Barcelona lectures April 2008.

4 times 2 hours.

$\boxed{\text{TALK 1}}$ = Presentation of the main results

- general intro [informal] 10
- one-dimensional case 15
- basics on the alg. structure of polynomial maps 45
- examples. 30
- main results. 10

(I) (10 min) . $F(x,y) = (P(x,y), Q(x,y)) : \mathbb{C}^k \to \mathbb{C}^k$ (mainly $k=2$)

→ study its dynamical properties

ie describe $\{F^n(z)\}_{n\geq 0}$ $F^{n+1} = F \circ F^n$.

two expected behaviours.

× "regular dynamics": perturb $z \rightsquigarrow z'$ does not affect the qualitative behaviour of the orbit

× "chaotic dynamics": small perturbation have drastic consequences. Sensitivity to initial conditions.

[not precise but we shall see in the 1D case what this means]

two zones

$\mathcal{F}$ = open sets of dyn. regular points

hope to describe exactly the behaviour of $F^n(z)$.

$\mathcal{J}$ = closed set where dyn. is unstable.

use stochastic / probabilistic methods to say what the behaviour of $F^n(z)$ is for a.e z w.r.t some measure. In most cases, there is a privileged measure [in particular in the complex case].

→ this is the program!

<u>in 1D</u> complete

<u>in 2D</u> very few results.

- × almost nothing on the regular part !!!
- × most efforts have been devoted to the construction of a nice invariant measure [ergodic] with some success [$h=2$ almost complete].

methods in 2D = analytic / potential theory
[psh fcts, positive closed currents]
but use in an <u>essential</u> way the algebraic nature of F

<u>Do here</u> no analytic tools → no construction of ergodic measures
[nice texts of Guedj–Sibony]

- present the "algebraic" results needed to apply their construction
- hoping you will find the results interesting in their own right.

2/ (15 min) (II) $k=1$ $\quad F(z) = z^d + a_1 z^{d-1} + \dots + a_d \quad d \geq 2$

Find a big open set where the dyn. is simple :

$|z| \gg 1 \quad |F(z)| \simeq |z|^d \to +\infty$

$$\frac{|F(z)|}{|z|^d} = \left| 1 + \frac{a_1}{z} + \dots + \frac{a_d}{z^d} \right|$$

R st. $\left| \frac{a_k}{R^k} \right| \leq \frac{1}{2d}$ & $|z| \geq R > 1$

$$\frac{3}{2} \geq \frac{|F(z)|}{|z|^d} \geq 1/2$$

$\boxed{\text{Prop}}$ $\exists\, C_0, C_1 \quad C_0 \geq \log \max\{1, |F(z)|\} - \frac{1}{d} \log \max\{1, |z|\} \geq C_1$

Consequences.

1. $\frac{1}{d^n} \log \max\{1, |F^n|\}$ cvg uniformly on $\mathbb{C}$ to g_F
2. g_F is $\mathcal{C}^0$, ≥ 0 $\quad g_F \circ F = d\, g_F \quad g_F - \log \max\{1, |z|\} = O(1)$

$\boxed{\text{Thm}}$ 3. $\Omega = \{g_F > 0\}$ open $\quad \ni z \quad |F^n(z)| \geq (g_F(z) - \varepsilon)^{d^n} \to +\infty$.
$K_F = \{g_F = 0\}$ filled-in Julia set compact
$\subset \{|F^n(z)| = O(1)\}$

Comments

$\mathcal{F}$ = regular part of the dyn. $= \Omega \cup \mathrm{Int}(K_F)$
$= \{z,\ \exists\, U \text{ nbd} \ni z \quad F^n|_U \text{ have bdd derivatives}\}$.
$\to$ classification Fatou Julia Sullivan

$\mathcal{J}$ = chaotic $= \partial K_F$

$\boxed{\text{Thm}}$ Brolin

1. $\mu = \Delta g_F$ is a proba. measure $\quad \mathrm{Supp}(\mu) = \mathcal{J}$.
2. for a.e. $z \in \mathrm{Supp}(\mu) \quad \frac{1}{N} \sum_0^{N-1} \delta_{F^n(z)} \to \mu$.
3. $\frac{1}{d^n} \sum_{F^n(z_i) = a} \delta_{z_i} \to \mu$

45 min

3/ (III). rest of the talks

$$F = (P, Q) : \mathbb{C}^2 \to \mathbb{C}^2 \qquad z = (x, y) \in \mathbb{C}^2$$

→ shall try to generalize the results of the previous part.

