Large time behavior of the Vlasov-Navier-Stokes system on the torus

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Abstract

We study the large time behavior of Leray solutions to the Vlasov-Navier-Stokes system set on $\mathbb{T}^d \times \mathbb{R}^d$, for $d = 2, 3$. Under the assumption that either (i) the viscosity in the Navier-Stokes equations is large enough or (ii) the initial so-called modulated energy is small enough, we prove that the distribution function converges to a Dirac mass in velocity, with sharp exponential rate. The proof is based on the fine structure of the system and on a bootstrap analysis allowing to get global bounds on moments.

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1 Introduction

We consider the Vlasov-Navier-Stokes system in $\mathbb{T}^d \times \mathbb{R}^d$, for $d = 2$ or 3:

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v[f(u - v)] = 0, \tag{1.1}$$

$$\partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla p = j_f - \rho_f u, \tag{1.2}$$

$$\mathrm{div}\, u = 0, \tag{1.3}$$

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where

$$\rho_f(t,x) := \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v,$$
$$j_f(t,x) := \int_{\mathbb{R}^d} vf(t,x,v)\,\mathrm{d}v.$$

This system of nonlinear PDEs describes the transport of particles (described by their density function f) within a fluid (described by its velocity u and its pressure p). It belongs to the large family of *fluid-kinetic systems*, which were historically introduced in the pionneer works of O'Rourke [21] and Williams [23]. Among all possible couplings (we refer to the introduction of [13] for other examples), the Vlasov-Navier-Stokes has been intensively studied because of both his physical relevance (see [5] for instance) and the mathematical challenges that it offers. The Vlasov-Navier-Stokes is fully coupled: both unknowns f and u depend on each other. This is due to the Brinkman force (the source term in the fluid equation) and the drag acceleration (the inertial term in the kinetic equation). We refer to [5] for the physical justification of these, and to [10, 2, 3, 17, 18] for the (partial) mathematical derivation of the former. The physical constants behind the Vlasov-Navier-Stokes are all normalized in (1.1) − (1.3), except the viscosity $\nu > 0$ which appears (only) in the fluid equation. However, a careful inspection reveals that both the Brinkman force and the drag acceleration (again, see [5]) shall also depend on ν. More precisely, the drag acceleration exerted on a particle of position x and velocity x is

$$a(t,x,v) = \frac{6\pi\nu r}{m}(u(t,x) - v),$$

while the Brinkman force which results from the whole spray is given by

$$F(t,x) = -m\int_{\mathbb{R}^d} f(t,x,v)a(t,x,v)\,\mathrm{d}v,$$

where $m = \frac{4}{3}\pi r^3\rho_p$ is the mass of the particles and r (resp. ρ_p) is their radius (resp. density). Therefore, the regime that we consider here corresponds to the scaling $r \sim 1/\nu$ and $\rho_p \sim \nu^3$. This particular setting affects only a limited number of results of our study, namely the ones resting on a "large ν assumption".

The mathematical analysis of the Vlasov-Navier-Stokes system has been for a long time focused on the existence of (weak or strong) solutions on rather academic domains [4, 9, 22] like the flat torus that we consider in this paper, or more realistic ones [15, 6]. Most of the previous results furnish global existence of weak solutions in the following sense: a Leray solution for the fluid equation and a renormalized one [11] for the kinetic equation (for a more precise definition, see Definition 1.3 below). These global weak solutions are all build by an approximation-compactness argument which is based on the kinetic energy dissipation of the system. More regular solutions can also be constructed. In 2D, thanks to the uniqueness result [16], they coincide with the weak solutions. In 3D regular solutions are only known to exist locally (see [9] for instance), this adverb refering either to the size of the initial data or to the lifetime of the solution. This issue is of course due to the Navier-Stokes component of the system.

Very few articles deal with the long time behavior of this system. At the formal level, one expects a monokinetic behavior for this system due to the damping of the fluid component and the friction term acting on the dispersed phase (see Subsection 1.1). This behavior however has never been completely proven for the Vlasov-Navier-Stokes system. The closest attempt is the paper [9] of Choi and Kwon in which a conditional theorem is proven: the monokinetic behavior is shown to happen under a boundedness assumption that has not been established for any global solution up to now. We intend to fill this gap by using the functional introduced by Choi and Kwon in [9] and proving that this boundedness property (in fact, a stronger one) indeed holds, for any weak solution of the Vlasov-Navier-Stokes system, under largeness assumption on the viscosity and/or appropriate smallness assumption on the initial data. Concerning the long-time behavior of fluid-kinetic systems, when a Fokker-Planck dissipation is added in the kinetic equation, the situation is less involved because the maxwellian gives a natural example of non-singular equilibrium that (at least locally) attracts smooth solutions. This has been investigated for instance in [14, 7]. Without this dissipation term, apart from [9], we can mention the work of Jabin [20] in which the Navier-Stokes is replaced by a stationary Stokes equation (and a different coupling term) and [13] in which a specific geometry is considered for the Vlasov-Navier-Stokes sytem, allowing for non-singular stationary solutions.

As far as our knowledge goes, the results that we present below constitutes the first complete and rigorous proof of asymptotic monokinetic behavior for the Vlasov-Navier-Stokes system.

The following energy functionals will play a crucial role in our study of this system.

Definition 1.1. *The* kinetic energy *of the system* (1.1) – (1.3) *is given by*

$$\mathrm{E}(t) := \frac{1}{2}\int_{\mathbb{T}^d} |u(t,x)|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^d\times\mathbb{R}^d} f(t,x,v)|v|^2 \,\mathrm{d}v\,\mathrm{d}x, \tag{1.4}$$

we define the modulated energy *as*

$$\mathscr{E}(t) := \frac{1}{2}\int_{\mathbb{T}^d\times\mathbb{R}^d} f(t,x,v)|v - \langle j_f(t,x)\rangle|^2 \,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^d} |u(t,x) - \langle u(t)\rangle|^2 \,\mathrm{d}x + \frac{1}{4}|\langle j(t)\rangle - \langle u(t)\rangle|^2. \tag{1.5}$$

where, for any integrable function g on the torus $\mathbb{T}^d$, we denote its average on $\mathbb{T}^d$ by $\langle g\rangle$. Finally, the dissipation *is defined as*

$$\mathrm{D}(t) := \int_{\mathbb{T}^d\times\mathbb{R}^d} f(t,x,v)|u(t,x) - v|^2 \,\mathrm{d}v\,\mathrm{d}x. \tag{1.6}$$

In the following, we shall sometimes write

$$\mathscr{E}(t) = \widetilde{\mathscr{E}}(t) + \frac{1}{2}\int_{\mathbb{T}^d} |u(t,x) - \langle u(t)\rangle|^2 \,\mathrm{d}x$$

with

$$\widetilde{\mathscr{E}}(t) := \frac{1}{2}\left[\int_{\mathbb{T}^d\times\mathbb{R}^d} f|v - \langle j_f\rangle|^2 \,\mathrm{d}v\mathrm{d}x + \frac{1}{2}|\langle j_f\rangle - \langle u\rangle|^2\right]. \tag{1.7}$$

The kinetic energy is known from the seminal papers on the Vlasov-Navier-Stokes sytem [15, 4]. The modulated one is more recent and was first introduced in [9].

Definition 1.2. *We shall say that (f_0, u_0) is an admissible initial condition if*

$$u_0 \in \mathrm{L}^2_{\mathrm{div}}(\mathbb{T}^d) = \{\mathrm{U} \in \mathrm{L}^2(\mathbb{T}^d),\ \mathrm{div}\,\mathrm{U} = 0\}, \tag{1.8}$$

$$0 \le f_0 \in \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{T}^d \times \mathbb{R}^d), \tag{1.9}$$

$$(x,v) \mapsto f_0(x,v)|v|^2 \in \mathrm{L}^1(\mathbb{T}^d \times \mathbb{R}^d), \tag{1.10}$$

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} f_0 \,\mathrm{d}v\,\mathrm{d}x = 1. \tag{1.11}$$

Remark 1.1. *The last condition does not play any role for what concerns the properties of existence, uniqueness and long time behavior that we are about to discuss. However, this normalization allows to simplify the formulas.*

Definition 1.3. *Consider an admissible initial data* (u_0, f_0) *in the sense of Definition 1.2. A Leray solution of the Vlasov-Navier-Stokes system with initial condition* (u_0, f_0) *is a pair* (u, f) *such that*

$$\begin{aligned} u &\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^2(\mathbb{T}^d)) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{H}^1_{\mathrm{div}}(\mathbb{T}^d)), \\ f &\in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathbb{T}^d \times \mathbb{R}^d)), \\ j_f - \rho_f u &\in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{H}^{-1}(\mathbb{T}^d)), \end{aligned}$$

with u *being a Leray solution of* (1.2) – (1.3) *(initiated by* u_0*) and* f *a solution of* (1.1) *(initiated by* f_0*), and such that the following energy estimate holds for almost all* $t \geq 0$

$$\mathrm{E}(t) + \int_0^t \mathrm{D}(s)\mathrm{d}s \leq \mathrm{E}(0), \tag{1.12}$$

where the functionals E *and* D *are the energy and dissipation introduced in Definition 1.1.*

The existence of Leray solutions (f, u) (in the sense of Definition 1.3) to the Vlasov-Navier-Stokes system have been is established in dimension $2, 3$, and even on general domains in the references aforecited. In the bidimensional case, uniqueness holds under an additional decay condition on the initial distribution function.

Definition 1.4. *We say that an initial condition satisfies the* pointwise decay assumption *of order* $q > 0$ *if*

$$(x, v) \longmapsto (1 + |v|^q) f_0(x, v) \in \mathrm{L}^\infty(\mathbb{T}^d \times \mathbb{R}^d), \tag{1.13}$$

and in that case we denote

$$N_q(f_0) := \sup_{x \in \mathbb{T}^d, v \in \mathbb{R}^d} (1 + |v|^q) f_0(x, v). \tag{1.14}$$

It has been established in [16] that under the pointwise decay assumption of order $q > 4$ (and in fact an even less stringent condition is sufficient), uniqueness holds for weak solutions of the Vlasov-Navier-Stokes system.

We finally introduce a useful notation for moments in velocity.

Definition 1.5. *For all* $\alpha \geq 0$ *and any measurable non-negative function* $\mathrm{f} : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}_+$*, we set*

$$m_\alpha \mathrm{f}(t, x) := \int_{\mathbb{R}^d} \mathrm{f}|v|^\alpha \, \mathrm{d}v, \tag{1.15}$$

$$M_\alpha \mathrm{f}(t) := \int_{\mathbb{T}^d \times \mathbb{R}^d} \mathrm{f}|v|^\alpha \, \mathrm{d}v\mathrm{d}x. \tag{1.16}$$

In this paper, we focus on the description of the long time behavior of Leray solutions to the Vlasov-Navier-Stokes system. To this end, it is enlighting to first study the linear Vlasov equation with friction.

