Observation:

$$J \subseteq \mathcal{I}(V(J), p)$$

$$V(\mathcal{I}_{(Z,x)}, x) = (Z, x)$$

This is exactly going to be the analogue of the usual Hilbert's Nullstellensatz.

Lecture 11 Tuesday, February 11

Def: $x \in \mathbb{C}^n$

A germ of analytic subset (Z, x) at x is irreducible if for any decomposition,

$$(Z, x) = (Z_1, x) \cup (Z_2, x)$$

then:

$$(Z_1, x) = (Z, x) \text{ or } (Z_2, x) = (Z, x)$$

Lemma:

(Z, x) is irreducible $\Longleftrightarrow$ $\mathcal{I}_{(Z,x)} = \{f \in \mathcal{O}_{\mathbb{C}^n, x},\ f|_Z \equiv 0\} \subseteq \mathcal{O}_{\mathbb{C}^n, x}$ is a prime ideal.

Proof ($\Rightarrow$) $fg \in I_{(Z,x)} \Leftrightarrow fg|_Z = 0$

$Z_1 = (f=0)$, $Z_2 = (g=0)$

$Z = Z_1 \cup Z_2$

Since Z is irreducible, WLOG $Z = Z_1$.

$\Rightarrow f \in I_{(Z,x)}$, so $I_{(Z,x)}$ is a prime ideal.

($\Leftarrow$) Assume (Z,x) is reducible.

So, $(Z,x) = (Z_1,x) \cup (Z_2,x)$

$Z_1 \subsetneq Z$, $Z_2 \subsetneq Z$.

$\exists f_1 \in I_{(Z,x)}$, $f_1|_{Z_2} \neq 0$.

$\exists f_2 \in I(Z,x)$, $f_2|_{Z_2} \neq 0$.

$f_1 \notin I_{(Z,x)}$, $f_2 \notin I_{(Z,x)}$

But $f_1 f_2 \in I_{(Z,x)}$

$\Rightarrow I_{(Z,x)}$ is not a prime ideal. $\square$

Theorem 5: Every germ of analytic subset (Z,x) admits a finite decomposition:

$$(Z,x) = \bigcup_{k=1}^{N} (Z_k, x) \qquad (*)$$

where (Z_k,x) are irreducible analytic subsets, and $(Z_k,x) \nsubseteq (Z_\ell,x)$ for any $k \neq \ell$, and this decomposition is unique up to re-ordering.

Proof: (Z,x) - irreducible, we stop.

If it is reducible, can write $(Z,x) = (Z_1,x) \cup (Z_2,x)$.
If (Z_1,x) and (Z_2,x) are irreducible, then we are done. otherwise decompose further.

The existence of the decomposition $(*)$ follows from the following claim: any decreasing sequence of germs of analytic subsets
$(Z_{k+1},x) \subseteq (Z_k,x)$ is stationary.

Since the ring $\mathcal{O}_{(\mathbb{C}^n,x)}$ is Noetherian, the sequence of corresponding ideals

$$I_k \subseteq I_{k+1} \subseteq I_{k+2} \subseteq \dots$$

is stationary: Here $I_k = I(Z_k,x)$

$$I_k = I_{k_0} \quad \text{for all } k \geq k_0$$

$$\Rightarrow (Z_k,x) = (Z_{k_0},x) \text{ for all } k \geq k_0.$$

Uniqueness: Say we had 2 decompositions:

$$(Z,x) = \bigcup_k (Z_k,x) = \bigcup_\ell (W_\ell,x)$$

$$\Rightarrow (Z_k,x) = \bigcup (Z_k \cap W_\ell, x)$$

irreducible $\Rightarrow \exists \ell \; (Z_k,x) = (Z_k \cap W_\ell, x) \subseteq (W_\ell, x)$

Same argument applied to W_ℓ $\Rightarrow (W_\ell,x) \subseteq (Z_{k'},x)$ for some k'