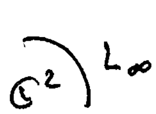
→ present basic objects / invariants attached to F come from alg. geometry, better to work with compact spaces.

1. Embed $\mathbb{C}^2 \subseteq \mathbb{P}^2(\mathbb{C}) \ni p = [x_0 : x_1 : x_2] \sim [t x_0 : t x_1 : t x_2]$
$t \in \mathbb{C}^*$.

$$(x, y) \longmapsto [x : y : 1]$$

$$L_\infty = \mathbb{P}^2 \setminus \mathbb{C}^2 = \{ [x : y : 0] \} \simeq \mathbb{P}^1(\mathbb{C})$$

line at infinity



2. $F : \mathbb{C}^2 \to \mathbb{C}^2$ extends to rational map
$\tilde{F}$ (or simply F) : $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$

$d = \deg(F) = \max \{ \deg(P), \deg(Q) \}$

$$\tilde{F} = \left[P\left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right) x_2^d : Q\left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right) x_2^d : x_2^d \right] = [\tilde{P} : \tilde{Q} : x_2^d]$$

indeterminacy set $I(F) = \{ \tilde{P} = \tilde{Q} = 0 \} \cap L_\infty$

finite set.

(Ex) $I(F) = \phi \iff P_d^{-1}(0) \cap Q_d^{-1}(0) = \{0\}$

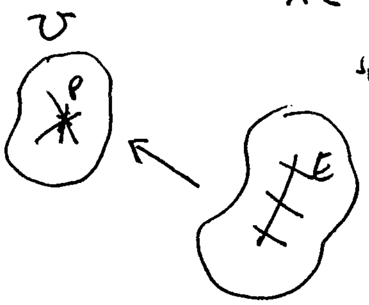
$P = P_d + \text{l.o.t.} \qquad Q = Q_d + \text{l.o.t.}$

(Ex) $F = (x^3 + y, x^2 + y^2) \qquad \tilde{F} = (x_0^3 + x_1 x_2^2, x_0^2 x_2 + x_1^2 x_2, x_2^3)$

$I = [0 : 1 : 0]$.

3. Blow-up of a point. $p \in X$ [complex surface smooth]

$$X \xleftarrow{\pi_p} \hat{X}$$



smooth
- hol. map isomorphism outside p.

  $\pi_p : \hat{X} \setminus \pi_p^{-1}(p) \xrightarrow{\sim} X \setminus \{p\}$.
- $\pi_p^{-1}(p) = E$ exceptional set $\approx \mathbb{P}^1(\mathbb{C})$ smooth

meaning = replace p by the complex directions passing through it.

<u>Fundamental procedure</u>

local coordinates $(x,y) \in \mathbb{C}^2$

$$\hat{X} \subseteq \{ (x,y) \times [z_0 : z_1] \in \mathbb{C}^2 \times \mathbb{P}^1(\mathbb{C}) \}$$
$$z \quad x \quad e$$

$\hat{X} =$ closure of $\{ z \in e \}$

Thm $\exists\, \mu : X \to \mathbb{P}^2$ composition of pt blow-ups.

$$\begin{array}{ccc} X & \xrightarrow{\hat{F}} & \mathbb{P}^2 \\ \mu \downarrow & \nearrow & \\ \mathbb{P}^2 & \overset{F}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

$\hat{F}$ is holomorphic.

Log: $p \in I(F)$ "$F(p)$" = union of rational curves.

4. **def** F is dominant iff $\exists z$ det $DF_z \not\equiv 0$.

(ex) non dominant $F(x,y)$ depends only on x.

