(\Omega)$ and define $u_\varepsilon = u * p_\varepsilon$

$$u_\varepsilon(z) = \int u(z+w)\, p_\varepsilon(w)\, d\,\mathrm{Leb}(w)$$

is $\mathcal{C}^\infty$, subharmonic (wherever it is defined)

on $\Omega_\varepsilon = \{ z \mid \operatorname{dist}(z, \partial\Omega) > \varepsilon \}$

Moreover,

$$\underbrace{u_\varepsilon(z) \searrow u(z)}_{\text{pointwise convergence (\underline{not} uniform)}} \quad \text{as} \quad \varepsilon \to 0$$

<u>Corollary</u>: $u \in SH(\Omega)$. For any $z \in \Omega$

$$r \to M(r,z) = \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})\, d\theta$$

is non-decreasing.

<u>Proof</u>: We know it is true for u_ε, and so can apply monotone convergence theorem.

(Assuming Theorem 6). $\square$

Proof of Thm 6

- $u_\varepsilon \in \mathcal{C}^\infty$, $\mu_\varepsilon \xrightarrow{L^1_{loc}} \mu$ convolution theory.
- $u_\varepsilon(z) = \frac{1}{2\pi}\iint_{\mathbb{D}(0,1)} u(z+\varepsilon w)\, p(|w|)\, d\mathrm{Leb}(w)$

$$= \frac{1}{2\pi}\int_{t=0}^{1} t\,dt \int_0^{2\pi} \underbrace{u(z+\varepsilon t e^{i\theta})}_{M_u(\varepsilon t,\, z)} p(t)\, d\theta$$

<u>Claim</u>: u_ε satisfies the sub-mean value inequality

- μ = measure ≥ 0 supported on $[0,\delta]$.
- $u(z) \leq \frac{1}{2\pi}\int d\mu(t) \int u(z+te^{i\theta})\,d\theta$
- $u \leq u * \varphi_\mu$ since $u \in SH(\Omega)$

$$\left.\right\}\ \varphi_\mu(z) = \hat{\varphi}_\mu(|z|) \text{ and } t\,\hat{\varphi}_\mu(t)\,dt = d\mu(t)$$

<u>Formal computation</u>

$$u * p_\varepsilon \leq (u * \varphi_\mu) * p_\varepsilon$$

because $(u - u*\varphi_u) * p_\varepsilon \geq 0$

By change of variables formula,

$$(u * \varphi_\mu) * \rho_\varepsilon = (u * \rho_\varepsilon) * \varphi_\mu$$

$$\Rightarrow u * \rho_\varepsilon \leq (u * \rho_\varepsilon) * \varphi_\mu$$

$$\Rightarrow u_\varepsilon \in SH(\Omega_\varepsilon)$$

$$u * \rho_\varepsilon * \rho_\eta = \int_0^1 M_{u_\varepsilon}(\eta t, z) \, t \, \rho(t) \, dt$$

true for $\eta > 0$.

Now, $t \longmapsto M_{u_\varepsilon}(\eta t, z)$ is non-decreasing.

and the right hand side above $\searrow u * \rho_\varepsilon$ as $\eta \to 0$. So we proved that

$$u * \rho_\varepsilon * \rho_\eta \searrow u * \rho_\varepsilon$$

as $\eta \downarrow 0$

$\Rightarrow$ For $\eta > 0$, $u * \rho_\varepsilon * \rho_\eta \searrow$ as $\varepsilon \downarrow 0$.

$\varepsilon \longmapsto u * \rho_\varepsilon * \rho_\eta$ is non-decreasing

Let $\eta \downarrow 0$, get $\varepsilon \longmapsto u * \rho_\varepsilon$ is non-decreasing.

$u \in SH(\Omega) \Rightarrow u * \rho_\varepsilon \geq u(z)$ by

Submean value inequality.

• Since u is upper semi-continuous.
for $\varepsilon \ll 1$, $u * \rho_\varepsilon(z) \leq u(z) + \eta$
for any $z, \eta > 0$.

$\underline{\eta > 0}$ $u(z) * \eta$

$\{u < u(z) * \eta\}$ open and contain z.
$\cup|$
$\mathbb{D}(z, \varepsilon)$ for some $\varepsilon > 0$.

<u>Theorem 7</u>: $u \in L^1_{loc}$ such that
$\Delta u \geq 0$ (distribution theory)
($\forall v \in \mathcal{C}_0^2(\Omega)$, $v \geq 0$

$\int u \Delta v \geq 0$)

then there exists a <u>unique</u> $\tilde{u} \in SH(\Omega)$
such that $\tilde{u} = u$ a.e.