1.1 Long time behavior for the linearized Vlasov equations with friction

The Vlasov equation (with friction) around the trivial equilibrium $(0, 0)$ reads

$$\partial_t g + v \cdot \nabla_x g - \mathrm{div}_v[gv] = 0 \tag{1.17}$$

Endowed with an initial condition g_0, this equation admits the explicit solution

$$g(t, x, v) = e^{dt} g_0(x - (e^t - 1)v, e^t v). \tag{1.18}$$

Definition 1.6. *For* $\mathrm{U} \in \mathbb{R}^d$*, we denote by* δ_{U} *the Dirac measure in velocity supported at* U*, defined by*

$$\langle \delta_{\mathrm{U}}, \varphi \rangle = \varphi(\mathrm{U}), \quad \forall \varphi : v \in \mathbb{R}^d \to \mathbb{R}.$$

The long time behavior of the solution to (1.17) is explicit, as we observe from (1.18) that

$$g(t,x,v) \rightharpoonup_{t \to +\infty} \left(\int_{\mathbb{R}^d} g_0(x-v,v)\,\mathrm{d}v \right) \otimes \delta_0.$$

More generally, given $\mathrm{U} \in \mathbb{R}^d$, for the equation

$$\partial_t g + v \cdot \nabla_x g + \mathrm{div}_v[g(\mathrm{U}-v)] = 0, \tag{1.19}$$

the long time behavior of the solution is also explicit and described by

$$g(t,x,v) - \left(\int_{\mathbb{R}^d} g_0(x-v-t\mathrm{U}, v+\mathrm{U})\,\mathrm{d}v \right) \otimes \delta_{\mathrm{U}} \rightharpoonup_{t \to +\infty} 0$$

The mechanism at stake in (1.17) and (1.19) is a competition between transport and friction. Friction always wins in the end, causing concentration to a Dirac mass in velocity. In view of this behavior, we may expect a similar concentration phenomenon in velocity for the full Vlasov-Navier-Stokes system.

1.2 Main result

Our main result provides a sharp description of the long time behavior of Leray solutions to the Vlasov-Navier-Stokes system. In order to quantitatively describe the evolution of the kinetic component of the system, we will make use of the Wasserstein distance W_1, whose definition and basic properties are recalled in an Appendix (see Section 9.1).

For the sake of readability, we distinguish between dimension 2 and dimension 3.

Theorem 1.1. *Let* $d = 2$*. There exists an universal nondecreasing onto function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that the following holds. Let* (f_0, u_0) *be an admissible initial condition such that* $N_q(f_0) < +\infty$ *for some* $q > 4$*. There is a unique Leray solution* (f, u) *of the Vlasov-Navier-Stokes system in the sense of Definition 1.3. Furthermore, if*

- *either the viscosity satisfies* $\nu > 2/c_P^2$*, where* c_P *stands for the best constant in the Poincaré-Wirtinger inequality on* $\mathbb{T}^2$*,*
- *or the initial modulated energy* $\mathscr{E}(0)$ *is small enough, in the sense that*

$$\psi\left(N_q(f_0) + \mathrm{E}(0) + \nu^{-1}\right) \mathscr{E}(0) < 1, \tag{1.20}$$

then the solution (f, u) *enjoys the following asymptotic behavior. For all* $0 < \lambda < 1$*, there exists* $\mathrm{C}_\lambda > 0$ *such that for all* $t \geq 0$*,*

$$\left\| u(t) - \frac{\langle u_0 + j_{f_0} \rangle}{2} \right\|_{\mathrm{L}^2(\mathbb{T}^d)} \leq \mathrm{C}_\lambda \exp\left(-\min(\lambda_0 \nu/2, \lambda) t\right), \tag{1.21}$$

$$\mathrm{W}_1\left(f(t), \rho(t) \otimes \delta_{\frac{\langle u_0 + j_{f_0} \rangle}{2}} \right) \leq \mathrm{C}_\lambda \exp\left(-\lambda t\right), \tag{1.22}$$

where λ_0 *is a universal positive constant appearing in Lemma 2.5.*

Theorem 1.2. *Let* $d = 3$*. There exists an universal nondecreasing onto function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that the following holds. Let* (f_0, u_0) *be an admissible initial condition such that* $N_q(f_0) < +\infty$ *for some* $q > 4$*,* $M_\alpha f_0 < +\infty$ *for some* $\alpha > 3$ *and* $u_0 \in \mathrm{H}^{1/2}(\mathbb{T}^3)$*. Then, for any Leray solution* (f, u) *of the Vlasov-Navier-Stokes system in the sense of Definition 1.3, if*

- *either the viscosity ν is large enough, in the sense that*

$$\nu > \psi\left(N_q(f_0) + M_\alpha f_0 + \mathrm{E}(0) + \|u_0\|_{\mathrm{H}^{1/2}(\mathbb{T}^3)}\right), \tag{1.23}$$

- *or $\|u_0\|_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)} < \nu^2/\mathrm{C}_\star^2$ where $\mathrm{C}_\star$ is the universal constant given in Proposition 4.3 and the initial modulated energy $\mathscr{E}(0)$ is small enough, in the sense that*

$$\psi\left(N_q(f_0) + M_\alpha f_0 + \mathrm{E}(0) + \|u_0\|_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \nu^{-1}\right)\mathscr{E}(0) < \min\left(1, \frac{\nu^2}{\mathrm{C}_\star^2} - \|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)}\right), \tag{1.24}$$

then (f,u) enjoys the following asymptotic behavior. For all $0 < \lambda < 1$, there exists $\mathrm{C}_\lambda > 0$ such that for all $t \geq 0$,

$$\left\|u(t) - \frac{\langle u_0 + j_{f_0}\rangle}{2}\right\|_{\mathrm{L}^2(\mathbb{T}^d)} \leq \mathrm{C}_\lambda \exp\left(-\min(\lambda_0\nu/2, \lambda)t\right), \tag{1.25}$$

$$\mathrm{W}_1\left(f(t), \rho(t) \otimes \delta_{\frac{\langle u_0 + j_{f_0}\rangle}{2}}\right) \leq \mathrm{C}_\lambda \exp\left(-\lambda t\right), \tag{1.26}$$

where λ_0 is a universal positive constant appearing in Lemma 2.5.

A series of remarks is in order.

Remark 1.2. *As already said, existence of Leray solutions both in dimensions* 2 *and* 3 *follows from [4] (note by the way that both the pointwise decay assumption and the higher order Sobolev assumption in dimension* 3 *is not relevant for this part). Uniqueness in dimension* 2 *is a consequence of [16].*

Remark 1.3. *In dimension* 3*, the fact that more stringent regularity assumptions are required is due to the well-known difficulties related to the resolution of the Navier-Stokes equations.*

Remark 1.4. *The rate of convergence is almost sharp considering that for the linearized equation* (1.19)*, one can check that*

$$\mathrm{W}_1\left(g(t,x,v), \left(\int_{\mathbb{R}^d} g_0(x - v - t\mathrm{U}, v + \mathrm{U})\,\mathrm{d}v\right) \otimes \delta_{\mathrm{U}}\right) \lesssim e^{-t}.$$

In (1.22) *and* (1.26)*, we reach the rate $e^{-\lambda t}$ for all $0 < \lambda < 1$.*

Remark 1.5. *The fact that the asymptotic state for the distribution function is a Dirac mass in velocity, and thus is singular, virtually forbids the use of standard PDE techniques, such as high order Sobolev energy estimates, to prove this result.*

Remark 1.6. *This result proves that for the Vlasov-Navier-Stokes system, the large time behavior on the torus is very different from that of the same equations set in domain with partially dissipative boundary conditions (and under adequate geometric control conditions): in [13], it is indeed proved that in the latter case there exist smooth non-trivial equilibria that are locally stable.*

Both theorems are consequences of the following result, bearing on the large time behavior of the modulated energy $\mathscr{E}(t)$. We refer to Lemmas 2.4 and 2.5 in Section 2 to see why Theorem 1.3 indeed implies Theorems 1.1 and 1.2.

Theorem 1.3. *Under the assumptions of Theorem 1.1 or Theorem 1.2, the following holds. For all* $0 < \lambda < 2$*, there exists* $\mathrm{C}_\lambda > 0$ *such that for all* $t \geq 0$*,*

$$\begin{aligned} \|u(t) - \langle u(t)\rangle\|^2_{\mathrm{L}^2(\mathbb{T}^d)} &\leq \mathscr{E}(0)e^{-\nu\lambda_0 t}, \\ \widetilde{\mathscr{E}}(t) &\leq \mathrm{C}_\lambda e^{-\lambda t}, \end{aligned} \tag{1.27}$$

where λ_0 *is a universal positive constant appearing in Lemma 2.5. Furthermore, we have the global bound*

$$\|\rho_f\|_{\mathrm{L}^\infty(0,+\infty;\mathrm{L}^\infty(\mathbb{T}^d))} < +\infty. \tag{1.28}$$

Remark 1.7. *The constants* C_λ *appearing in Theorems 1.1, 1.2 and 1.3 are actually uniform with respect to the various (semi-)norms of* u_0 *and* f_0 *that appear in the assumptions.*

It is possible to show the existence of an asymptotic local density and therefore to make the long time behavior of the distribution function more explicit than in (1.22) and (1.26). This is the content of the following result.

Corollary 1.4. *Under the assumptions of Theorem 1.1 or Theorem 1.2, there exists* $\overline{\rho} \in \mathrm{L}^\infty(\mathbb{T}^d)$ *such that the solution to the Vlasov-Navier-Stokes system enjoys the following asymptotic behavior. For all* $0 < \lambda < 1$*, there exists* $\mathrm{C}_\lambda > 0$ *such that for all* $t \geq 0$*,*

$$\mathrm{W}_1\left(f(t), \overline{\rho}\left(x - t\frac{\langle u_0 + j_{f_0}\rangle}{2}\right) \otimes \delta_{\frac{\langle u_0 + j_{f_0}\rangle}{2}}\right) \leq \mathrm{C}_\lambda e^{-\lambda t}. \tag{1.29}$$

We deduce that when $\langle u_0 + j_{f_0}\rangle = 0$, the distribution function $f(t)$ weakly converges to a stationary solution. This implies that in this case the Leray solution (u, f) converges to a stationary solution of the Vlasov-Navier-Stokes system. When $\langle u_0 + j_{f_0}\rangle \neq 0$, the asymptotic behavior is that of a travelling wave.

It is actually even possible to further describe the asymptotic local density $\overline{\rho}$ and prove modified scattering, at the expense of asking that the viscosity parameter ν is large enough *and* that the initial modulated energy $\mathscr{E}(0)$ is small enough (with possibly more stringent constraints than for Theorems 1.1 and 1.2); we postpone such a statement and refer to Corollary 8.1 in Section 8.

There are mainly two stabilization mechanisms at stake in the large time dynamics of solutions to the Vlasov-Navier-Stokes system. The first one is due to *friction* in the Vlasov equation, that forces the distribution function to concentrate in velocity.

The second stabilization mechanism comes from the *dissipation* in the Navier-Stokes equations. There is a competition in the Navier-Stokes equations between this dissipation and the possible growth of the non-linearity and the Brinkman force $F = j_f - \rho_f u$. As is well known, the non-linearity is (energy-)subcritical in dimension 2 and supercritical in dimension 3, which explains the higher Sobolev regularity assumption needed in the latter case. Loosely speaking, the various smallness assumptions we make allow to tame the influence of the forcing.

