__proof of the claim__ $K = \mathbb{Q} e_1 \oplus \dots \oplus \mathbb{Q} e_s$ $\quad e_i \in K$.

~~The two norms $|\cdot|_K$ and the product norm on~~

~~$\mathbb{Q}_p e_1 \oplus \dots \oplus \mathbb{Q}_p e_p \subseteq \hat{K}$~~

Look at $V = \mathrm{Vect}_{\mathbb{Q}_p}(e_1, \dots, e_s) \subseteq \hat{K}$

$\mathbb{Q}_p e_1 + \dots + \mathbb{Q}_p e_s$

Two norms on V : the norm from $\hat{K}$ and a product norm given by a basis $(e_{i_1}, \dots, e_{i_t})$ of V. They are equivalent (see proof of thm on unique extension)

Hence V is complete and $V = \overline{K} \Rightarrow \dim_{\mathbb{Q}_p} \hat{K} \leq \dim_{\mathbb{Q}} K < \infty$ $\square$

__Product formula__ : for $x \in K$

$$\prod_{v \in M_K} |x|_v^{n_v} = 1$$

proof)

$p \in M_{\mathbb{Q}}$ fixed $\quad \prod_{v \in M_{K,p}} |x|_v^{n_v} = |N_{K/\mathbb{Q}}(x)|_p$

$$\prod_{v \in M_K} |x|_v^{n} = \prod_{p \in M_{\mathbb{Q}}} |N_{K/\mathbb{Q}}(x)|_p = 1$$

$\square$