Prop F is not dominant iff.
1) either F is a constant map
2) or $\exists$ irreducible curve $C \subseteq \mathbb{C}^2$
   $F(\mathbb{C}^2) \subseteq C$.

proof $\hat{F} : X \longrightarrow \mathbb{P}^2$ hol. $\hat{F}(X) = Z$ compact subspace of $\mathbb{P}^2$
irreducible

$k = \mathrm{rk}\, D\hat{F}_z$ z generic

~~Stein factorization~~

$$\begin{array}{ccc} X & \xrightarrow{\hat{F}} & \mathbb{P}^2 \\ & g \searrow & \nearrow \\ & & Z \end{array}$$

connected fibers. finite

4/ ~~[illegible] generic~~

~~[illegible] generic~~

~~[illegible] $k=2$~~

$k=2$ then locally $\hat{F}$ is invertible $Z=\mathbb{P}^2$.

$k=1$ then locally $\hat{F}$ is a projection $(x,y)\mapsto x$ $\dim(Z)=1$

$k=0$ locally $\hat{F}=$ cte $\Rightarrow$ $\hat{F}\equiv$ cte $\square$.

<u>rmk</u> = normalization of $C_i \simeq \mathbb{C}$.

$\boxed{\text{Prop}}$ F dominant $\Rightarrow$ $F(\mathbb{C}^2)\supseteq \mathbb{C}^2\setminus Z$ Z affine curve.

<u>proof</u> $\hat{F}: X\to\mathbb{P}^2$ hol. surjective

$$Z = \hat{F}(L_\infty)\setminus L_\infty = \hat{F}(L_\infty)\cap\mathbb{C}^2. \quad \square.$$

Ex $(x,y)\longmapsto(x,xy)$ $F(\mathbb{C}^2)=\mathbb{C}^*\times\mathbb{C}\cup\{(0,0)\}$

5. topological degree $=e$.

$\boxed{\text{Prop}}$ $\left[\begin{array}{l} F \text{ dominant} \\ \exists\, Z \text{ affine curve } \forall p\in\mathbb{C}^2\setminus Z \quad \# F^{-1}\{p\}=e\geq 1\end{array}\right.$

<u>proof</u> $\hat{F}: X\to\mathbb{P}^2$ $\quad J\hat{F}=$ critical set of $\hat{F}$

$$\hat{F}: X\setminus \hat{F}^{-1}\hat{F}(J\hat{F}) \to \mathbb{P}^2\setminus\hat{F}(J\hat{F})$$

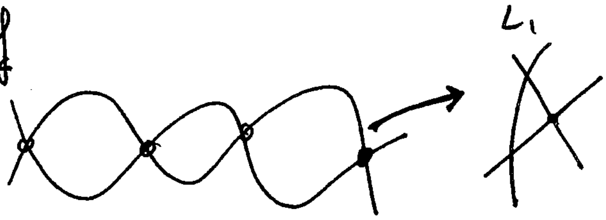
unramified <u>finite</u> cover

because p generic and $\#\hat{F}^{-1}(p)=+\infty$

$\Rightarrow$ $\hat{F}$ has 1D fibers. $\square$

$\boxed{\text{Prop}}$ $e\leq \deg(F)^2$

<u>proof</u>



L_1 L_2 2 lines

$p=L_1\cap L_2$ $\quad$ p generic

$\square$

$\underline{\text{rmk}}$ $$e = d^2 - \sum_{p \in I(F)} \mu_p$$ not that easy to compute.

6. Asymptotic degree.

F dominant

$\boxed{\text{Prop}}$ $$\boxed{\begin{array}{l} e(F^n) = e(F)^n \quad \forall n \\ \deg(F^{n+m}) \leq \deg(F^n)\deg(F^m) \quad \forall n,m \end{array}}$$

proof $F^n = [\tilde{P}_n : \tilde{Q}_n : z^{d_n}]$

$F^n(F^m) = F^{n+m} = [\tilde{P}_n(\tilde{P}_m, \tilde{Q}_m, z^{d_m}) : \quad , z^{d_n d_m}]$.

may have z divides the first factors. $\square$

def $$\lambda(F) = \lim \deg(F^n)^{1/n}$$

$\boxed{\text{Prop}}$ $$\boxed{\begin{array}{l} 1 \leq e \leq \lambda(F)^2 \\ \forall \phi \text{ birational} \qquad \lambda(\phi \circ F \circ \phi^{-1}) = \lambda(F). \end{array}}$$

proof $\deg(F \circ G) \leq \deg(F) \times \deg(G)$ $\square$.