But then $(Z_k,x) \subseteq (W_\ell,x) \subseteq (Z_{k'},x)$

The minimality condition $(Z_m,x) \not\subseteq (Z_n,x)$ for $m \neq n$ now forces:

$$(Z_k,x) = (Z_{k'},x), \text{ i.e. } k = k'$$

Thus, $(Z_k,x) = (W_\ell,x)$. Now remove both of these from both decompositions, and keep going inductively. $\square$

Analytic Nullstellensatz Theorem

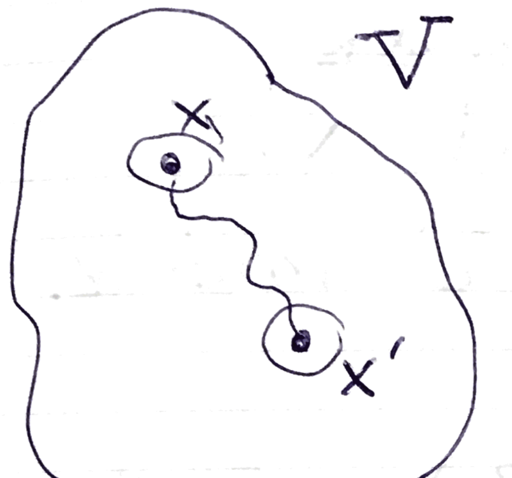
$\mathfrak{a} \subseteq \mathcal{O}_{(\mathbb{C}^n,0)}$ an ideal

Then $\mathcal{I}_{(V(\mathfrak{a}),0)} = \sqrt{\mathfrak{a}} = \{f \mid f^k \in \mathfrak{a}$ for some $k \in \mathbb{N}\}$

§2.5 Oka Coherence Theorem

V = connected complex manifold of dimension n.

$x \in V \rightsquigarrow \mathcal{O}_{V,x}$ = structure of this ring



"principle of analytic continuation" vaguely says that the two rings $\mathcal{O}_{V,x_1}$ and $\mathcal{O}_{V,x_2}$ talk to each other.

$\rightsquigarrow$ read introduction of Grauert-Remmert's book on "Coherent analytic sheaves"!

<u>Set-up</u>: Sheaves of $\mathcal{O}_V$-modules

A sheaf of $\mathcal{O}_V$-modules is a family $\{\mathcal{F}_x\}_{x \in V}$ of $\mathcal{O}_{(V,x)}$-modules such that

$\mathcal{F} = \bigcup_{x \in V} \mathcal{F}_x$ is endowed with a topology such that the map $\pi : \mathcal{F} \longrightarrow V$ sending $\mathcal{F}_x$ to x is a <u>local homeomorphism</u>.

If $\Omega \subseteq V$ open subset,

$\mathcal{F}(\Omega) = \{\text{sections of } \mathcal{F} \text{ over } \Omega\}$

$= \{s : \Omega \to \mathcal{F},\ \mathcal{C}^0,\ \pi \circ s = id_\Omega\}$

<u>Terminology</u>: $\mathcal{F}_x$ is the <u>stalk</u> of $\mathcal{F}$ at x.

<u>Observation</u>: $\mathcal{F}(\Omega)$ is $G(\Omega)$-module.

<u>Equivalent definition</u>:

Ω open set $\longrightarrow \mathcal{F}(\Omega)$ a $G(\Omega)$-module.

$\Omega' \subseteq \Omega$ restriction operators

$$r_{\Omega',\Omega} : \mathcal{F}(\Omega) \to \mathcal{F}(\Omega')$$

$r_{\Omega,\Omega} = id$ and for $\Omega'' \subseteq \Omega' \subseteq \Omega$, we must have

$$r_{\Omega'',\Omega} = r_{\Omega'',\Omega'} \circ r_{\Omega',\Omega}$$

+ 2 extra conditions:

<u>locality</u>: Ω = open, $\Omega = \bigcup_i \Omega_i$

$\sigma, \tilde{\sigma} \in \mathcal{F}(\Omega)$. If $\sigma|_{\Omega_i} = \tilde{\sigma}|_{\Omega_i}\ \forall i$ then $\sigma = \tilde{\sigma}$.

