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# (5) Archimedean and non-Archimedean norms

⤳ aim is to prove thm (B). To do so we shall need to introduce the machinery of heights.

⤳ need to recall some facts on places of number fields and normed fields.

~~Def~~ Metrized field = $K$ field

$$|\cdot| : K \to \mathbb{R}_+ \quad \begin{cases} |x+y| \leq |x| + |y| \\ |xy| = |x| \cdot |y| \\ |x| = 0 \text{ iff } x = 0. \end{cases}$$

example = trivial norm $K$ any field

$|x|_0 = 1$ if $x \neq 0$ $\quad |x|_0 = 0$

example = archimedean norms

for all $x \neq 0$ $\exists n \in \mathbb{N}$ $|n \cdot x| > 1$.

$K = \mathbb{R}$ or $\mathbb{C}$ $\quad |\cdot|_\infty =$ euclidean norm

**Thm** If $K$ complete metrized field $|\cdot|$ archimedean. Then $(K, |\cdot|) \simeq (\mathbb{R}, |\cdot|_\infty^\varepsilon)$

or $(\mathbb{C}, |\cdot|_\infty^\varepsilon)$ $0 < \varepsilon \leq 1$

indication of proof = archimedean $\Leftrightarrow$ $\sup_{n \in \mathbb{N}} |n \cdot 1| > 1$

in particular car $(K) = 0$. Prime field $= \mathbb{Q}$.

Ostrowski $\quad |\cdot|\Big|_{\mathbb{Q}} = |\cdot|_\infty^\varepsilon$ for some $0 < \varepsilon \leq 1$

then $(K, |\cdot|)$ normed extension of $(\mathbb{R}, |\cdot|_\infty^\varepsilon)$.

Gelfand $\Rightarrow$ $K = \mathbb{R}$ or $\mathbb{C}$ ///

$(K, |\cdot|)$ normed field. TFAE

① $|\cdot|$ is not archimedean

② $|\cdot|$ is non-archimedean i.e. $|x+y| \leq \max\{|x|, |y|\}$

<u>proof</u> ② $\Rightarrow$ $|nx| \leq |x|$ by induction $\Rightarrow$ ①.

① $\Rightarrow$ ② we know $|n| \leq 1$ for all $n \in \mathbb{N}$.

$$|x+y| = |(x+y)^n|^{1/n} = \left|\sum \binom{n}{k} x^k y^{n-k}\right|^{1/n}$$
$$\leq \left(\sum |x|^k |y|^{n-k}\right)^{1/n}$$
$$\leq \max\{|x|, |y|\} \, n^{1/n}$$

///

<u>example</u> $p$ prime $|\cdot|_p$ = $p$-adic norm on $\mathbb{Q}$.

$$\left|p^r \frac{a}{b}\right|_p = p^{-r} \qquad a \wedge p = b \wedge p = 1 \qquad r \in \mathbb{Z}.$$

$\mathbb{Q}_p$ = completion of $\mathbb{Q}$.

<u>Fact</u>: $K, |\cdot|$ non-Archimedean metrized field.

$K^\circ = \{x \in K, |x| \leq 1\}$ is a ring having a unique maximal ideal $K^{\circ\circ} = \{x \in K, |x| < 1\}$.

<u>def</u> $K^\circ$ = ring of integers of $K$

$\tilde{K} = K^\circ / K^{\circ\circ}$ residue field $\pi : K^\circ \to \tilde{K}$.

<u>ex)</u> $K^\circ = \mathbb{Z}_p$ $K^{\circ\circ} = p\mathbb{Z}_p$. $\tilde{K} = \mathbb{F}_p$.

<u>proof of the fact</u>

$K^\circ$ stable by product & addition thanks to ultrametric property.

