8/ Suppose two embeddings define the same norm

$$\begin{tikzcd} & \mathbb{C} \\ K \arrow[ur, hook, "\sigma"] \arrow[dr, hook, "\sigma'"'] & \\ & \mathbb{C} \end{tikzcd}$$

then ~~we~~ ~~[illegible]~~ the identity map on K induce a field automorphism on the completions

$$\hat{K}_\sigma \xrightarrow{\varphi} \hat{K}_{\sigma'}$$ which is continuous (isometric)

$\Rightarrow$ if $\hat{K}_\sigma = \mathbb{R}$ then $\hat{K}_{\sigma'} = \mathbb{R}$ and $\varphi = id$

if $\hat{K}_\sigma = \mathbb{C}$ then $\hat{K}_{\sigma'} = \mathbb{C}$ and $\varphi = id$ or $z \mapsto \bar{z}$.

$M_{K,p} = \{ |.|$ multiplicative norms on K, $|.||_{\mathbb{Q}} = |.|_p \}$ p prime

$\rightsquigarrow$ same analysis $P = \prod_{i=1}^{r_p} P_i$ P_i irreducible over $\mathbb{Q}_p$

$\deg(P_i) = n_i$.

<u>Thm</u>

- $M_{K,p}$ has exactly r_p elements.
- For each $v \in M_{K,p}$ let $n_v = [K_v : \mathbb{Q}_v]$

then $$\prod_{v \in M_{K,p}} |x|_v^{n_v} = |N_{K/\mathbb{Q}}(x)|_p \quad \forall x \in K.$$

<u>obs</u>: $[K_v : \mathbb{Q}_v]$ can be arbitrary large.

<u>proof</u>. $\mathbb{Q} \subseteq \mathbb{Q}_p \subseteq \mathbb{Q}_p^{alg \text{ - fixed}} \subseteq \mathbb{C}_p$. zeros of $P = \{x_1 \dots x_n\}$

$|.| \in M_{K,p}$ $\quad K \subseteq \hat{K}$

<u>claim</u>: $\hat{K}/\mathbb{Q}_p$ is a finite extension

- Pick any inclusion $\hat{K} \subseteq \mathbb{Q}_p^{alg}$ respecting $\mathbb{Q}_p$: maybe several but they are all isometric since $\hat{K}$ admit a <u>unique</u> metric

$\Rightarrow \exists\, \sigma : K \hookrightarrow \mathbb{C}_p$ $|x| = |\sigma(x)|_p$ $\sigma = \sigma_i$ for some i $\sigma_i(T) = x_i$

- if $\sigma_i, \sigma_j : K \hookrightarrow \mathbb{C}_p$ $|\sigma_i(x)|_p = |\sigma_j(x)|_p$ then $\hat{K}_i \simeq \hat{K}_j$ isometric field automorphism identity on K

$\Rightarrow \exists\, \varphi \in Gal(\mathbb{Q}_p^{alg}/\mathbb{Q}_p)$ $\sigma_i = \varphi\, \sigma_j$.

$\Leftrightarrow$ x_i & x_j are zeros of the same irreducible factor of P over $\mathbb{Q}_p$ $\hookrightarrow$ TSVP