Gluing: If $\sigma_i \in \mathcal{F}(\Omega_i)$ such that $r_{\Omega_i \cap \Omega_j, \Omega_i}(\sigma_i) = r_{\Omega_i \cap \Omega_j, \Omega_j}(\sigma_j)$ $\forall i, j$ then $\exists\, \sigma \in \mathcal{F}(\Omega)$ such that $r_{\Omega_i, \Omega}(\sigma) = \sigma_i$ $\forall i$.

Start with definition 2.

$$\rightsquigarrow \mathcal{F}_x = \varinjlim_{\Omega \ni x} \mathcal{F}(\Omega)$$

topology: basis of open sets on $\mathcal{F} = \bigcup_{x \in V} \mathcal{F}_x$.

Pick $\mathcal{U} \underset{\text{open}}{\subseteq} V$, pick $f \in \mathcal{F}(\Omega)$

$$\Theta(\Omega, f) = \{ f_x \in \mathcal{F}_x,\ x \in \Omega \}$$

Examples:

① structure sheaf: $\{ \mathcal{O}_{V,x} \}_{x \in V}$

for any $p \geq 1$, $\{ \mathcal{O}_{V,x}^{\oplus p} \}_{x \in V}$ $\mathcal{O}_{V,x}$-module (locally) free.

② Ideal sheaves: $\mathcal{F}_x \subseteq \mathcal{O}_{V,x}$ for all x

Ex: Z analytic subset of V

$\rightsquigarrow \left(\mathcal{I}_{(Z,x)}\right)_{x\in V}$

③ $\Omega \subsetneq V$ open subset

~~$\Omega' \subseteq \Omega$, $\mathcal{F}(\Omega') = \mathcal{O}(\Omega$~~

$\mathcal{F}_x = \mathcal{O}_{\Omega,x}$, $x \in \Omega$

$\mathcal{F}_x = 0$, if $x \notin \Omega$.

Def: A sheaf $\mathcal{F}$ (of $\mathcal{O}_V$-modules) is locally finitely generated (of finite type) if for $x \in V$, there exists $\Omega \ni x$ open, and global sections $f_1, \dots, f_N \in \mathcal{F}(\Omega)$ such that for all $y \in \Omega$

$$\mathcal{F}_y = (f_1)_y \cdot \mathcal{O}_{V,y} + (f_2)_y \mathcal{O}_{V,y} + \dots + (f_N)_y \mathcal{O}_{V,y}$$
$$= \langle f_1, f_2, \dots, f_N \rangle.$$

Ex: $\mathcal{O}_V^{\oplus p}$

<u>Def</u>: A sheaf $\mathcal{F}$ of $\mathcal{O}_V$-modules is <u>coherent</u> if:

- it is of finite type.
- for any sheaf morphism $\mathcal{O}_V^{\oplus p} \to \mathcal{F}$ the sheaf of relations is also of finite type.

$\varphi = \{\varphi_x\}_{x \in V}$, $\varphi_x : \mathcal{O}_{V,x}^{\oplus p} \to \mathcal{F}_x$

(continuous, morphism of $\mathcal{O}_{V,x}$-modules).

For any $x \in V$, there exists an open neighborhood $\Omega \ni x$ and ~~$f_1, f_2, \ldots, f_p \in \mathcal{F}(\Omega)$~~ $s_1, s_2, \ldots, s_p \in \mathcal{F}(\Omega)$

$y \in \Omega$,

$$\varphi_x(\underbrace{f_1, \ldots, f_p}_{\in\, \mathcal{O}_{V,x}}) = \sum_{i=1}^{p} (s_i)_x f_i$$