$\to$ important property: even with $\phi \in \mathrm{Aut}(\mathbb{C}^2)$

$\deg(\phi \circ F \circ \phi^{-1}) \neq \deg(F)$!

$\underline{\text{Main quest}}$

$\to$ describe $\{\deg(F^n)\}_{n \geq 0}$ and $\lambda(F)$.

$\underline{\text{Motivation}}$. beside its intrinsic interest.

$|F^n(z)| \simeq |z|^{d_n}$ for fixed n and $|z| \gg R_n$

but R_n might tend to ∞.

$\boxed{\text{Thm}}$ suppose $d \overset{?}{=} d(F^n) = \deg(F)^n \quad \forall n$

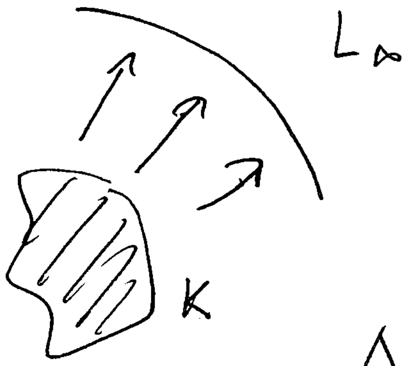
then $\frac{1}{d^n} \log \max \{1, |F^n(z)|\} \longrightarrow g_F \not\equiv 0$.

<u>Hope</u> control of $\deg(F^n)$ yields enough information to show $\frac{1}{\deg(F^n)} \log \max \{1, |F^n|\}$ converges.

6/ (IV) Examples. (30 min)

1. Holomorphic maps of $\mathbb{P}^2$

$F = (P_d, Q_d) + \text{lot}$ $\qquad$ $F_\infty = [P_d : Q_d]$ hol. map of degree d on L_∞



$e = d \qquad \lambda = d^2$

$$g_F = \lim \frac{1}{d^n} \operatorname{Log} \max\{1, |F^n|\}$$

→ same thm is true as in 1D.

⚠ chaotic set $\neq \partial\{g_F = 0\}$ as it contains chaotic set of F_∞.

$$\mu = (dd^c)^2 g_F \qquad \operatorname{Supp}(\mu) \subseteq \partial\{g_F = 0\}.$$

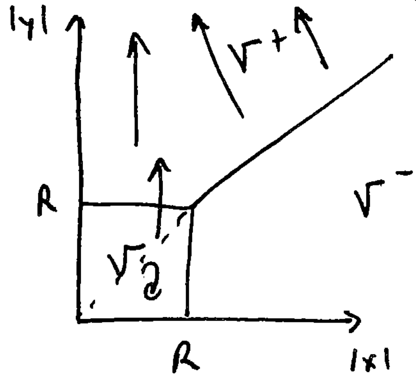
2. Automorphisms of $\mathbb{C}^2$

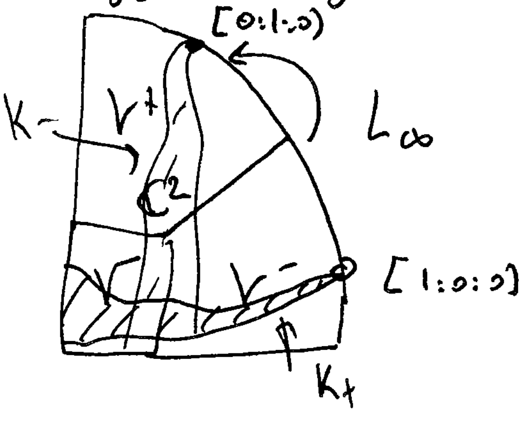
$F \circ F^{-1} = id \qquad F^{-1} \in \operatorname{Aut}(\mathbb{C}^2)$

Well-known Structure thm Jung → rephrased Friedland-Milnor

$$(x,y) \longmapsto (y, ax + P(x)) \qquad \deg P \geq 2 \qquad a \neq 0.$$
$$= (x_1, y_1)$$