<u>Proof</u>: Take $u_\varepsilon = u * \rho_\varepsilon$
$\Rightarrow \int u_\varepsilon \Delta v \geq 0 \quad \forall v \in \mathcal{C}_0^2$
$v \geq 0$

Integration by Parts

$\Longrightarrow \quad \Delta u_\varepsilon \geq 0$

$\overset{\text{thm 5}}{\Longrightarrow} \quad u_\varepsilon \in SH(\Omega)$

$\Longrightarrow \quad u_\varepsilon \downarrow$ as $\varepsilon \downarrow 0$.

from proof of thm 6.

Now define $\tilde{u} = \lim_{\varepsilon \to 0} u_\varepsilon$

- $\tilde{u} = u$ a.e. $\qquad u \in L^1_{loc}$.
- $\tilde{u}$ is u.s.c.
  (because it is decreasing limit of continuous functions)
- $\tilde{u} \in SH(\Omega)$ $\leftarrow$ already did this couple lectures ago. (maybe last lecture).

This shows the existence part.

<u>Uniqueness</u>: If $u = v$ a.e. and $u, v \in SH(\Omega)$

$u_\varepsilon = u * \rho_\varepsilon = v * \rho_\varepsilon = v_\varepsilon$

$\downarrow$ as $\varepsilon \to 0$ $\qquad \downarrow$ as $\varepsilon \to 0$

$u_\varepsilon \searrow u \qquad v_\varepsilon \downarrow v \qquad \rightarrow u = v.$

Theorem 8: $f: \Omega \to \Omega'$ hol.

If $u \in SH(\Omega')$, then $u \circ f \in SH(\Omega)$

(or $u \circ f \equiv -\infty$)

Proof: Reduce to the case $u \in \mathcal{C}^\infty$

using regularization $u_\varepsilon \searrow u$.

We need to show that the Laplacian of $u \circ f$ is ≥ 0.

$$\Delta(u \circ f) = \partial \bar{\partial}(u \circ f) = \partial(\bar{\partial} u \circ f \cdot \bar{\partial} \bar{f})$$

$$= \partial \bar{\partial} u \circ f \cdot |\partial f|^2 \geq 0$$

as desired.

Lecture 18 Thursday, March 12

Plurisubharmonic Functions (p.s.h).

(This is in §3.2. of the book).

$\Omega \subseteq \mathbb{C}^n$ connected, open domain

Def: $u : \Omega \to [-\infty, +\infty)$

We say that u is psh if $u \not\equiv -\infty$ and

1) u is usc (upper semi-continuous)

2) the restriction of u to any $\mathbb{C}$-line is either $\equiv -\infty$ or subharmonic.

For any $z \in \Omega$, any vector $v \in \mathbb{C}^n$,

$t \mapsto u(z + tv)$ is subharmonic.

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta} v)\, d\theta$$

for all $r < \operatorname{dist}(z, \partial\Omega)$

Observation: $n=1 \rightsquigarrow$ psh functions = subharmonic functions.

Basic Properties:

$PSH(\Omega) = \{ u \mid \text{psh on } \Omega \}$

(1) $u, v \in PSH(\Omega)$, $c > 0$ $\Rightarrow$ $cu + v \in PSH(\Omega)$

$\max\{u, v\} \in PSH(\Omega)$

(2) $(u_i)_{i \in I} \in PSH(\Omega)$

$$u = \sup_{i \in I} u_i$$

Suppose that u is usc, and $u < +\infty$, then $u \in PSH(\Omega)$.

(3) $u_n \searrow u$ $\quad u_n \in PSH(\Omega)$

Then: either $u \equiv -\infty$ or $u \in PSH(\Omega)$.

(4) $\varphi : \mathbb{R}^p \longrightarrow \mathbb{R}$ convex, increasing in each variable.