Thanks to the fine structure of the system, there happens to be a modulated energy/dissipation identity that follows from the energy identity and the conservation laws of the system, as exhibited by Choi and Kwon [9]. This identity somehow reflects the two stabilization mechanisms we have just discussed.

1.3 Outline of the proof and organisation of the paper

To conclude this introduction, let us provide a (non-technical) outline of the proof of Theorem 1.3. This also gives the opportunity to describe how this paper is organized.

We first gather in Section 2 conservation laws for the Vlasov-Navier-Stokes system, that turn out to be crucial in view of the description of long time behavior of its solutions.

In particular, we state the remarkable modulated energy/dissipation identity of Choi and Kwon [9] that formally reads

$$\mathscr{E}(t) + \int_0^t \mathrm{D}(s)\mathrm{d}s = \mathrm{E}(0), \quad \forall t \geq 0. \tag{1.30}$$

Two important consequences are deduced:

- first that the fluid velocity tends to homogenize exponentially fast, with a rate proportional to the viscosity ν, more precisely we get

$$\|u(t) - \langle u(t)\rangle\|^2_{\mathrm{L}^2(\mathbb{T}^d)} \leq \mathscr{E}(0)e^{-\nu\lambda_0 t}, \quad \forall t \geq 0, \tag{1.31}$$

  where $\lambda_0 > 0$ is an universal constant ;

- second that up to a control of the L^∞ norm (in time and space) of the local density $\rho_f = \int_{\mathbb{R}^d} f \,\mathrm{d}v$, the dissipation $\mathrm{D}(t)$ essentially controls the modulated energy $\mathscr{E}(t)$, so that by (1.30), $\mathscr{E}(t)$ decays exponentially fast, yielding Theorem 1.3. This observation was already made in [9].

As a consequence the main task to complete the proof of Theorem 1.3 is to obtain a global L^∞ bound for ρ_f, that is

$$\|\rho_f\|_{\mathrm{L}^\infty(0,+\infty;\mathrm{L}^\infty(\mathbb{T}^d))} < +\infty. \tag{1.32}$$

In Section 3, we present the main tools we rely on to obtain bounds on moments. They are based on the method of characteristics, which allows, considering the characteristics curves (X, V) solving the system

$$\begin{aligned} &\dot{\mathrm{X}}(s;t,x,v) = \mathrm{V}(s;t,x,v), \\ &\dot{\mathrm{V}}(s;t,x,v) = u(s, \mathrm{X}(s;t,x,v)) - \mathrm{V}(s;t,x,v), \end{aligned} \tag{1.33}$$

with $(\mathrm{X}(t;t,x,v), \mathrm{V}(t;t,x,v)) = (x,v)$, to write solutions to the Vlasov equation as

$$f(t,x,v) = e^{dt} f_0(\mathrm{X}(0;t,x,v), \mathrm{V}(0;t,x,v)). \tag{1.34}$$

We deduce that

$$\rho_f(t,x) = e^{dt} \int_{\mathbb{R}^d} f_0(t, \mathrm{X}(0;t,x,v), \mathrm{V}(0;t,x,v))\,\mathrm{d}v. \tag{1.35}$$

In order to study (1.35), we rely on two different changes of variables in velocity:

- we call the first one the *affine* change of variables and corresponds to

$$v \mapsto e^t \left(v - \int_0^t e^{s-t} \langle u(s)\rangle \,\mathrm{d}s \right).$$

  The outcome is the estimate

$$\|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^d)} + \|j_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^d)} \lesssim \left(1 + \left(\int_0^t e^s \|u(s) - \langle u(s)\rangle\|_{\mathrm{L}^\infty(\mathbb{T}^d)} \,\mathrm{d}s \right)^d \right). \tag{1.36}$$

  This is useful to obtain a short time control ρ_f; however for (1.36) to be meaningful for large times, we need to prove that $\|u(s) - \langle u(s)\rangle\|_{\mathrm{L}^\infty(\mathbb{T}^d)}$ decays faster than the exponential. We will be able to prove such a statement in the case $\nu \gg 1$.

- We call the second one the *straightening* change of variables; this corresponds to $v \mapsto \mathrm{V}(0; t, x, v)$. However to prove that this indeed a diffeomorphism, we need a condition, that is
$$\int_0^t \|\nabla_x u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\,\mathrm{d}s \leq \mathrm{c}(d), \tag{1.37}$$
where $\mathrm{c}(d) > 0$ is a universal constant. The outcome is the estimate
$$\|\rho_f\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty(\mathbb{T}^d))} \lesssim N_q(f_0).$$
Compared to the previous change of variables, we need here a higher order control on u but with a milder constraint on the decay rate. We will able to obtain such a control in the case $\mathscr{E}(0)$ small. Similar bounds for $j_f = \int_{\mathbb{R}^d} vf\,\mathrm{d}v$ can be obtained as well.

  This change of variables is inspired by that used by Bardos and Degond [1] for the study of global small solutions to the Vlasov-Poisson system on $\mathbb{R}^3$.

Observe that the use of the method of characteristics is actually only formal, as we a priori do not have enough smoothness on u to solve the system of characteristics. However, as we are only interested in obtaining estimates, we can rely on an approximation argument and DiPerna-Lions weak stability theory [11]. This remark also prevails for Section 2.

At this stage of the proof, the main task is therefore to prove global bounds for u and $\nabla_x u$, that correspond to
$$\int_0^{+\infty} e^s \|u(s) - \langle u(s)\rangle\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\,\mathrm{d}s < +\infty, \qquad \text{for } \nu \gg 1 \tag{1.38}$$
$$\text{or} \quad \int_0^{+\infty} \|\nabla_x u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\,\mathrm{d}s \leq \mathrm{c}(d), \qquad \text{for } \mathscr{E}(0) \ll 1. \tag{1.39}$$

Sections 4 to 7 are dedicated to this task. The general strategy is as follows. We shall obtain regularity estimates for $u - \langle u\rangle$ using higher order energy estimates for the Navier-Stokes equations and maximal parabolic estimates for the Stokes equations. Such bounds are not relevant in terms of decay in time but the idea is to interpolate them with the exponential decay in L^2 bounds already proved in (1.31). More precisely, we use the Gagliardo-Nirenberg-Sobolev interpolation inequalities to obtain
$$\|u(s) - \langle u(s)\rangle\|_{\mathrm{L}^\infty(\mathbb{T}^d)} \lesssim \|\mathrm{D}^2 u(s)\|_{\mathrm{L}^2(\mathbb{T}^2)}^{\alpha(d)} \|u(s) - \langle u(s)\rangle\|_{\mathrm{L}^2(\mathbb{T}^2)}^{1-\alpha(d)} \tag{1.40}$$
where $\alpha(d) \in (0,1)$ depends on the dimension d; we argue similarly for the control of $\nabla_x u$.

In Section 4 we discuss $\mathrm{H}^{1/2}(\mathbb{T}^d)$ and $\mathrm{H}^1(\mathbb{T}^d)$ energy inequalities for the Navier-Stokes system with source and explain how to use them obtain

- a short time control of ρ_f, using the aforementioned affine change of variables ;
- $\mathrm{L}^\infty_t \mathrm{H}^1_x \cap \mathrm{L}^2_t \mathrm{H}^2_x$ estimates for u, on time intervals *away* from zero, that is to say
$$\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^d)} + \nu \int_{t_0}^t \|\Delta u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^d)}\,\mathrm{d}s \lesssim_{0,\nu} 1 + \sup_{[t_0,t]} \|\rho_f\|_{\mathrm{L}^\infty(\mathbb{T}^d)}. \tag{1.41}$$
for some $t_0 > 0$.

Here, we shall need to distinguish between dimension 2 and 3. In dimension 3, we introduce the notion of *strong existence times* in order to be able to propagate regularity.

In Section 5, we obtain the global bound (1.38) in the case $\nu \gg 1$ by interpolating between (1.41) and (1.31) (using the Gagliardo-Nirenberg inequality (1.40)). We end up with the following inequalities.

- In dimension 2, for all $t \geq 0$,

$$\sup_{[0,t]} \|\rho_f\|_{\mathrm{L}^\infty(\mathbb{T}^2)} \lesssim \mathscr{E}(0)^{1/2}(1 + \sup_{[0,t]} \|\rho_f\|^{1/2}_{\mathrm{L}^\infty(\mathbb{T}^2)}), \tag{1.42}$$

- in dimension 3, for all strong existence times $t \geq 0$,

$$\sup_{[0,t]} \|\rho_f\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \lesssim 1 + \frac{\mathscr{E}(0)^{3/8}}{\nu^{9/8}}(1 + \sup_{[0,t]} \|\rho_f\|_{\mathrm{L}^\infty(\mathbb{T}^3)})^{9/8}. \tag{1.43}$$

Observe that for dimension 2, (1.42) is a *sub-linear* inequality, which allows to conclude straightaway that (1.32) holds. On the contrary, in dimension 3, (1.43) is *super-linear*, and we rely on a bootstrap argument, requiring to take ν large enough. We finally have to check that all $t \geq 0$ are strong existence times.

In Section 6, the aim is to obtain (1.39). We directly implement the interpolation strategy, relying this time on higher order maximal parabolic estimates for the Stokes equation. The outcome is the following inequality:

$$\|\nabla u\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\mathbb{T}^d))} \lesssim_{0,\nu} \mathscr{E}(0)^{\gamma_{d,1}} \left(1 + \left(\sup_{[0,t]} \|\rho_f\|_{\mathrm{L}^\infty(\mathbb{T}^d)} + \sup_{[0,t]} \|j_f\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\right)^{\gamma_{d,2}}\right), \tag{1.44}$$

for some universal constants $\gamma_{d,1}, \gamma_{d,2} > 0$.

Then Section 7 is dedicated to the proof of the global bound (1.32) in the case $\mathscr{E}(0) \ll 1$. Combining (1.44) with the straightening change of variables, the result is that is $\mathscr{E}(0)$ is small enough, then the global bound (1.32) can be obtained. In dimension 3, we also check that all $t \geq 0$ are strong existence times.

This finally concludes the proof of Theorem 1.3 for all cases.

In Section 8, we provide some sharper statements about the asymptotic state ultimately reached by the distribution function f.

We first prove Corollary 1.4, that is we prove the existence of a density $\overline{\rho}(x) \in \mathrm{L}^\infty(\mathbb{T}^d)$, such that

$$\mathrm{W}_1\left(f(t), \overline{\rho}\left(x + t\frac{\langle u_0 - j_{f_0}\rangle}{2}\right) \otimes \delta_0\right) \underset{t\to+\infty}{\longrightarrow} 0, \tag{1.45}$$

with sharp exponential decay.