$K^{\circ\circ}$ ideal, maximal (since units of $K^\circ$ are elements $|x| = 1$)

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## p-adic fields

**Thm** Let $K/\mathbb{Q}_p$ be any finite extension.
Then there exists a unique norm on $K$ extending $|\cdot|_p$.

obs . . $\tilde{K}$ is a finite extension of $\tilde{\mathbb{Q}}_p = \mathbb{F}_p$

$$f_K = [\tilde{K} : \mathbb{F}_p].$$

. $|K^\times| = p^{\mathbb{Z}/e_K}$ for some $e_K \in \mathbb{N}^\times$.

One can show $e_K f_K = [K : \mathbb{Q}_p]$

exercise : prove $\leq$.

proof relies on several important principles.

x uniqueness. $|\cdot|_1$ and $|\cdot|_2$ two norms
fix a basis $K = \mathbb{Q}_p e_1 + \dots + \mathbb{Q}_p e_n \simeq \mathbb{Q}_p^n, \|\cdot\|_\infty$

$$|x|_1 \leq (\max |e_i|_1) \max |x_i|_p \quad x = \sum x_i e_i.$$

$\|x\|_\infty$

hence $x \longmapsto |x|_1$ is continuous for the product norm $\|\cdot\|_\infty$.

$\alpha = \inf_{\|x\|_\infty = 1} |x|_1 > 0$ because $\{\|x\|_\infty = 1\} \subseteq \mathbb{Z}_p^n$ is compact.

$$|x|_1 \geq \alpha \|x\|_\infty.$$

$\Rightarrow$ $|\cdot|_1$ & $|\cdot|_2$ are equivalent

$$\Rightarrow \quad |x|_1 = |x^n|_1^{1/n} \leq C^{1/n} |x^n|_2^{1/n}$$

$$\Rightarrow \quad |x|_1 = |x|_2$$

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17/ * existence

recall the definition of the relative norm

$$N_{K/\mathbb{Q}_p} : K^\times \to \mathbb{Q}_p^\times .$$

$$N_{K/\mathbb{Q}_p}(x) = \det(y \mapsto xy)$$

$$= \prod_{\sigma \in Gal(K/\mathbb{Q}_p)} \sigma(x) .$$

if $K/\mathbb{Q}_p$ Galois

if $x \in \mathbb{Q}_p$, $N_{K/\mathbb{Q}_p}(x) = x^n$ $\quad n = [K : \mathbb{Q}_p]$.

set $f(x) = |N_{K/\mathbb{Q}_p}(x)|_p^{1/n} : K^\times \to \mathbb{R}_+$ $\quad f(0) = 0$.

- $f = 0$ iff $x = 0$
- $f(xy) = f(x) f(y)$

Before $K = e_1 \mathbb{Q}_p + \dots + e_n \mathbb{Q}_p$ $\quad \|x\| = \max |x_i|$

exercise : $f$ is continuous (for $\|\cdot\|$) ~~...~~.

by compacity $\exists C > 1$ $\frac{1}{C} \leq f(x)$ for all $\|x\| = 1$

and $f(x) \leq C$ for all $\|x\| \leq 1$.

$$f(x+y) \underset{\|y\| \geq \|x\|}{=} f(y) f\left(1 + \frac{x}{y}\right) \leq C' f(y)$$

$$f(x) = f\left(\frac{x}{y}\right) f(y) \leq C' f(y)$$

$$f(x+y) \leq C'' \max\{f(x), f(y)\}$$

use $f(x+y) = f((x+y)^n)^{1/n}$ to conclude

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observation for any $\sigma \in Gal(K/\mathbb{Q}_p)$ $\quad |\sigma(x)| = |x|$. by uniqueness of the extension.

R/ **the field $\mathbb{C}_p$** = p-adic complex numbers

recall $\mathbb{Q} \xrightarrow[\text{completion}]{} \mathbb{R} \xrightarrow[\text{alg closure}]{} \mathbb{C}$ both complete and alg. closed.

p prime $\mathbb{Q} \xrightarrow[\text{completion}]{} \mathbb{Q}_p \xrightarrow[\text{alg closure}]{} \mathbb{Q}_p^{alg}$

~~[illegible]~~ **observation** : there exists one and only one norm on $\mathbb{Q}_p^{alg}$ which extends $|\cdot|_p$.