Extend φ by continuity to:

$$[-\infty, +\infty)^p \longrightarrow [-\infty, \infty)$$

$$u_1, u_2, \ldots, u_p \in PSH(\Omega)$$

$$\varphi(u_1, u_2, \ldots, u_p) \in PSH(\Omega)$$

<u>Examples</u>: $f \in \mathcal{O}(\Omega)$ (hol, and $\not\equiv 0$)

$\operatorname{Log}|f| \in PSH(\Omega) \rightarrow \mathcal{E}^0$ $\quad \Omega \rightarrow [-\infty, +\infty)$

<u>More examples</u> : $f_1, f_2, \dots, f_p \in \mathcal{O}(\Omega)$

$$u = \max_{i=1}^{p} \{ \operatorname{Log} |f_i| \} \in PSH(\Omega)$$

$$\{ u = -\infty \} = \bigcap \{ f_i = 0 \}$$

$\uparrow$ analytic subset

- $\operatorname{Log}\left(\sum |f_i|^{\alpha_i} \right) \in PSH(\Omega)$ $\alpha_i > 0$

Take $\varphi = \operatorname{Log}\left(\sum |t_i|^{\alpha_i} \right)$

- $\varphi = |t|^{\alpha}$ in particular $|f|^2 \in PSH(\Omega)$
  $\alpha > 1$ $f \in \mathcal{O}(\Omega)$

<u>Aim</u> : Characterize psh functions in terms of $\partial\bar{\partial}$-operator.

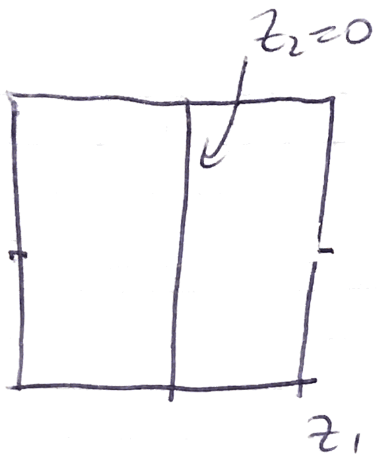
<u>Thm</u>: $u \in PSH(\Omega)$

$\Rightarrow u \in L^1_{loc}(\Omega)$

<u>Proof</u> : By Fubini $(n=2)$

$u(0) > -\infty$ We prove $u \in L^1(\bar{\mathbb{D}})$

$\bar{\mathbb{D}}$ = polydisk centered at $0 \Subset \Omega$.



$u(z_1,0) \in L^1_{loc}(|z_1| \leq 1)$

for a.e. z_1 we have $u(z_1,0) > -\infty$.

$$0 \geq \int_{|z_1|\leq 1} u(z_1;z_2)\, d\mathrm{leb}(z_2) \geq u(z_1,0) > -\infty.$$

Fubini: $\int u(z_1,z_2)\, d\mathrm{leb}(z_1)\, d\mathrm{leb}(z_2)$

$$\underset{\text{Fubini}}{\geq} \int u(z_1,0)\, d\mathrm{leb}(z_1) \geq u(0) > -\infty.$$

Regularization procedure for psh functions

Take a smoothing kernel:

- $\rho : \mathbb{R}^n \longrightarrow \mathbb{R}_+ \quad \mathcal{C}^\infty$

  $\mathrm{Supp}(\rho) \subseteq [-1,+1]^m$

$\rho \equiv +1$ in a neighborhood of 0

$$\int_{\mathbb{R}^n_+} \rho = C > 0.$$

$$\rho_\varepsilon(z_1, \dots, z_n) = \frac{1}{\varepsilon^{2n}} \rho\left(\frac{|z_1|}{\varepsilon}, \dots, \frac{|z_n|}{\varepsilon}\right) \cdot \frac{1}{C \cdot \mathrm{Vol}(S^{2n-1})}$$

$\mathcal{C}^\infty$, $\mathrm{Supp}(\rho_\varepsilon) \subseteq \mathbb{D}^n(0, \varepsilon)$

$$\int \rho_\varepsilon \, d\mathrm{Leb}(z) = +1.$$

<u>Thm</u>: $u \in PSH(\Omega)$, Define

$$u_\varepsilon = u * \rho_\varepsilon$$

Then $u_\varepsilon \in PSH(\Omega_\varepsilon) \cap \mathcal{C}^\infty(\Omega_\varepsilon)$,

and $u_\varepsilon \downarrow u$ as $\varepsilon \downarrow 0$ pointwise.