Next, at the expense of taking ν large enough and $\mathscr{E}(0)$ small enough, we are able to prove a modified scattering result, giving a more precise description of the function $\overline{\rho}$. We show the existence of *limit characteristics* $(\widetilde{\mathrm{X}}_{s,\infty}(x,v), \widetilde{\mathrm{V}}_{s,\infty}(x,v))$ satisfying an explicit system of integral equations (see (8.2)), such that setting

$$\widetilde{\rho}(t,x) := \int_{\mathbb{R}^d} f_0\left(\widetilde{\mathrm{X}}_{0,\infty}(x,v) - t\frac{\langle u_0 + j_{f_0}\rangle}{2}, \widetilde{\mathrm{V}}_{0,\infty}(x,v)\right) \mathrm{d}v, \tag{1.46}$$

we have

$$\mathrm{W}_1\left(f(t), \widetilde{\rho}(t,x) \otimes \delta_{\frac{\langle u_0 + j_{f_0}\rangle}{2}}\right) \underset{t\to+\infty}{\longrightarrow} 0. \tag{1.47}$$

with sharp exponential decay (of course this yields $\overline{\rho} = \widetilde{\rho}$). It turns out that the characteristics $(\widetilde{\mathrm{X}}_{s,\infty}(x,v), \widetilde{\mathrm{V}}_{s,\infty}(x,v))$ are small perturbations of the vector field

$$(x,v) \mapsto \left(x - v - (t-1)\frac{\langle u_0 + j_{f_0}\rangle}{2}, v \right),$$

which corresponds to the linearized dynamics.

To conclude the paper, Section 9 is an Appendix where we provide some reminders (in particular, we shortly review the W^1 distance) and finally justify H^1 energy estimates for the Navier-Stokes equations with source.

2 Conservation laws, energy dissipation identities and consequences

We gather in this section several identities and a priori estimates for the Vlasov-Navier-Stokes system, which are valid in all dimensions:

- we first give basic conservation laws and the energy and modulated energy dissipation identities which are the key algebraic features of the Vlasov-Navier-Stokes system in view of the description of its long time behavior;
- several consequences are deduced: first that the fluid velocity approaches its average in space in $\mathrm{L}^2(\mathbb{T}^d)$ norm exponentially fast (Lemma 2.5), and second that equipped with a global control of the L^∞ norm (in time and space) of the kinetic phase density ρ_f, it is possible to deduce Theorem 1.3 (Lemma 2.6). Obtaining this global bound will be our main task which we will focus on in the following of the paper.

2.1 Conservation laws

We discuss here some conservations laws for the Vlasov-Navier-Stokes system. We start by describing some basic ones in a first lemma: the first two ones come from the structure of the Vlasov equation alone, while the third one is a consequence of the fine structure of the complete system.

Lemma 2.1. *Any Leray solution (in the sense of Definition* (1.3)*) satisfies the following conservations laws. For almost all $t \geq 0$,*

$$f(t) \geq 0, \quad \text{for almost all } (x,v) \in \mathbb{T}^d \times \mathbb{R}^d, \tag{2.1}$$

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} f(t)\,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{T}^d \times \mathbb{R}^d} f_0\,\mathrm{d}v\,\mathrm{d}x = 1, \tag{2.2}$$

$$\langle u + j_f\rangle(t) = \langle u_0 + j_{f_0}\rangle. \tag{2.3}$$

Proof. Considering the results of [4], the only item to prove is (2.3). Let us assume that both u and f are smooth functions. Integrating the Vlasov equation against v, the conservation law satisfied by j_f reads

$$\partial_t j + \mathrm{div}\left(\int_{\mathbb{R}^d} f v \otimes v \, dv \right) = \rho_f u - j_f, \tag{2.4}$$

so that $\langle j_f \rangle$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle j_f\rangle = \langle \rho_f u - j_f\rangle.$$

On the other hand, from (1.2), $\langle u\rangle$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u\rangle = \langle j_f - \rho_f u\rangle,$$

from which we deduce $\frac{\mathrm{d}}{\mathrm{d}t}\langle u + j_f\rangle = 0$, and consequently (2.3).

In the general case, we use an approximation argument relying on DiPerna-Lions theory [11] for linear transport equations. With the same approximation procedure developed in [4], we consider a sequence of nonnegative distribution functions $(f_n)_n$ solving the Vlasov equation with regularized vector fields $(u_n)_n$ and regularized and truncated initial conditions $(f_{0,n})_n$, and such that for all $n \geq 1$ and all $t \geq 0$,

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f_n|v|^2 \,\mathrm{d}v\mathrm{d}x \lesssim 1.$$

By the DiPerna-Lions theory, f is the (strong) limit of $(f_n)_n$ in $\mathrm{L}^\infty(0,T;\mathrm{L}^1(\mathbb{T}^d\times\mathbb{R}^d))$ for any $T>0$. We infer that for almost all $t \geq 0$,

$$|\langle j_f - j_{f_n}\rangle| \leq \left(\int_{\mathbb{T}^d\times\mathbb{R}^d} |f-f_n||v|^2\,\mathrm{d}v\mathrm{d}x\right)^{1/2} \|f_n - f\|_{\mathrm{L}^1(\mathbb{T}^d\times\mathbb{R}^d)} \to_{n\to+\infty} 0.$$

The result follows by letting n go to infinity in $(2.3)_n$. $\square$

A straightforward consequence of (2.3) in Lemma 2.1 is the following formula:

Lemma 2.2. *For almost all $t \geq 0$:*

$$\frac{1}{4}|\langle j\rangle - \langle u\rangle|^2 = \left|\langle j\rangle - \frac{\langle u_0 + j_{f_0}\rangle}{2}\right|^2 = \left|\langle u\rangle - \frac{\langle u_0 + j_{f_0}\rangle}{2}\right|^2. \tag{2.5}$$

We shall use in this paper several times the DiPerna-Lions theory [11], in the same fashion as in the proof of Lemma 2.1. Thanks to the property of strong stability of renormalized solutions, this allows to systematically argue as if both f and u are smooth when looking to establish estimates for the kinetic phase. The argument, as already outlined in the proof of Lemma 2.1, is the following:

- consider an approximating sequence $(u_n)_n$ for u and $(f_n)_n$ the associated solution to the Vlasov equation, with a regularized initial condition;
- prove the desired estimate for the solution f_n (without using the higher regularity of f_n or u_n);
- pass to the limit using the strong stability property of renormalized solutions (and Fatou's lemma).

In the following, for brevity, *we will never write down this argument explicitly* but will indicate by the symbol ♠ the proofs where it is needed.

2.2 The modulated energy of Choi and Kwon

As already said in the introduction, in [9], Choi and Kwon introduced the following *modulated* energy[1]

$$\mathscr{E}(t) := \frac{1}{2}\left[\int_{\mathbb{T}^d\times\mathbb{R}^d} f(t)|v - \langle j_f(t)\rangle|^2\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{T}^d} |u(t) - \langle u(t)\rangle|^2\,\mathrm{d}x + |\langle j_f(t)\rangle - \langle u(t)\rangle|^2\right] \tag{2.6}$$

[1] As a matter of fact, they consider the more general Vlasov-*inhomogeneous* Navier-Stokes system but we recover the system (1.1)–(1.3) as soon we stick to the case of constant density.

and using the conservation laws (2.2) and (2.3) they proved the following formal identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(t) + \mathrm{D}(t) = 0. \tag{2.7}$$

At the level of Leray solutions, we are only able to obtain the inequality version of (2.7), as stated in the next lemma.

Lemma 2.3. *Any Leray solution (in the sense of Definition* (1.3)*) satisfies the following modulated energy/dissipation inequality. For almost all $t \geq 0$,*

$$\mathscr{E}(t) + \int_0^t \mathrm{D}(s)\,\mathrm{d}s \leq \mathscr{E}(0). \tag{2.8}$$

Proof. Combining the energy inequality (1.12) and the conservation laws (2.2) and (2.3) in Lemma 2.1, we get that for almost all $t \geq 0$,

$$\begin{aligned}
\mathscr{E}(t) &= \mathrm{E}(t) + \frac{1}{2}\left[\left(\int f\,\mathrm{d}v\,\mathrm{d}x\right)\langle j_f\rangle^2 - 2\langle j_f\rangle^2 - \langle u\rangle^2 + \frac{1}{2}|\langle j_f\rangle - \langle u\rangle|^2\right] \\
&\leq -\int_0^t \mathrm{D}(s)\,\mathrm{d}s + \mathrm{E}(0) - \frac{1}{2}|\langle j_f\rangle - \langle u\rangle|^2 \\
&\leq -\int_0^t \mathrm{D}(s)\,\mathrm{d}s + \mathrm{E}(0) - \frac{1}{2}|\langle j_{f_0}\rangle - \langle u_0\rangle|^2 \\
&\leq -\int_0^t \mathrm{D}(s)\,\mathrm{d}s + \mathrm{E}(0) \\
&\qquad + \frac{1}{2}\left[\left(\int f_0\,\mathrm{d}v\,\mathrm{d}x\right)\langle j_{f_0}\rangle^2 - 2\langle j_{f_0}\rangle^2 - \langle u_0\rangle^2 + \frac{1}{2}|\langle j_{f_0}\rangle - \langle u_0\rangle|^2\right] \\
&= -\int_0^t \mathrm{D}(s) - \mathscr{E}(0),
\end{aligned}$$

hence concluding the proof. ∎

The modulated energy is interesting in view of the expected long time monokinetic dynamics for the kinetic phase, because of the following control.

Lemma 2.4. *With the previous notations, we have that for all $t \geq 0$,*

$$\begin{aligned}
&\mathrm{W}_1\left(f(t), \rho(t)\otimes\delta_{\frac{\langle u_0 + j_{f_0}\rangle}{2}}\right) \lesssim (\widetilde{\mathscr{E}}(t))^{1/2}, \\
&\left\|u(t) - \frac{\langle u_0 + j_{f_0}\rangle}{2}\right\|_{\mathrm{L}^2(\mathbb{T}^d)} \lesssim \max\left((\widetilde{\mathscr{E}}(t))^{1/2}, \|u(t) - \langle u(t)\rangle\|_{L^2(\mathbb{T}^d)}\right),
\end{aligned} \tag{2.9}$$

where we recall $\widetilde{\mathscr{E}}$ is defined in (1.7)*.*

Proof. By the Monge-Kantorovich duality for the W_1 distance (see Proposition 9.2 in the Appendix), the Cauchy-Schwarz inequality, (2.1) and (2.2), we have

$$\begin{aligned}
\mathrm{W}_1\left(f(t), \rho(t)\otimes\delta_{\frac{\langle u_0 + j_{f_0}\rangle}{2}}\right) &= \sup_{\|\nabla_{x,v}\phi\|_\infty \leq 1}\left\{\int_{\mathbb{T}^d}(f(t,x,v)\phi(x,v) - \rho(t,x)\phi(x,\langle j\rangle))\mathrm{d}x\right\} \\
&= \sup_{\|\nabla_{x,v}\phi\|_\infty \leq 1}\left\{\int_{\mathbb{T}^d} f(t,x,v)(\phi(x,v) - \phi(x,\langle j\rangle))\mathrm{d}v\mathrm{d}x\right\} \\
&\lesssim \left(\int_{\mathbb{T}^d\times\mathbb{R}^d} f|v - \langle j\rangle|^2\,\mathrm{d}v\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{T}^d\times\mathbb{R}^d} f\,\mathrm{d}v\mathrm{d}x\right)^{1/2} \\
&\lesssim \left(\int_{\mathbb{T}^2\times\mathbb{R}^2} f|v - \langle j\rangle|^2\,\mathrm{d}v\mathrm{d}x\right)^{1/2} \\
&\lesssim (\mathscr{E}(t))^{1/2}.
\end{aligned}$$