The sheaf of relations $\mathcal{R}_{\varphi,x} = \ker(\varphi)_x \subseteq \mathcal{O}_{V,x}^{\oplus p}$

$$= \{ (f_1, \ldots, f_p) \in \mathcal{O}_{V,x}^{\oplus p}, \sum_{i=1}^{p} (s_i)_x f_i = 0 \}$$

Coherent $\Leftrightarrow$ 1) locally, so, for $x \in V$, $\exists \Omega \ni x$, $\exists p \geq 1$, and a surjective morphism $\mathcal{O}_\Omega^{\oplus p} \to \mathcal{F} \to 0$

and

2) for any such morphism

$$\mathcal{O}_\Omega^{\oplus p} \to \mathcal{F} \to 0$$

$\exists\, q \geq 1$ such that the sequence

$$\mathcal{O}_\Omega^{\oplus q} \longrightarrow \mathcal{O}_\Omega^{\oplus p} \to \mathcal{F} \longrightarrow 0$$

is exact.

Theorem 6 (Oka)

For all $p \geq 1$, $\mathcal{O}_V^{\oplus p}$ is a coherent sheaf.

Hard part: We need to show that for any sheaf morphism

$$f : \mathcal{O}_V^{\oplus q} \longrightarrow \mathcal{O}_V^{\oplus p}$$

the kernel (f) is of finite type.

Let's look at the 3 examples of sheaves we saw

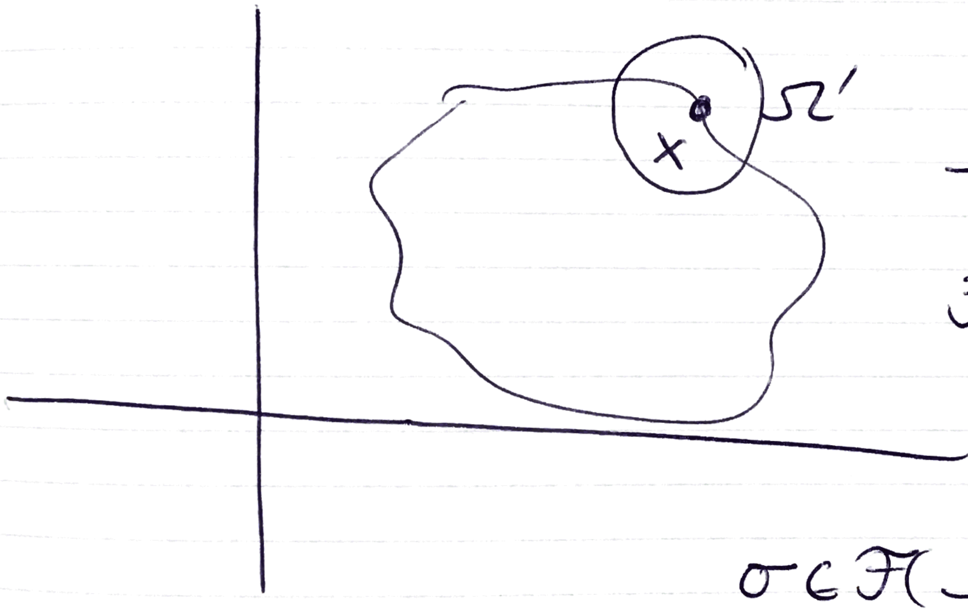
Ex ① finite type (easy)
② Coherent (Oka)

② $(\mathcal{I}_{(Z,x)})$ finite type (Cartan-Oka)

$\Downarrow$ (because it is an ideal sheaf).

Coherent

③ it is not of finite type.



Ω

$\mathcal{F}(\Omega') = (0)$.

$\downarrow$

$\sigma \in \mathcal{F}(\Omega')$

$$\mathcal{O}_{V,y} \cdot (\sigma_y) = \mathcal{F}_y$$

for some $y \in \Omega \cap \Omega'$

but this will not be possible.

Lecture 12 — Thursday, February 13

$V = \mathbb{C}$-manifold.

$\mathcal{F}$ = sheaf of $\mathcal{O}_V$-modules.