"Fubini".

$$u_\varepsilon(z) = \int u(z - \varepsilon w) \, \rho(|w_1|, \dots, |w_n|) \, d\mathrm{Leb}(w)$$

Let's just do the case $n=2$

(for simplicity)

$$u_\varepsilon(z) = \int u(z - \varepsilon w)\, p(|w_1|, |w_2|)\, d\mathrm{Leb}(w)$$

$$= \int d\mathrm{Leb}(w) \Big[\int u(z_1 - \varepsilon w_1, z_2 - \varepsilon w_2)$$

$$p(|u_1|, |u_2|)\, d\mathrm{Leb}(w_2) \Big]$$

using Thm 6 from last time

Take $\varepsilon' \leq \varepsilon$

$$\geq \int u(z_1 - \varepsilon w_1, z_2 - \varepsilon' w_2)$$

$$p(|u_1|, |u_2|)\, d\mathrm{Leb}(w_2)$$

$$\geq \int u(z_1 - \varepsilon' w_1, z_2 - \varepsilon' w_2)$$

$$p(|u_1|, |u_2|)\, d\mathrm{Leb}(w_2)$$

$u_\varepsilon \in PSH(\Omega_\varepsilon)$

$$\int u(z - \varepsilon w)\, p(|w|)\, d\mathrm{Leb}(w_2)$$

$$\geq u(z_1 - \varepsilon w_1, 0) \cdot p(|w_1|, 0)$$

We proved $\varepsilon \mapsto u_\varepsilon(z)$ $\nearrow$

and $u_\varepsilon(z) \geq u(z)$

- By using u.s.c, we got $\lim_{\varepsilon \downarrow 0} u_\varepsilon(z) \le u(z)$.

$\Rightarrow u_\varepsilon(z) \downarrow u(z)$ pointwise.

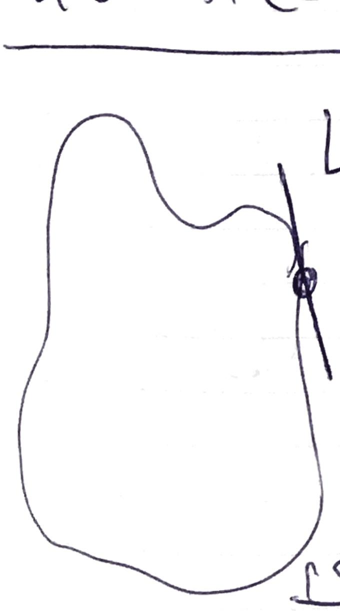
- $u_\varepsilon \in PSH(\Omega_\varepsilon) \rightsquigarrow$ check the submean value inequality on any $\mathbb{C}$-lines.

$$\underset{\substack{\| \\ u * \rho_\varepsilon(z)}}{u_\varepsilon(z)} \underset{(*)}{\le} \frac{1}{2\pi} \int_0^{2\pi} \underset{\substack{\| \\ u * \rho_\varepsilon}}{u_\varepsilon(z + re^{i\theta} v)}\, d\theta$$

Since u satisfies submean value inequality, we get $(*)$ holds.

<u>Thm</u>: $u \in \mathcal{C}^2(\Omega)$

$u \in PSH(\Omega)$ <u>iff</u> $\displaystyle\sum_{1 \le i, j \le n} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \lambda_i \bar{\lambda}_j \ge 0$ for any $\vec{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{C}^n$



$\partial\Omega$

Levi condition

domain of hol.

v.s.

convexity at the boundary.

Proof of Thm: $u \in PSH(\Omega)$

$t \mapsto u(z+t\lambda)$ is $\mathcal{C}^\infty$ and subharmonic.

$\partial_t \overline{\partial_t} u(z+t\lambda) \geq 0$

$$\overline{\partial_t}(u(z+t\lambda)) = \sum_{j=1}^{n} \frac{\partial u}{\partial \overline{z_j}}(z+t\lambda)\overline{\lambda_j}$$

$$\partial_t \overline{\partial_t}(u(z+t\lambda)) = \sum_{1 \leq i,j \leq n} \frac{\partial^2 u}{\partial z_i \partial \overline{z_j}}(z+t\lambda)\lambda_i \overline{\lambda_j}$$

The reverse implication is the same (just follow the steps in back).