Likewise, using (2.1) and (2.2) and the identity (2.5), we have

$$
\begin{aligned}
\mathrm{W}_1\Big(\rho(t)\otimes\delta_{\langle j\rangle},\rho(t)\otimes\delta_{\frac{\langle u_0+j_{f_0}\rangle}{2}}\Big)
&= \sup_{\|\nabla_{x,v}\phi\|_\infty\le 1}\int_{\mathbb{T}^d}\rho(t,x)\Big(\phi(x,\langle j\rangle)-\phi\left(x,\langle u_0+j_{f_0}\rangle/2\right)\Big)\mathrm{d}x\\
&\le \int_{\mathbb{T}^d}\rho(t,x)\left|\langle j\rangle-\frac{\langle u_0+j_{f_0}\rangle}{2}\right|\mathrm{d}x\\
&\le \left(\int_{\mathbb{T}^d\times\mathbb{R}^d} f(t)\,\mathrm{d}v\,\mathrm{d}x\right)\left|\langle j\rangle-\frac{\langle u_0+j_{f_0}\rangle}{2}\right|\\
&\lesssim (\mathscr{E}(t))^{1/2}.
\end{aligned}
$$

We therefore deduce

$$
\mathrm{W}_1\left(f(t),\rho(t)\otimes\delta_{\frac{\langle u_0+j_{f_0}\rangle}{2}}\right)\lesssim(\mathscr{E}(t))^{1/2}.
$$

On the other hand, using again (2.5), we can also estimate

$$
\begin{aligned}
\left\|u(t)-\frac{\langle u_0+j_{f_0}\rangle}{2}\right\|_{\mathrm{L}^2(\mathbb{T}^d)} &\le \|u(t)-\langle u(t)\rangle\|_{\mathrm{L}^2(\mathbb{T}^d)}+\left\|\langle u(t)\rangle-\frac{\langle u_0+j_{f_0}\rangle}{2}\right\|_{\mathrm{L}^2(\mathbb{T}^d)}\\
&\le \|u(t)-\langle u(t)\rangle\|_{\mathrm{L}^2(\mathbb{T}^d)}+\frac{1}{4}|\langle j\rangle-\langle u\rangle|^2,
\end{aligned}
$$

hence the result. □

We now describe in the two following subsections two key consequences of the modulated energy/dissipation identity.

2.3 Long time behavior for the fluid

A straightforward application of the modulated energy/dissipation identity is a first description of the behavior of the Navier-Stokes part.

Lemma 2.5. *There exists a constant $\lambda_0=\lambda_0(d)>0$ such that*

$$
\|u(t)-\langle u(t)\rangle\|^2_{\mathrm{L}^2(\mathbb{T}^d)}\le\mathscr{E}(0)e^{-\nu\lambda_0 t},\quad\forall t\ge 0. \tag{2.10}
$$

Proof. The Poincaré-Wirtinger inequality yields

$$
\|g-\langle g\rangle\|_{\mathrm{L}^2(\mathbb{T}^d)}\le c_P(d)\|\nabla g\|_{\mathrm{L}^2(\mathbb{T}^d)},\quad\forall g\in(\mathrm{H}^1(\mathbb{T}^d))^d, \tag{2.11}
$$

where the positive real number $c_P(d)>0$ is the Poincaré constant. Using (2.8) and the definition of $\mathscr{E}$ and D we infer

$$
\begin{aligned}
\|u(t)-\langle u(t)\rangle\|^2_{\mathrm{L}^2(\mathbb{T}^d)} &\le \mathscr{E}(0)-\nu\int_0^t\|\nabla_x u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^d)}\,\mathrm{d}s\\
&\le \mathscr{E}(0)-\nu c_P^2(d)\int_0^t\|u(s)-\langle u(s)\rangle\|^2_{\mathrm{L}^2(\mathbb{T}^d)}\,\mathrm{d}s.
\end{aligned}
$$

Then (2.10) follows by Gronwall's lemma with the choice $\lambda_0(d):=c_P(d)^2$. □

Remark 2.1. *Observe that we did not specify the limit of $t\mapsto\langle u(t)\rangle$ as $t\to+\infty$, so that even if we are interested only in the long time behavior of the fluid part, it is not yet possible to conclude.*

Lemmas 2.4 and 2.5 explain why Theorem 1.2 indeed implies Theorems 1.1 and 1.2.

2.4 Conditional long time behavior for the kinetic phase

The following result relating the dissipation and the modulated energy is a variant of [9, Theorem 1.2].

Lemma 2.6. *Let $T > 0$ such that*

$$\|\rho_f\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{T}^d))} < +\infty. \tag{2.12}$$

Then for all $t \in [0,T]$ and all $\alpha \in (0,1)$,

$$\mathrm{D}(t) \geq 2(1-\alpha)\widetilde{\mathscr{E}}(t) - \frac{1-\alpha}{\alpha}\|\rho\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{T}^d))}\|u(t) - \langle u(t)\rangle\|^2_{\mathrm{L}^2(\mathbb{T}^d)}, \tag{2.13}$$

where we recall $\widetilde{\mathscr{E}}(t)$ is defined in (1.7). As a result, we have for all $t \in [0,T]$,

$$\widetilde{\mathscr{E}}(t) \leq \mathscr{E}(0)\left(1 + \frac{1-\alpha}{\alpha}\frac{1}{2\lambda_0}\|\rho\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{T}^d))}\right)e^{-2(1-\alpha)t}, \tag{2.14}$$

for λ_0 given in Lemma 2.5.

Proof ♠. We shall use the following identities

$$\begin{aligned} |v-u|^2 &= |v - \langle u\rangle|^2 + 2(v - \langle u\rangle)\cdot(\langle u\rangle - u) + |\langle u\rangle - u|^2, \\ |v - \langle u\rangle|^2 &= |v - \langle j\rangle|^2 + 2(v - \langle j\rangle)\cdot(\langle j\rangle - \langle u\rangle) + |\langle j\rangle - \langle u\rangle|^2. \end{aligned} \tag{2.15}$$

Indeed, combining them and integrating after multiplication by f yields

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} f|v-u|^2\,\mathrm{d}v\mathrm{d}x = \int_{\mathbb{T}^d\times\mathbb{R}^d} f|v - \langle j\rangle|^2\,\mathrm{d}v\mathrm{d}x + |\langle j\rangle - \langle u\rangle|^2 \tag{2.16}$$

$$+ \int \rho|\langle u\rangle - u|^2\mathrm{d}x + 2\int_{\mathbb{T}^d\times\mathbb{R}^d} f(v - \langle u\rangle)\cdot(\langle u\rangle - u)\mathrm{d}v\mathrm{d}x, \tag{2.17}$$

as

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} f(v - \langle j\rangle)\cdot(\langle j\rangle - \langle u\rangle)\mathrm{d}v\mathrm{d}x = 0.$$

Now note that for any $\alpha \in (0,1)$, Young's inequality entails that

$$\begin{aligned} &2\int_{\mathbb{T}^d\times\mathbb{R}^d} f(\langle u\rangle - v)\cdot(u - \langle u\rangle)\,\mathrm{d}v\mathrm{d}x \\ &\quad\geq -\alpha\int_{\mathbb{T}^d\times\mathbb{R}^d} f|v - \langle u\rangle|^2\,\mathrm{d}v\mathrm{d}x - \frac{1}{\alpha}\int_{\mathbb{T}^d\times\mathbb{R}^d} f|u - \langle u\rangle|^2\,\mathrm{d}v\mathrm{d}x \\ &\quad= -\alpha\left(\int_{\mathbb{T}^d\times\mathbb{R}^d} f|v - \langle j\rangle|^2\mathrm{d}v\mathrm{d}x + |\langle j\rangle - \langle u\rangle|^2\right) - \frac{1}{\alpha}\int_{\mathbb{T}^d\times\mathbb{R}^d} f|u - \langle u\rangle|^2\,\mathrm{d}v\mathrm{d}x, \end{aligned}$$

according to (2.15). Henceforth, as $1 - \frac{1}{\alpha} < 0$, we deduce that

$$\begin{aligned} \int_{\mathbb{T}^d\times\mathbb{R}^d} f|v-u|^2\,\mathrm{d}v\mathrm{d}x &\geq (1-\alpha)\left(\int_{\mathbb{T}^d\times\mathbb{R}^d} f|u - \langle j\rangle|^2\,\mathrm{d}v\mathrm{d}x + |\langle u\rangle - \langle j\rangle|^2\right) \\ &\quad - \left(\frac{1}{\alpha} - 1\right)\|\rho\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{T}^d))}\|u - \langle u\rangle\|^2_{\mathrm{L}^2(\mathbb{T}^d)}, \end{aligned}$$

whence (2.13) follows. In addition, by definition of $\widetilde{\mathscr{E}}$ and using (2.8), one has for all $t \geq 0$

$$\begin{aligned} \widetilde{\mathscr{E}}(t) &\leq \mathscr{E}(t) \\ &\leq \mathscr{E}(0) - \int_0^t \mathrm{D}(s)\mathrm{d}s \\ &\leq \mathscr{E}(0) - 2(1-\alpha)\int_0^t \widetilde{\mathscr{E}}(s)\mathrm{d}s - \frac{1-\alpha}{\alpha}\|\rho\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{T}^d)}\int_0^t \|u(s) - \langle u(s)\rangle\|^2_{\mathrm{L}^2(\mathbb{T}^d)}\mathrm{d}s. \end{aligned}$$

Henceforth, applying Lemma 2.5 and using Gronwall's lemma, we deduce (2.14).

□

Lemma 2.6 shows that in order to prove Theorem 1.2 it is thus sufficient to focus on proving the global bound (2.12) for $T = +\infty$.

3 Changes of variables and L^∞ bounds on moments

In this section we aim at establishing tools for obtaining bounds on the moments ρ_f and j_f for the kinetic part. We firstly use some interpolation estimates to guarantee some rough unconditional integrability for ρ_f and j_f. Next, using some adequate change of variables in velocity, we shall prove refined estimates proving that ρ and j can be controlled along the flow in the following way:

- firstly, we have

$$\|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^d)} + \|j_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^d)} \lesssim \left(1 + \left(\int_0^t e^s \|u(s) - \langle u(s)\rangle\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\,\mathrm{d}s\right)^d\right),$$

  which leads on the first hand to *short time* estimates, while on the other hand this requires a fast decay of the fluid part to be meaningful for *long times*, and will turn out to be useful in the case where the viscosity ν is large enough (but $\mathscr{E}(0)$ can be arbitrary);

- second, up to a control of $\|\nabla_x u\|_{\mathrm{L}^1(0,t;\mathrm{L}^\infty(\mathbb{T}^d))}$, we have

$$\|\rho_f\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty(\mathbb{T}^d))} \lesssim 1,$$
$$\|j_f\|_{\mathrm{L}^\infty(0,t;\mathrm{L}^\infty(\mathbb{T}^d))} \lesssim \left(\int_0^t \|u(s) - \langle u(s)\rangle\|_\infty\,\mathrm{d}s + e^{-t}\left(1 + \int_0^t e^s|\langle u(s)\rangle|\mathrm{d}s\right)\right),$$

  where we recall $N_q(f_0)$ is defined in (1.14). The control on $\nabla_x u$ will be obtained in the regime where $\mathscr{E}(0)$ is small (but $\nu > 0$ can be arbitrary).