$e = 1 \qquad \lambda = \deg(P) \qquad \deg(F^n) = \deg(F)^n.$



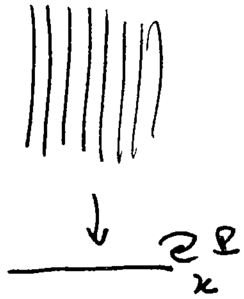


using H^{-1} $\qquad K_+ = \{F^n(z) \notin V^+ \text{ for } n \geq 0\}$
$\qquad = \{|F^n(z)| = O(1) \quad n \geq 0\}$

→ NOT COMPACT [although take ...]

3. Skew products

$$F(x,y) = (P(x),\ Q(x,y)) \quad \text{map which preserves the fibration}$$
$$= (x^p + o(x^{p+1}), \qquad \{x = \text{cte}\}.$$
$$A_q(x)\, y^q + \dots + A_0(x))$$



$e = p \times q$

<u>Fact</u>: $\lambda = \max\{p, q\}$

<u>proof</u> $F^n = (P^n, Q_n)$

$$F^{n+1} = (P^{n+1}, Q_{n+1}) = F(P^n, Q_n)$$
$$Q_{n+1}(x,y) = Q(P^n(x), Q_n(x,y))$$
$$\deg_y Q_{n} = q^n \qquad B_{q^n}(x)\, y^{q^n} + \dots + B_0(x)$$
$$\deg_x Q_n \leq \max\{\deg A_i\} \times p^n + q \times \deg_x(Q_n)$$
$$\leq D\,(p^n + p^{n-1} q + \dots + q^n)$$
$$\deg_x(Q_n) \geq \deg A_q \times p^n + q \times \deg A_q. \qquad \square.$$

<u>$e = \lambda^2$</u> $\iff p = q$

$$e < \lambda^2 \Rightarrow \deg(F^n) \sim \lambda^n$$
$$e = \lambda^2 \Rightarrow \deg(F^n) \sim \lambda^n \text{ if } \deg A_q = 0$$
$$\deg(F^n) \sim n\lambda^n \text{ if } \deg A_q > 0.$$

$g_Q \rightsquigarrow$ on $\{g_Q > 0\}$ $|F^n(z)| \simeq c^{d_Q(F^n)}$

$\rightsquigarrow$ on $\{g_Q = 0\}$

show that $g_Q = \lim_n \frac{1}{q^n} \log \max\{1, |Q_n|\}$

- in general the situation is delicate might have some x for which $g_Q(x, \cdot) \equiv 0$.
- for all x [wrt Δg_Q] $g_Q(x, \cdot) \not\equiv 0$.
  - $\to$ $g_Q > 0 \Rightarrow |F^n| \to \infty$
  - $\to$ ⚠ if $\{g_Q = 0\}$ nothing can be said.

$$\int f(x,y)\, d\mu = \int_{x \in J_Q} \left[\int_{y \in \mathbb{C}} f(x,y)\, \Delta g_Q(x,y) \right] \Delta g_Q(x)$$

Thm
1) μ is ergodic
2) $\mathrm{Supp}(\mu)$ is compact $\Leftrightarrow$ $A_q^{-1}(0) \cap J_Q = \emptyset$

$\to$ if $A_q^{-1}(0) \cap J_Q \neq \emptyset$ $\exists$ ~~points~~ z
Cluster set of $\{F^n(z)\}$ is a closed and <u>unbdd set</u> of $\mathbb{C}^2$
<u>OSCILLATION</u>

4. Monomial maps.

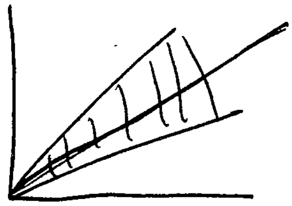
$$(x,y) \xmapsto{F_M} (x^a y^b, x^c y^d) = z^M \qquad M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$F_M^n = F_{M^n}$$

$e = |\det(M)|$ [look at what happens on $S^1 \times S^1$]