Thm: $f : \Omega' \to \Omega$ holomorphic map

$\Omega' \subseteq \mathbb{C}^{n'}$, and $\Omega \subseteq \mathbb{C}^n$ open and connected (domain)

Pick $u \in PSH(\Omega)$. Then:

either $u \circ f \equiv -\infty$, or $u \circ f \in PSH(\Omega')$

Proof : Reduces to $u \in \mathcal{C}^\infty$ by regulization + stability of psh functions along $\downarrow$ sequences!

$$\frac{\partial^2 (u \circ f)}{\partial z_i \partial \bar{z}_j} = \sum_{k,\ell} \frac{\partial^2 u}{\partial w_k \partial \bar{w}_\ell} \circ f \cdot \frac{\partial f_k}{\partial z_i} \cdot \frac{\overline{\partial f_\ell}}{\partial z_j} \geq 0$$

$1 \leq i, j \leq n'$

$(z_1, \dots, z_{n'}) \in \mathbb{C}^{n'}$

$1 \leq k, \ell \leq n$

$(w_1, \dots, w_\ell) \in \mathbb{C}^n$

purely linear algebra argument

Change of basis (at least when $n = n'$)

Complements

(1) $u \in L^1_{loc}$, define $\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}$ as:

$$\left(\frac{\partial^2 u}{\partial z_i \partial_j}, \varphi \right) = \int u \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}$$

linear form

$\varphi \in \mathcal{C}^\infty_0$

Thm : $u \in PSH(\Omega)$, $\lambda \in \mathbb{C}^n$

$$H(u)\cdot\lambda = \sum \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \lambda_i \overline{\lambda_j} \text{ is}$$

a positive measure.

$\forall \varphi \in \mathcal{C}_0^\infty$, $\varphi \geq 0$

$$(H(u)\cdot\lambda, \varphi) \geq 0.$$

Observation : $\frac{\partial^2 u}{\partial z_j \partial \bar{z}_j}$ are positive measures.

diagonal terms

Thm : $v \in L^1_{loc}(\Omega)$ and $H(v)\cdot\lambda \geq 0$

$\Rightarrow \exists u \in PSH(\Omega)$, $v = u$ a.e.

Proof : Define $v_\varepsilon = v * \rho_\varepsilon$ check that $v_\varepsilon \in PSH(\Omega)$, and $v_\varepsilon \downarrow$ as $\varepsilon \downarrow 0$. set $u = \lim_{\varepsilon \to 0} v_\varepsilon$.

$v_\varepsilon \downarrow v$ in L^1_{loc}

$\Rightarrow$ $u = v$ a.e.

There is a natural topology on p.s.h functions.

$PSH(\Omega) \subseteq L^1_{loc}(\Omega)$

is a convex cone.

(a) $PSH(\Omega)$ is closed in $L^1_{loc}(\Omega)$.

$PSH(\Omega) \ni u_n \xrightarrow{L^1_{loc}} u \implies u = \tilde{u}$ a.e.

$\tilde{u} \in PSH(\Omega)$.

(b) Every bounded subset is relatively compact.

$u_n \in PSH(\Omega)$. Suppose that:

$\forall K \subset \Omega$ compact $\quad \sup_n \|u_n\|_{L^1(K)} < +\infty$

$\exists\, u_{n_k} \xrightarrow{L^1_{loc}} u \in PSH(\Omega)$

Sketch of proof when $\Omega \subseteq \mathbb{C}$

$\mu_n = \Delta u_n \geq 0$ on Ω

Fix $\omega \subset \bar{\omega} \subseteq \Omega$

Take $\varphi \in \mathcal{C}^\infty_0 \quad \varphi|_\omega \equiv 1$

We want to check:

$\int \varphi \, d\mu_n$ is bounded.

$$\int \varphi \, d\mu_n = \int \varphi \, \Delta u_n = \int u_n \Delta \varphi$$

But $\int \varphi \, d\mu_n \geq \mu_n(\omega) > 0$.

And for the upper bound,

$$\int u_n \Delta \varphi \leq \int |u_n| \sup_{\Omega} |\Delta \varphi|$$

$$\leq \int_{\mathrm{Supp}(\varphi)} |u_n| \cdot C < C(\varphi) < +\infty.$$

$$\varphi_{\mu_{n_k}} \longrightarrow \mu$$

Suppose ω is a disk.

$$g_\mu(z) = \int \log|z-w| \, d\mu(w)$$

$\Delta g_\mu = \mu \leftarrow$ This is one of the important claims.

$$g_{\varphi_{\mu_{n_k}}} \longrightarrow g_\mu$$

$h_k = g_{\varphi \mu_{n_k}} - \mu_{n_k}$ = harmonic on ω with uniformly bounded L^1-norm.

Then we get bounds on L^∞-norms on h_k and its derivatives

$\Rightarrow$ Pick equicontinuous family and then pick a further subsequence.

$h = h * \rho_\varepsilon$

L^1 norm

we get $\mathcal{C}^k$ bound for $k \geq 1$

$\Rightarrow \mathcal{C}^\infty$-bound.