2 approaches:

- $\pi : \mathcal{F} \to V$ local homeo with fiber
  $\mathcal{F}_x$ = stalk at $x = \pi^{-1}(x)$ (for $x \in V$).
- $\Omega \rightsquigarrow \mathcal{F}(\Omega)$ which is $\mathcal{O}(\Omega)$-module.
  open set of V
  + restriction $r_{\Omega',\Omega} : \mathcal{F}(\Omega) \to \mathcal{F}(\Omega')$
  - compatibilities (presheaf) $\Omega' \subseteq \Omega$
  +
  - locality & gluing

In this second approach, stalk is $\mathcal{F}_x = \varinjlim_{\Omega \ni x} \mathcal{F}(\Omega)$.

$\mathcal{F}, \mathcal{G}$ sheaves on V.

$\varphi : \mathcal{F} \to \mathcal{G}$ morphism of sheaves of $\mathcal{O}_V$-modules

is the data of $\varphi_\Omega : \mathcal{F}(\Omega) \to \mathcal{G}(\Omega)$

- φ_Ω is a morphism of $\mathcal{O}(\Omega)$-modules.
- $$\begin{array}{ccc} \mathcal{F}(\Omega) & \xrightarrow{\varphi_\Omega} & \mathcal{G}(\Omega) \\ {\scriptstyle r_{\Omega',\Omega}}\downarrow & & \downarrow{\scriptstyle r_{\Omega',\Omega}} \\ \mathcal{F}(\Omega') & \xrightarrow[\varphi_{\Omega'}]{} & \mathcal{G}(\Omega') \end{array}$$

$\rightsquigarrow \varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ morphism of $\mathcal{O}_{V,x}$-modules.

<u>Ex</u>: $\mathcal{F} = \mathcal{O}_V^{\oplus p}$ for $p \geq 1$.

$\mathcal{F}$ is "globally generated" by $(e_1, \dots, e_p)$ over V.

$e_i \in \mathcal{F}(V)$, $e_i = (0, 0, \dots, \underset{\uparrow}{1[i]}, 0, \dots, 0)$

i-th slot (that is what [i] indicates)

$\varphi : \mathcal{O}_V^{\oplus p} \to \mathcal{G}$

$\varphi(e_i) \in \mathcal{G}(V)$

$f_x \in \mathcal{F}_x$

$$f_x = \sum_{i=1}^{p} \underbrace{f_{i,x}}_{\in \mathcal{O}_{V,x}} \cdot e_i$$

$$\varphi(f_x) = \sum_{i=1}^{p} f_{i,x} \, \varphi(e_i)$$

$\varphi : \mathcal{F} \to \mathcal{G}$ morphism of sheaves

- $\ker(\varphi) \subseteq \mathcal{F}$

  $\ker(\varphi)(\Omega) = \ker(\varphi_\Omega : \mathcal{F}(\Omega) \to \mathcal{G}(\Omega))$

  $\Longrightarrow$ is a sheaf.

  $\ker(\varphi)_x = \ker(\varphi_x : \mathcal{F}_x \to \mathcal{G}_x)$

- $\mathrm{Im}(\varphi) \subseteq \mathcal{G}$, $\mathrm{Im}(\varphi)(\Omega) = \mathrm{Im}(\varphi_\Omega : \mathcal{F}(\Omega) \to \mathcal{G}(\Omega))$

  Warning: is a presheaf but is not a sheaf in general!

Ex: $\exp: \mathcal{C}^0(S^1, \mathbb{R}) \to \mathcal{C}^0(S^1, \mathbb{R}_+^\times)$
provides an example where the map is surjection on stalks, but is not surjective on global sections.

The way to make the image a sheaf is the following:

$$\operatorname{Im}(\varphi)_x = \operatorname{Im}(\varphi_x : \mathcal{F}_x \to \mathcal{G}_x)$$

and $\operatorname{Im}(\varphi) = \bigcup_{x \in V} \operatorname{Im}(\varphi)_x \subseteq \mathcal{G}$ with the induced topology.