$\deg(F^n) = \max\left\{(1,1)\, M^n \binom{1}{0} ,\ (1,1)\, M^n \binom{0}{1}\right\}$

<u>case 1</u>



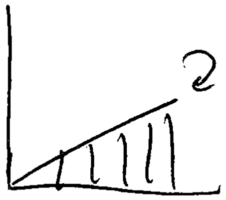
M^2

Perron – Frobenius

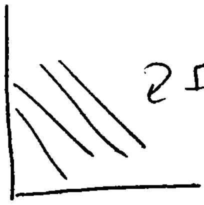
$\exists !$ eigenvector $u \in \mathbb{R}^2_+$

$Sp(M) = \{\lambda > \lambda'\}$. $\quad Mu = \lambda u$

<u>case 2</u>



$\sim M^2$

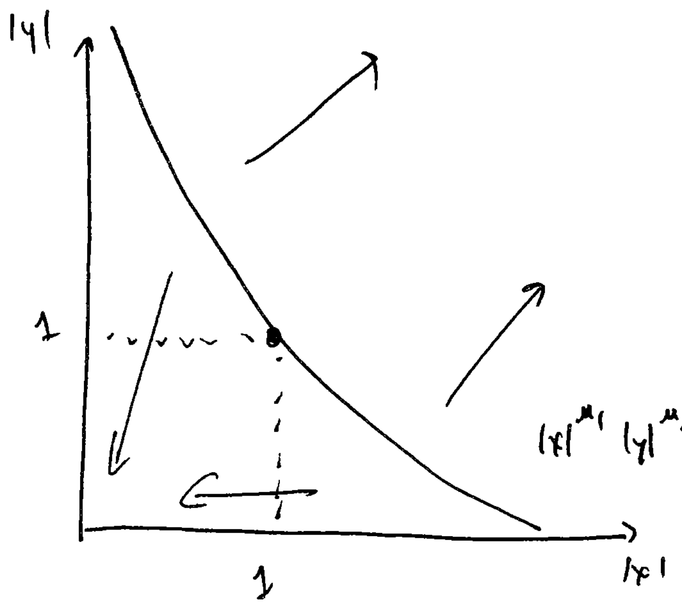


$\sim M^2$

$M = d \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\qquad M^2 = \begin{pmatrix} d & 0 \\ 0 & d' \end{pmatrix}$

<u>Focus on case 1</u> $[\Leftarrow a, b, c, d > 0]$

$\deg(F^n) \simeq \lambda^n$ [quadratic integer]



$|x|^{\mu_1} |y|^{\mu_2} = 1$ $\qquad \phi \circ F_M = \phi^\lambda$.

v = other eigenvector

$\lambda' > 1$

$\lambda' < 1$

a $\lambda' = 1$ even $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\lambda = 3$ $\lambda' = 1$!

$\rightsquigarrow$ different speed of convergence towards ∞

phenomenon described by Dinh – Dujardin – Sibony

Vigny .

8/ (10 min)

(V) Main theorem.

(1) $F : \mathbb{C}^2 \to \mathbb{C}^2$ dominant

- λ is a quadratic integer.
- $\left\{ \begin{array}{l} \deg(F^n) \simeq \lambda^n \\ \text{or } \deg(F^n) \simeq n\lambda^n \end{array} \right.$ and F is a skew product for a suitable choice of coordinates.
- $\deg(F^{n+k}) = \sum_0^{k-1} a_\ell \deg(F^{n+\ell}) \quad \forall n \quad a_\ell \in \mathbb{Z}$.

(2) $g_F = \lim \frac{1}{d^n} \operatorname{Log}^+ |F^n| \not\equiv 0$ exists $[e < d^2]$

$\left[\begin{array}{l} \times \text{ if } e < \lambda \quad g_F \text{ is } \mathcal{C}^0 \text{ and} \\ \quad |F^n(z)| \leq c^{(e+\varepsilon)^n} \quad \text{if } z \in \{g_F = 0\}. \end{array} \right.$

2 types of methods $\left\langle \begin{array}{l} \text{talk 2} = \text{action of F on a valuative space} \\ \text{talk 3} = \text{———————— cohomological space} \\ \qquad H^{1,1} \end{array} \right.$