Many proofs in this section rely on the representation using characteristics of the solution to the Vlasov equation, which holds at least when u is a smooth vector field.

Definition 3.1. *Assume u is smooth (say $\mathscr{C}^1$). We define the characteristic curves $\mathrm{X}(s;t,x,v)$ and $\mathrm{V}(s;t,x,v)$ associated to u as the solution to the system of ODEs*

$$\begin{aligned}\dot{\mathrm{X}}(s;t,x,v) &= \mathrm{V}(s;t,x,v),\\ \dot{\mathrm{V}}(s;t,x,v) &= u(s,\mathrm{X}(s;t,x,v)) - \mathrm{V}(s;t,x,v),\end{aligned} \tag{3.1}$$

with the initial condition $(\mathrm{X}(t;t,x,v),\mathrm{V}(t;t,x,v)) = (x,v)$.

By the method of characteristics, for a smooth vector field u, we can write the solution f to the Vlasov equation as

$$f(t,x,v) = e^{dt} f_0(\mathrm{X}(0;t,x,v),\mathrm{V}(0;t,x,v)). \tag{3.2}$$

As already explained, we then rely on DiPerna-Lions theory to ensure that the estimates we are able to prove with this representation formula still hold even if u is not smooth enough.

For instance, a rough bound on the L^∞ norm of f can be directly deduced from (3.2).

Lemma 3.1. *For almost all $t \geq 0$,*

$$\|f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^d\times\mathbb{R}^d)} \leq \|f_0\|_{\mathrm{L}^\infty(\mathbb{T}^d\times\mathbb{R}^d)} e^{dt}. \tag{3.3}$$

3.1 Rough local bounds on moments

We recall the notations M_α and m_α introduced in Definition 1.5.

Lemma 3.2. *Assume $M_\alpha f_0 < \infty$ for some $\alpha \geq 1$ and that $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^{\alpha+d} \cap \mathrm{W}^{1,1}(\mathbb{T}^d))$. Then $M_\alpha f(t) < \infty$ and for all $t > 0$ and*

$$M_\alpha f(t) \lesssim_{\alpha,d} \left(M_\alpha f_0 + e^{\frac{dt}{\alpha+d}} \int_0^t \|u(s)\|_{\alpha+d} \,\mathrm{d}s \right)^{\alpha+d}. \tag{3.4}$$

Proof ♠. Multiplying the Vlasov equation by $|v|^\alpha$ and integrating over $\mathbb{T}^d \times \mathbb{R}^d$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} M_\alpha f(t) + \alpha M_\alpha f(t) = \alpha \int_{\mathbb{T}^d} u(t,x) \cdot m_{\alpha-1}(t,x) \,\mathrm{d}x. \tag{3.5}$$

For $0 \leq \ell \leq k$ recall the interpolation estimate for any measurable function $g : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}_+$

$$\|m_\ell g\|_{\mathrm{L}^{\frac{k+d}{\ell+d}}(\mathbb{T}^d)} \lesssim_d (M_k g)^{\frac{\ell+d}{k+d}} \|g\|_{\mathrm{L}^\infty(\mathbb{T}^d)}^{\frac{k-\ell}{k+d}}. \tag{3.6}$$

In particular for $(\ell, k) = (\alpha - 1, \alpha)$ we get

$$\|m_{\alpha-1} g\|_{\mathrm{L}^{\frac{\alpha+d}{\alpha+d-1}}(\mathbb{T}^d)} \lesssim_d (M_\alpha g)^{\frac{\alpha+d-1}{\alpha+d}} \|g\|_{\mathrm{L}^\infty(\mathbb{T}^d)}^{\frac{1}{\alpha+d}}.$$

We can control $\|g\|_\infty$ by Lemma 3.1, so that using Hölder's inequality in (3.5), we infer

$$\frac{\mathrm{d}}{\mathrm{d}t} M_\alpha f(t)^{\frac{1}{\alpha+d}} + \frac{\alpha}{\alpha+d} M_\alpha f(t)^{\frac{1}{\alpha+d}} \lesssim_{\alpha,d} e^{\frac{dt}{\alpha+d}} \|u(t)\|_{\mathrm{L}^{\alpha+d}(\mathbb{T}^d)},$$

from which we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\{ e^{\frac{\alpha t}{\alpha+d}} M_\alpha f(t)^{\frac{1}{\alpha+d}} \right\} \lesssim_{\alpha,d} e^t \|u(t)\|_{\mathrm{L}^{\alpha+d}(\mathbb{T}^d)},$$

from which (3.4) follows. □

Lemma 3.3. *For $d = 2$, we have $\rho_f, j_f \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^\infty(\mathbb{T}^2))$. In the case $d = 3$, assuming $M_3 f_0 < +\infty$, we have the following*

(i) $M_3 f \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+)$;

(ii) $\rho_f \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^2(\mathbb{T}^3))$;

(iii) $j_f \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^{5/3}(\mathbb{T}^3))$.

Proof. For the case $d = 2$, we refer to [16, Lemma 3].

We therefore focus on $d = 3$. By Lemma 3.2, we have

$$M_3 f(t) \lesssim \left(M_3 f_0 + e^{\frac{t}{2}} \int_0^t \|u(s)\|_{\mathrm{L}^6(\mathbb{T}^3)} \,\mathrm{d}s \right)^6.$$

But, using the Sobolev embedding $\mathrm{H}^1(\mathbb{T}^3) \hookrightarrow \mathrm{L}^6(\mathbb{T}^3)$ and the Poincaré-Wirtinger inequality and the energy estimate (1.12), we infer

$$\begin{aligned}
\int_0^t \|u(s)\|_{\mathrm{L}^6(\mathbb{T}^3)} \,\mathrm{d}s &\leq \int_0^t \|u(s) - \langle u(s) \rangle\|_{\mathrm{L}^6(\mathbb{T}^3)} \,\mathrm{d}s + \sqrt{t}\mathrm{E}(0)^{1/2} \\
&\lesssim \sqrt{t} \left(\int_0^t \|\nabla u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)} \,\mathrm{d}s \right)^{1/2} + \sqrt{t}\mathrm{E}(0)^{1/2} \\
&\lesssim \sqrt{t}\mathrm{E}(0)^{1/2}.
\end{aligned}$$

This concludes (i). Using the interpolation estimate (3.6) for $(\ell,k)=(0,3)$ and $(\ell,k)=(1,3)$ we have

$$\|\rho_f(t)\|_{\mathrm{L}^2(\mathbb{T}^3)} = \|m_0 f(t)\|_{\mathrm{L}^2(\mathbb{T}^3)} \lesssim M_3 f(t)^{1/2}\|f(t)\|_{\mathrm{L}^\infty}^{1/2},$$
$$\|j_f(t)\|_{\mathrm{L}^{3/2}(\mathbb{T}^3)} \leq \|m_1 f(t)\|_{\mathrm{L}^{3/2}(\mathbb{T}^3)} \lesssim M_3 f(t)^{2/3}\|f(t)\|_{\mathrm{L}^\infty}^{1/3}.$$

We therefore obtain (ii) and (iii) using (i) and Lemma 3.1. ☐

3.2 The affine change of variables

We discuss in this section an affine change of variables in velocity, which will be crucial in the following in order to

- obtain *short time* estimates for the L^∞ norm of ρ_f and j_f (in any set of assumptions that we consider);
- obtain *long time* estimates for the L^∞ norm of ρ_f and j_f in the case ν large, $\mathscr{E}(0)$ arbitrary.

Lemma 3.4. *Consider a vector field $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+;\mathrm{H}^1\cap\mathrm{L}^\infty(\mathbb{T}^d))$ and $f_0 \in \mathrm{L}^1\cap\mathrm{L}^\infty(\mathbb{T}^d\times\mathbb{R}^d)$ satisfying* (1.13) *for some $q>d+1$. Then, if $f\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathrm{L}^1\cap\mathrm{L}^\infty(\mathbb{T}^d\times\mathbb{R}^d))$ is the renormalized solution of*

$$\partial_t f + v\cdot\nabla_x f + \nabla_v\cdot[f(u-v)] = 0,$$
$$f_{|t=0} = f_0,$$

one has the following estimates for almost all $t\geq 0$:

$$\|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^d)} + \|j_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^d)} \lesssim_{d,q} N_q(f_0)\left(1+\left(\int_0^t e^s\|u(s)-\langle u(s)\rangle\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\,\mathrm{d}s\right)^d\right), \tag{3.7}$$

where we recall $N_q(f_0)$ is defined in (1.14)*.*

Remark 3.1. *We note that in view of long time estimates, the estimate* (3.7) *is useful only if one can prove that $\|u(s)-\langle u(s)\rangle\|_{\mathrm{L}^\infty(\mathbb{T}^d)}$ decays faster that e^s. Such a decay will be achieved by interpolation with the L^2 decay estimate* (2.10)*obtained in Lemma 2.5, thanks to the modulated energy identity. However such a procedure will require ν to be large enough, which explains why Lemma 3.4 is useful in this regime only.*

Proof ♠. Let $(\mathrm{X}(s;t,x,v),\mathrm{V}(s;t,x,v))$ be the characteristics (3.1) associated to u. By the method fo characteristics, we write

$$\rho_f(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v = e^{dt}\int_{\mathbb{R}^d} f_0(t,\mathrm{X}(0;t,x,v),\mathrm{V}(0;t,x,v))\,\mathrm{d}v,$$

and a similar formula for j_f from which we infer, using the pointwise decay assumption on the initial data,

$$\rho_f(t,x)+|j_f(t,x)| \leq N_q(f_0)e^{dt}\int_{\mathbb{R}^d}\frac{1+|\mathrm{V}(0;t,x,v)|}{1+|\mathrm{V}(0;t,x,v)|^q}\,\mathrm{d}v. \tag{3.8}$$

Now, thanks to the differential equation satisfied by $s\mapsto \mathrm{V}(s;t,x,v)$ we have

$$\begin{aligned}\mathrm{V}(0;t,x,v) &= e^t v - \int_0^t e^s u(s,\mathrm{X}(0;s,x,v))\,\mathrm{d}s \qquad (3.9)\\ &= e^t\left(v-\int_0^t e^{s-t}\langle u(s)\rangle\,\mathrm{d}s\right) - \int_0^t e^s\Big(u(s,\mathrm{X}(0;s,x,v)) - \langle u(s)\rangle\,\mathrm{d}s\Big).\end{aligned}$$

We consider now, for a fixed $(t,x) \in \mathbb{R}_+ \times \mathbb{T}^d$, the following change of variable

$$w(v) := e^t \left(v - \int_0^t e^{s-t} \langle u(s) \rangle \, \mathrm{d}s \right), \tag{3.10}$$

and notice that for all $v \in \mathbb{R}^d$

$$|\mathrm{V}(0;t,x,v) - w(v)| \leq \int_0^t e^s \|u(s) - \langle u(s) \rangle\|_{\mathrm{L}^\infty(\mathbb{T}^d)} \, \mathrm{d}s.$$