[Remarks: $\mathcal{F}$ any presheaf $\rightsquigarrow$ we can cook up a sheaf $\mathcal{F}^+$ associated to $\mathcal{F}$ which satisfies the following universal property]

$$\begin{array}{ccc} \text{presheaf } \mathcal{F} & \longrightarrow & \mathcal{F}^+ \text{ sheaf} \\ & \searrow & \downarrow \exists! \\ & & \mathcal{G} \text{ sheaf} \end{array}$$

$\varphi: \mathcal{F} \to \mathcal{G} \rightsquigarrow \ker(\varphi), \operatorname{Im}(\varphi), \operatorname{coker}(\varphi_x) = \mathcal{G}_x / \operatorname{Im}(\varphi)_x$.

Exact sequence

$0 \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ $\quad \ker(\varphi) = 0$

$\iff \forall x, \ \varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective

$\iff \forall \Omega, \ \varphi_\Omega : \mathcal{F}(\Omega) \to \mathcal{G}(\Omega)$ is injective.

$\mathcal{F} \xrightarrow{p} \mathcal{G} \to 0$, $\mathrm{Im}(p) = \mathcal{G}$

$\iff \forall x$, $p_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective.

Theorem 6 (Oka)

For any complex manifold V, the sheaves $\mathcal{O}_V^{\oplus q}$ are coherent for all $q \geq 1$.

Recall that:

- $\mathcal{F}$ is coherent if ① it is of finite type: for all $x \in V$, $\exists \Omega \ni x$, $\exists p \geq 1$

$$\mathcal{O}_V^{\oplus p}\big|_{\Omega} \xrightarrow{p} \mathcal{F}\big|_{\Omega} \to 0$$

and ② for any exact sequence

$$\mathcal{O}_V^{\oplus p}\big|_{\Omega} \xrightarrow{p} \mathcal{F}\big|_{\Omega} \to 0$$

The sheaf of relations $\ker(p)$ is of finite type.

i.e.

$\exists \Omega' \subseteq \Omega$, $\exists r$ such that
$\Omega' \ni x$

$$\mathcal{O}_V^{\oplus r}\big|_{\Omega'} \longrightarrow \mathcal{O}_V^{\oplus p}\big|_{\Omega'} \xrightarrow{p} \mathcal{F}\big|_{\Omega} \to 0$$

Such that the sequence is exact.

Since being coherent is a local property, one only needs to prove Theorem 6 for $V \subseteq \mathbb{C}^n$ (can even assume that V = polydisk).

- $\mathcal{O}_V^{\oplus q}$ is always of finite type

(indeed, just take the identity map

$$\mathcal{O}_V^{\oplus q} \xrightarrow{id} \mathcal{O}_V^{\oplus q} \to 0).$$

So, theorem 6 $\Leftrightarrow$ $\forall \varphi : \mathcal{O}_V^{\oplus p} \to \mathcal{O}_V^{\oplus q} \to 0$, $\ker(\varphi)$ is of finite type

(in fact, we will prove this fact for any morphism φ).

$(\mathcal{S}_q^n)$: For all $\Phi_1, \ldots, \Phi_p \in \mathcal{O}(V)^{\oplus q}$ $\dim(V) = n$, for all $x \in V$, $\exists W \ni x$ open $\exists F_1, \ldots, F_r \in \mathcal{O}(W)^{\oplus p}$ such that

$$R_{x'} = \{(f_1, \ldots, f_p) \in \mathcal{O}_{V,x'}^{\oplus p}, \sum_{i=1}^{p} f_i \Phi_i = 0\}$$

$$= \langle F_1, F_2, \ldots, F_r \rangle$$

$$= \mathcal{O}_{V,x'} F_1 + \mathcal{O}_{V,x'} F_2 + \ldots + \mathcal{O}_{V,x'} F_r.$$

for all $x' \in W$.