**Fact** = $\mathbb{Q}_p^{alg}$ is not complete

**exercice** $F_n = \{x \in \mathbb{Q}_p^{alg} \mid \deg(x) = n\}$ closed, non-empty, and has empty interior (contradicts Baire theorem).

~ Koblitz p-adic numbers § III.4.

**Thm** | $\mathbb{C}_p$ = completion of $\mathbb{Q}_p^{alg}$ is algebraically closed.

**observation** × $|\mathbb{C}_p^{\times}| \underset{\uparrow}{=} |\mathbb{Q}_p^{alg\,\times}| \underset{\uparrow}{=} p^{\mathbb{Q}}$.

$\left(|x-y|<|x| \Rightarrow |x|=|y|\right)$ $\quad$ $\left(a_n x^n + \dots + a_0 = 0 \Rightarrow |a_i x^i| = |a_j x^j|\right)$.

× $\widetilde{\mathbb{C}_p} \underset{\uparrow}{=} \widetilde{\mathbb{Q}_p^{alg}} = \mathbb{F}_p^{alg}$ × $\mathbb{C}_p$ is not locally compact.

density of $\mathbb{Q}_p^{alg}$ in $\mathbb{C}_p$ (exercice).

**proof**. $x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$. $a_i \in \mathbb{C}_p$.

$\mathbb{Q}_p^{alg} \ni a_i^{(j)} \to a_i$ $\quad P_j = x^n + \sum_{i \le n-1} a_i^{(j)} x^i$. $x_1^{(j)} \dots x_n^{(j)}$ roots

(claim) $\exists A$ $|x_i^{(j)}| \le A$ for all $i, j$.

→ build by induction a sequence $x_{i_j}^{(j)}$ s.t.

19/ $\left|P_{j+1}\left(x^{(j)}_{i_j}\right)\right| = \left|\prod_{i\geq 1}^{n} x^{(j)}_{i_j} - x^{(j+1)}_{i}\right| = \left|P_j - P_{j+1}\left(x^{(j)}_{i_j}\right)\right|$

$$\leq \varepsilon_j A^n$$

where $\varepsilon_j = \max_i |a^{(j)}_i - a^{(j+1)}_i|$.

$\rightsquigarrow$ choose $x^{(j+1)}_{i_{j+1}}$ s.t. $|x^{(j)}_{i_j} - x^{(j+1)}_{i_{j+1}}| \leq \varepsilon_j^{1/n} A$

proof of the claim: $A \geq \sup\{1, |a_i^{(j)}|\}$ if $J = \{j, |x_i^{(j)}| \geq A\} \neq \emptyset$, symmetric poly of order $J$ of $x_i^{(j)}$ has norm $\geq A$. absurd. ///

Final remarks (*) one can show that $\mathbb{C}_p$ is isomorphic to $\mathbb{C}$ (as a field) see Roberts §3.3-5.

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## 6 Norms on number fields

$K = \mathbb{Q}$.

Thm (Ostrowski)
Any (multiplicative) norm on $\mathbb{Q}$ is of the following form

- $|\cdot|_0$ trivial norm
- $|\cdot|_\infty^\varepsilon$ archimedean norm $0 < \varepsilon \leq 1$
- $|\cdot|_p^t$ non-archimedean norm $p$ prime $t > 0$.

! Koblitz §.1-2.

Indication of proof when $|\cdot|$ is non trivial and non-archimedean

i.e $|n| \leq 1$ for all $n \in \mathbb{N}^\times$ (non-arch), $\exists n_0 \in \mathbb{N}^\times$ $|n_0| < 1$ (non-trivial)

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