In particular, using the change of variable (3.10) in the integral appearing in (3.8) we get to the following inequality

$$\rho_f(t,x) + |j_f(t,x)| \leq N_q(f_0) \int_{\mathbb{R}^d} \frac{1 + |w - \mathrm{A}(t,x,w)|}{1 + |w - \mathrm{A}(t,x,w)|^q} \, \mathrm{d}w, \tag{3.11}$$

where

$$\mathrm{A}(t,x,w) := \int_0^t e^s \left(u \left(s, \mathrm{X} \left(0; s, x, e^t w + \int_0^t e^{s-t} \langle u(s) \rangle \, \mathrm{d}s \right) \right) - \langle u(s) \rangle \right) \mathrm{d}s$$

satisfies

$$\|\mathrm{A}(t)\|_{\mathrm{L}^\infty(\mathbb{T}^d \times \mathbb{R}^d)} \leq \int_0^t e^s \|u(s) - \langle u(s) \rangle\|_{\mathrm{L}^\infty(\mathbb{T}^2)} \, \mathrm{d}s. \tag{3.12}$$

If $|w| > 2|\mathrm{A}(t,x,w)|$, one has by reverse triangular inequality

$$|w - \mathrm{A}(t,x,w)| \geq |w| - |\mathrm{A}(t,x,w)| \geq |w| - \|\mathrm{A}(t)\|_{\mathrm{L}^\infty(\mathbb{T}^d \times \mathbb{R}^d)} > \frac{|w|}{2},$$

and therefore,

$$\mathbf{1}_{|w|>2|\mathrm{A}(t,x,w)|} \frac{1 + |w - \mathrm{A}(t,x,w)|}{1 + |w - \mathrm{A}(t,x,w)|^q} \lesssim_q \frac{1 + |w|}{1 + |w|^q},$$

which is an integrable function of w on $\mathbb{R}^d$ because $q > d+1$. In particular, splitting the integral in (3.11) for $|w| \gtrless |\mathrm{A}(t,x,w)|$, we have

$$\rho_f(t,x) + |j_f(t,x)| \lesssim_{d,q} \|\mathrm{A}(t)\|^d_{\mathrm{L}^\infty(\mathbb{T}^d \times \mathbb{R}^d)} + 1.$$

Thanks to the control (3.12) on A, we eventually get (3.7). $\square$

3.3 The straightening change of variable

We discuss in this section another type of change of variables in velocity that will allow us to treat the other case $\mathscr{E}(0)$ small, ν arbitrary. The idea is to come down to the "free" case (that is to say to the characteristics associated to the vector field $(x,v) \to (v,-v)$ here), by using an appropriate diffeomorphism in velocity. We will see that a control bearing on $\|\nabla_x u\|_{\mathrm{L}^1_t \mathrm{L}^\infty(\mathbb{T}^d)}$ will allow us to reach this aim.

This change of variables is close in spirit to that employed by Bardos-Degond in [1] for the study of small data solutions to the Vlasov-Poisson system on $\mathbb{R}^3$. Note however that the stabilisation mechanisms are very different as for Vlasov-Poisson on $\mathbb{R}^3$, it is based on dispersion due to the free transport operator, while for the Vlasov-Navier-Stokes on $\mathbb{T}^3$, it will be based on the dissipation in the Navier-Stokes equations.

We also mention that similar ideas were recently used in the context of the inertialess limit of the Vlasov-Stokes system in [19].

Lemma 3.5. *There exists a numerical constant* $\mathrm{c}(d)$*, depending only on the dimension, such that the following holds. Fix* $t \in \mathbb{R}_+$ *and* $x \in \mathbb{R}^d$ *and consider the map*

$$\Gamma_{t,x} : v \mapsto \mathrm{V}(0; t, x, v).$$

Assume that

$$\int_0^t \|\nabla_x u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)} \,\mathrm{d}s \leq \mathrm{c}(d). \tag{3.13}$$

Then one has

(i) For all $v \in \mathbb{R}^d$*,* $|\det \mathrm{D}_v \Gamma_{t,x}(v)| \geq e^{dt}/4$*;*

(ii) $\Gamma_{t,x}$ *is a* $\mathscr{C}^1$*-diffeomorphism from* $\mathbb{R}^d$ *to itself.*

Remark 3.2. *The constant* $\mathrm{c}(d)$ *can be explicitly computed from the proof below. For instance, we can take* $\mathrm{c}(2) = \frac{1}{5}$*.*

Proof ♠. The proof follows the main lines of the arguments outlined in [1, Proposition 1 and Corollary 1].
(i) Consider a generic vector-valued flow $\mathrm{Z}^s_{t,z} := \mathrm{Z}(s; t, z)$ associated to a smooth vector field $w(t, z)$ defined on $\mathbb{R}_+ \times X$ and assume that $\|\mathrm{D}_z w(t)\|_{\mathrm{L}^\infty(X)} \leq 1 + \psi(t)$, for some function $\psi \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+)$. We have $\partial_s \mathrm{Z}^s_{t,z} = w(s, \mathrm{Z}^s_{t,z})$ which after differentiation with respect to z (introducing $\Theta^s_{t,z} := \mathrm{D}_z \mathrm{Z}^s_{t,z}$) leads to

$$\partial_s \Theta^s_{t,z} = \mathrm{D}_z w(s, \mathrm{Z}^s_{t,z}) \cdot \Theta^s_{t,z},$$

from which we get by Gronwall's inequality for $s \leq t$

$$\|\Theta^s_{t,z}\|_{\mathrm{L}^\infty(X)} \leq \|\Theta^t_{t,z}\|_{\mathrm{L}^\infty(X)} \exp\left(\int_s^t \|\mathrm{D}_z w(\sigma)\|_{\mathrm{L}^\infty(X)} \mathrm{d}\sigma\right) \leq e^{t-s} \exp\left(\int_s^t |\psi(\sigma)| \,\mathrm{d}\sigma\right), \tag{3.14}$$

where we used $\Theta^t_{t,z} = \mathrm{Id}$.

Now, let us get back to our system. Introducing the state variable $z := (x, v)$ which belongs to $X = \mathbb{T}^d \times \mathbb{R}^d$, the vector field $w(t, z) := (v, u(t, x) - v)$ satisfies the assumption for the above abstract result, since $\|\mathrm{D}_z w(t)\|_{\mathrm{L}^\infty(X)} \leq 1 + \|\nabla_x u\|_{\mathrm{L}^\infty(\mathbb{T}^d)}$. If we denote by $(\mathrm{X}(s; t, z), \mathrm{V}(s; t, z))$ the characteristics associated to u, integrating the equation defining $s \mapsto \mathrm{V}(s; t, z)$ we have

$$\mathrm{V}(0; t, z) = e^t v - \int_0^t e^s u(s, \mathrm{X}(s; t, z)) \,\mathrm{d}s, \tag{3.15}$$

which leads to

$$\mathrm{D}_v \mathrm{V}(0; t, z) - e^t \mathrm{Id} = -\int_0^t e^s \nabla_x u(s, \mathrm{X}(s; t, z)) \mathrm{D}_v \mathrm{X}(s, t; z) \,\mathrm{d}s.$$

We thus infer from (3.14) with $\psi = \|\nabla_x u\|_{\mathrm{L}^\infty(\mathbb{T}^d)}$ that

$$\|\mathrm{D}_v \mathrm{X}(s; t, z)\|_{\mathrm{L}^\infty(\mathbb{T}^d \times \mathbb{R}^d)} \leq e^{t-s} \exp\left(\int_s^t \|\nabla_x u(\tau)\|_{\mathrm{L}^\infty(\mathbb{T}^d)} \,\mathrm{d}\tau\right)$$

and thus that

$$\|e^{-t} \mathrm{D}_v \mathrm{V}(0; t, z) - \mathrm{Id}\|_{\mathrm{L}^\infty(\mathbb{T}^d \times \mathbb{R}^d)} \leq \exp\left(\int_0^t \|\nabla_x u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)} \mathrm{d}s\right) \int_0^t \|\nabla_x u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)} \mathrm{d}s.$$

Let us now assume for simplicity that $d = 2$. The case $d = 3$ can be treated with similar considerations. Assume that $\int_0^t \|\nabla_x u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\,\mathrm{d}s \le 1/5$. From $(5/4)^5 \ge 3 \ge e$ we infer $e^{1/5}/5 \le 1/4$, so that the assumption of the Lemma implies

$$\|e^{-t}\mathrm{D}_v\mathrm{V}(0;t,z) - \mathrm{Id}\|_{\mathrm{L}^\infty(\mathbb{T}^2\times\mathbb{R}^2)} \le \frac{1}{4}.$$

If H is a 2×2 matrix, one has

$$\det(\mathrm{Id} + H) = 1 + \mathrm{Tr}(H) + \det(H),$$

from which we conclude that

$$\det(\mathrm{D}_v\mathrm{V}(0;t,z)) = e^{2t}\det(\mathrm{Id} + e^{-t}\mathrm{D}_v\mathrm{V}(0,t;z) - \mathrm{Id}) \ge e^{2t}\left(1 - \frac{2}{4} - \frac{2}{4^2}\right) \ge \frac{e^{2t}}{4}.$$

(ii) We use Hadamard's global inversion Theorem (recalled in Theorem 9.3 in the appendix): it suffices to prove that $\Gamma_{t,x}$ is proper, that is $|v| \to +\infty \Rightarrow |\Gamma_{t,x}(v)| \to +\infty$. Since $\Gamma_{t,x}(v) = \mathrm{V}(0;t,x,v)$, one can use the formula (3.15) above to write

$$\Gamma_{t,x}(v) = e^t v - \int_0^t e^s u(s, \mathrm{X}(s;t,x,v)\,\mathrm{d}s,$$

and the properness follows since u belongs to $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+;\mathrm{L}^\infty(\mathbb{T}^d))$.

□

Thanks to the change of variables of Lemma 3.5, we deduce the following control on moments.

Lemma 3.6. *With the same assumption* (3.13) *as in Lemma 3.5, we have for almost all $t \ge 0$,*

$$\|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^d)} \le 4 I_q N_q(f_0), \tag{3.16}$$

$$\|j_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^d)} \le 4 I_q e^{-t}\left(\int_0^t e^s\|u(s)\|_\infty\,\mathrm{d}s + 1\right) N_q(f_0), \tag{3.17}$$

where $N_q(f_0)$ is given by (1.14) *and*

$$I_q := \int_{\mathbb{R}^d} \frac{1+|v|}{1+|v|^q}\,\mathrm{d}v.$$

Proof ♠. Let $(\mathrm{X}(s,t;x,v), \mathrm{V}(s,t;x,v))$ be the characteristics (3.1) associated to u. We start again from the representation formula

$$\rho_f(t,x) = e^{dt}\int_{\mathbb{R}^d} f_0(\mathrm{X}(0;t,x,v), \mathrm{V}(0;t,x,v))\,\mathrm{d}v,$$

and use point (ii) of Lemma 3.5 to perform the change of variables $w = \Gamma_{t,x}(v) = \mathrm{V}(0;t,x,v)$ end deduce

$$\rho_f(t,x) = e^{dt}\int_{\mathbb{R}^d} f_0(\mathrm{X}(0;t,x,\Gamma_{t,x}^{-1}(w)), w)\left|\mathrm{Jac}(\Gamma_{t,x}^{-1})(w)\right|\,\mathrm{d}w,$$

which, by point (i) of Lemma 3.5 implies

$$\|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^d)} \le 4 N_q(f_0) I_q. \tag{3.18}$$

For j_f we proceed similarly and write the representation formula (valid for the same reasons)

$$j_f(t,x) = e^{dt} \int_{\mathbb{R}^d} \Gamma_{t,x}^{-1}(w) f_0(\mathrm{X}(0;t,x,\Gamma_{t,x}^{-1}(w)), w) \left|\mathrm{Jac}(\Gamma_{t,x}^{-1})(w))\right| \,\mathrm{d}w.$$

By definition of $\Gamma_{t,x}^{-1}(w)$, the following identity holds.

$$w = e^t \Gamma_{t,x}^{-1}(w) - \int_0^t e^s u(s, \mathrm{X}(s;t,x,\Gamma_{t,x}^{-1}(w)))\,\mathrm{d}s, \tag{3.19}$$

from which we deduce

$$|\Gamma_{t,x}^{-1}(w)| \leq e^{-t}\left[|w| + \int_0^t e^s \|u(s)\|_\infty \,\mathrm{d}s\right],$$

hence the claimed result.