We will do double induction

(A) Induction on q.

$$(\mathcal{S}^n_{q-1}) \ \& \ (\mathcal{S}^n_1) \Longrightarrow (\mathcal{S}^n_q)$$

easy.

(B) Prove $(\mathcal{S}^n_1)$ by strong induction, by proving $(\mathcal{S}^{n-1}_q) \ \forall q \Longrightarrow (\mathcal{S}^n_1)$.

(reduce the proof over $\mathcal{O}_{\mathbb{C}^n,0}$ to $\mathcal{O}_{\mathbb{C}^{n-1},0}[w]$ using Weierstrass theory).

(A) Take $\Phi_1, \dots, \Phi_p \in \mathcal{O}(V)^{\oplus q}$

$V = \mathbb{D}(0,1)^n$, $x = 0$.

$(f_1, \dots, f_p) \in \Omega_{x'}$

$$f_1 \begin{pmatrix} \Phi_{11} \\ \vdots \\ \Phi_{1q} \end{pmatrix} + f_2 \begin{pmatrix} \Phi_{21} \\ \vdots \\ \Phi_{2q} \end{pmatrix} + \dots + f_p \begin{pmatrix} \Phi_{p1} \\ \vdots \\ \Phi_{pq} \end{pmatrix} = \vec{0}$$

$\in \mathcal{O}_{V,x'}$

Obs: $(f_1, \dots, f_p) \in \ker \left(\mathcal{O}_V^{\oplus p} \xrightarrow[(\Phi_{11}, \dots, \Phi_{p1})]{} \mathcal{O}_V \right)$

$(\mathcal{S}_1^n) \Rightarrow F_1, \dots, F_\nu \in \mathcal{O}(W)^{\oplus p}$, $0 \in W \subseteq V$

$$(f_1, f_2, \dots, f_p) = \sum_{\ell=1}^{\nu} \varphi_\ell F_\ell$$

$F_\ell = (F_{\ell_1}, \dots, F_{\ell_p})$.

$$(f_1, f_2, \dots, f_p) \in R_{x'} \Leftrightarrow \sum_{i=1}^{p} \Big(\sum_{\ell=1}^{\nu} \varphi_\ell \cdot F_{\ell_i} \Big) \Phi_{i,j} = 0$$

$(f_1, \dots, f_p) = \sum \varphi_\ell F_\ell$ for all $j = 2, 3, \dots, q$.

$\Leftrightarrow (\varphi_1, \dots, \varphi_\nu) \in$ kernel of a morphism

$$\mathcal{O}_V^{\oplus \nu} \to \mathcal{O}_V^{\oplus q-1}.$$

Apply (J^n_{q-1}) to conclude.

B Prove $(\mathcal{S}^{n-1}_q)$ $\forall q \Rightarrow (\mathcal{S}^n_1)$

$\Phi_1, \Phi_2, \dots, \Phi_p \in \mathcal{O}(V)$, V = polydisk $\ni 0$

WLOG, Φ_i are distinguished in w of order $d_i \le D$

$z = (z', w)$.

(where $D = \max(d_1, \dots, d_p)$)

$$\Phi_i(z', w) = w^{d_i} + \sum_{j=0}^{d_i - 1} \varphi_{ij}(z') w^j$$

+Weierstrass theorem.

<u>Lemma</u>: for all $x \in V$,

R_x is generated by polynomials in w with coefficients hol. in z' of degree $\le D$.

$$R_x = \{(f_1, \dots, f_p) \in \mathcal{O}_{V,x}^{\oplus p},\ \sum_{i=1}^{p} f_i \Phi_i = 0\}.$$

$f \in R_x$. By lemma,

$$f = (f_1, \dots, f_p)$$

$$f = \left(\sum_{j=0}^{D} w^j f_{1j}(z'), \dots, \sum_{j=0}^{D} w^j f_{pj}(z') \right)$$

f_{ij} = holomorphic in z'.

$$\sum f_i \Phi_i = 0$$

$$\left(\sum_{j=0}^{D} w^j f_{1,j}(z') \right) \left(w^{d_1} + \sum_{j=0}^{d_1 - 1} \varphi_{1j}(z') w^j \right)$$

$$+ \dots + \left(\sum_{j=0}^{D} w^j \varphi_{pj}(z') \right) \cdot \left(w^{d_p} + \sum_{j=0}^{d_p - 1} \varphi_{pj}(z') w^j \right)$$

$$\| \quad \text{equal to zero.}$$

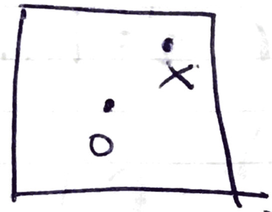
$$0.$$

This is a huge polynomial in w, the coefficients of w^j vanishes for $j = 0, \dots, 2D$ linear in $f_{ij}(z')$. Now apply statement $(\wp_q^{n-1})$ for $q = 2D - 1$ (?)

Proof of the Lemma: $D = \max d_i$

$x = (x', y) \qquad z = (z', w)$.

- Apply Weierstrass preparation theorem to Φ_p at x (for the coordinates $z' - x'$, $w - y$).



$$\Phi_p = ((w-y)+y)^{d_p} + \sum_{j=0}^{d_p - 1} \varphi_{pj}((z'-x')+x') \cdot ((w-y)+y)^j$$

$\Phi_p(x', w) =$ polynomial in $(w-y)$ of degree $= d_p$, and it is monic

It is distinguished in $(w-y)$ of order μ.

$\Phi_p = \varphi \cdot \psi \leftarrow$ unit

$\uparrow$ Weierstrass polynomial in $(w-y)$ of degree μ.

Lemma: φ Weierstrass $f \in \mathcal{O}_{(\mathbb{C}^{n-1},0)}[w]$

$\varphi | f$ in $\mathcal{O}_{(\mathbb{C}^n,0)}$ iff $\varphi | f$ in $\mathcal{O}_{(\mathbb{C}^{n-1},0)}[w]$

$\Rightarrow \psi$ is a polynomial in $(w-y)$ of degree $\nu = d_p - \mu$.

Take $(f_1, \dots, f_p) \in \mathcal{R}_x$

Weierstrass $(f_i)_x = (\Phi_{p,x})\, g_i + r_i$

where $\deg_{w-y}(r_i) < \mu$.
for $i = 1, \ldots, p-1$.

$$\begin{pmatrix} f_1 \\ \vdots \\ f_p \end{pmatrix} = \sum_{i=1}^{p-1} g_i \begin{pmatrix} 0 \\ \vdots \\ \Phi_{p,x} \\ \vdots \\ -\Phi_{i,x} \end{pmatrix} + \begin{pmatrix} r_1 \\ \vdots \\ \boxed{r_p} \end{pmatrix}$$

i-th slot

hasn't been defined yet.

We want the relation ~~to be~~ satisfied
$\Longrightarrow r_p \in \mathcal{R}_x$

$(r_1, \ldots, r_p) \in R_x$

$$\Phi_{p,x} \cdot r_p + \sum_{i=1}^{p-1} \Phi_{i,x} \cdot r_i = 0.$$

$\underset{\mu}{\varphi} \cdot \underset{\nu}{\Psi}$

poly. in $(w-y)$

$< d_i + \mu$.

$\Psi \cdot \varphi r_p$ polynomial of degree $\leq D + \mu$ in $(w-y)$

$\overset{\text{lemma}}{\Longrightarrow}$ $\Psi \cdot r_p$ is a poly in $w-y$ of degree $\leq D + \mu - \mu = D$.

$$\begin{pmatrix} r_1 \\ \vdots \\ r_p \end{pmatrix} = \frac{1}{\psi} \begin{pmatrix} \varphi r_1 \\ \varphi r_2 \\ \vdots \\ \varphi r_p \end{pmatrix}$$

unit + poly. of degree $\nu \leq D$ in $(w-y)$.