□

In the next lemma, we study how the pointwise decay condition of Definition 1.4 can be locally propagated.

Lemma 3.7. *Let $t_0 > 0$. If f_0 satisfies* (1.13) *and $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{H}^1 \cap \mathrm{L}^\infty(\mathbb{T}^d))$, then $f_{t_0} := f(t_0)$ satisfies also* (1.13) *and*

$$N(f_{t_0}) \lesssim_{q,d} (1 + \|u\|^q_{\mathrm{L}^1(0,t_0;\mathrm{L}^\infty(\mathbb{T}^d))}) N_q(f_0).$$

Proof ♠. We write

$$f(t_0,x,v) = e^{dt_0} f_0(\mathrm{X}(0;t_0,x,v), \mathrm{V}(0;t_0,x,v)),$$

where thanks to (3.9) we have

$$|v| \leq |V(0;t_0,x,v)| + \int_0^{t_0} \|u(s)\|_\infty \,\mathrm{d}s,$$

and therefore

$$(1+|v|^q) f(t_0,x,v) \lesssim_q e^d (1 + \|u\|^q_{\mathrm{L}^1(0,t_0;\mathrm{L}^\infty(\mathbb{T}^d))}) N_q(f_0).$$

□

This allows to obtain another version of Lemma 3.6 but with a control like (3.13) starting only from some time $t_0 > 0$.

Lemma 3.8. *Let $t_0 > 0$. With the same assumptions and notations as in Lemma 3.5, except that we replace* (3.13) *by*

$$\int_{t_0}^t \|\nabla_x u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)} \,\mathrm{d}s \leq \mathrm{c}(d), \tag{3.20}$$

we have

$$\|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^d)} \lesssim N_q(f_0)(1 + \|u\|^q_{\mathrm{L}^1(0,t_0;\mathrm{L}^\infty(\mathbb{T}^d))}), \tag{3.21}$$

$$\|j_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^d)} \lesssim e^{-t}\left(\int_0^t e^s \|u(s)\|_\infty \,\mathrm{d}s + 1\right) N_q(f_0)(1 + \|u\|^q_{\mathrm{L}^1(0,t_0;\mathrm{L}^\infty(\mathbb{T}^d))}). \tag{3.22}$$

Proof ♠. We can reproduce Lemma 3.4, Lemma 3.5 and Lemma 3.6 replacing the initial time $t = 0$ by $t = t_0$ and thus f_0 by $f(t_0)$. Using Lemma 3.7, we obtain the claimed estimate. □

4 Regularity estimates for solutions of the Vlasov-Navier-Stokes system

In this section, given an admissible initial condition (u_0, f_0), we *fix* an associated Leray solution (u, f), which satisfies all the estimates of Section 2 and 3.

In dimension $d = 3$, we shall also assume that $M_\alpha f_0 < \infty$ for some $\alpha > 3$ and $u_0 \in \mathrm{H}^{1/2}(\mathbb{T}^3)$.

This section is devoted to the following two tasks:

- obtaining a precise short time control for the L^∞ norm of ρ_f and j_f (relying on Lemma 3.4);
- obtaining $\mathrm{L}^\infty_t \mathrm{H}^1_x \cap \mathrm{L}^2_t \mathrm{H}^2_x$ estimates for u, on time intervals *away* from zero, as developed in Propositions 4.2 and 4.3.

Such estimates will be the key to prove Theorem 1.3 in the case where ν is large enough, a proof of which will be provided right after this section. However, this will not be sufficient to handle the other case $\mathscr{E}(0)$ small but $\nu > 0$ arbitrary: to this end, higher order estimates will be derived in Section 6.

We shall also introduce in this section the notion of *strong existence times* (see Definition 4.2) that concerns $d = 3$. Loosely speaking, this corresponds to times t for which the solution u of the Navier-Stokes equation is *strong* on the interval of time $[0, t]$, which means in this context with $\mathrm{H}^{1/2}(\mathbb{T}^d)$ regularity. A smallness criterion bearing both on u and on the Brinkman force $j_f - \rho_f u$ (see (4.9)) will be used.

Notation 4.1. *We introduce the following notations:*

$$F := j_f - \rho_f u,$$
$$S := F - (u \cdot \nabla)u.$$

We start with a first bound for F coming from the energy dissipation.

Lemma 4.1. *If $\rho_f \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^\infty(\mathbb{T}^d))$, one has for almost all $t \geq 0$*

$$\int_0^t \|F(s)\|^2_{\mathrm{L}^2(\mathbb{T}^d)} \, \mathrm{d}s \leq \min(\mathrm{E}(0), \mathscr{E}(0)) \sup_{s \in [0,t]} \|\rho_f(s)\|_\infty.$$

Proof. By Cauchy-Schwarz's inequality, we have a.e.,

$$|F| = \left| \int_{\mathbb{R}^d} f(v - u) \, \mathrm{d}v \right| \leq \rho_f^{1/2} \left(\int_{\mathbb{R}^d} f|v - u|^2 \, \mathrm{d}v \right)^{1/2},$$

from which we infer for almost all $s \geq 0$,

$$\|F(s)\|^2_{\mathrm{L}^2(\mathbb{T}^d)} \leq \|\rho_f(s)\|_\infty \mathrm{D}(s),$$

where D is the dissipation introduced in (1.6). The estimate follows thus from the energy (1.12) and modulated energy (2.8) estimates. □

4.1 Higher order energy estimates for the Navier-Stokes equations with a source term

This section is dedicated to H^1 energy estimates for the Navier-Stokes system in 2D and 3D. Such higher order energy estimates for the Navier-Stokes estimate seem to be folklore; yet since we need precise enough versions keeping track of several parameters and for the sake of completeness, we provide a proof of these statements in an appendix (see Section 9.4).

Proposition 4.2. *Fix $u_0 \in \mathrm{L}^2_{\mathrm{div}}(\mathbb{T}^2)$ and $F \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+;\mathrm{L}^2(\mathbb{T}^2))$. The Leray solution of the Navier-Stokes system with source term F, initiated by u_0 satisfies $u \in \mathscr{C}^0(\mathbb{R}_+^*;\mathrm{H}^1(\mathbb{T}^2))$ and $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+^*;\mathrm{H}^2(\mathbb{T}^2))$ and furthermore for $t \geq 1$*

$$\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \nu \int_1^t \|\Delta u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^2)}\,\mathrm{d}s \leq \varphi\left(\mathrm{A}(t) + \frac{1}{\nu}\right)\left(1 + \int_0^t \|F(s)\|^2_{\mathrm{L}^2(\mathbb{T}^2)}\,\mathrm{d}s\right),$$

where φ and A are given in Proposition 9.5.

Proposition 4.3. *There exists a universal constant $\mathrm{C}_\star > 0$ such that the following holds. Consider $\nu > 0$, $u_0 \in \mathrm{H}^{1/2}_{\mathrm{div}}(\mathbb{T}^3)$, $F \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+;\mathrm{H}^{-1/2}(\mathbb{T}^3))$ and $T > 0$ such that*

$$\|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \frac{\mathrm{C}_\star}{\nu}\int_0^T \|F(s)\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)}\,\mathrm{d}s \leq \frac{\nu^2}{\mathrm{C}_\star^2}. \tag{4.1}$$

Then, there exists on $[0,T]$ a unique Leray solution of the Navier-Stokes system with source F and with initial data u_0. This solution u belongs to $\mathscr{C}^0([0,T];\mathrm{H}^{1/2}(\mathbb{T}^3)) \cap \mathrm{L}^2(0,T;\mathrm{H}^{3/2}(\mathbb{T}^3))$ and satisfies for all $0 \leq t \leq T$

$$\|u(t)\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \nu\int_0^t \|\nabla u(s)\|^2_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)}\,\mathrm{d}s \leq \|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \frac{\mathrm{C}_\star}{\nu}\int_0^t \|F(s)\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)}\,\mathrm{d}s.$$

Furthermore, for some onto nondecreasing continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ independent of ν, u_0, F, T, we have for $0 \leq t \leq T$

$$\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^3)} + \nu\int_0^t \|\Delta u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s \leq \psi\left(\mathrm{A}(t) + \frac{1}{\nu}\right)\left(1 + \int_0^t \|F(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s\right), \tag{4.2}$$

where A is given by (9.6).

4.2 The 2D case

The 2D case is well-understood thanks to Proposition 4.2 and to [16].

Proposition 4.4. *Let $d = 2$. For all $t_0 > 0$, the following estimates hold.*

$$\|u\|_{\mathrm{L}^1(0,t_0;\mathrm{L}^\infty(\mathbb{T}^2))} \leq \mathrm{C}_0, \tag{4.3}$$

$$\sup_{[0,t_0]}\left\{\|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^2)} + \|j_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^2)}\right\} \leq \varphi\left(N_q(f_0) + \mathrm{E}(0) + \frac{1}{\nu}\right), \tag{4.4}$$

for $\mathrm{C}_0 > 0$ depending on the initial condition and t_0 and some onto nondecreasing continuous function φ, and for almost all $t \geq t_0$:

$$\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \nu\int_{t_0}^t \|\Delta u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^2)}\,\mathrm{d}s \leq \varphi\left(\mathrm{E}(0) + \mathscr{E}(0) + \nu^{-1}\right)(1 + \sup_{s\in[0,t]}\|\rho_f(s)\|_\infty), \tag{4.5}$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is the (universal) onto nondecreasing continuous function given in Corollary 4.2

Proof. The regularity estimates (4.3) and (4.4) are directly extracted from [16, Lemma 3 and Corollary 1]. As for the H^1 energy estimate (4.5), we apply Proposition 4.2 and Lemma 4.1. ☐

4.3 The 3D case

The three dimensional case is more involved and requires a specific analysis on which we focus until the end of this section. We first show how to obtain the same estimates as (4.3) and (4.4) in dimension 2.

Proposition 4.5. *Let $t_0 > 0$. We have $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^\infty(\mathbb{T}^3))$ and both ρ_f and j_f belong to $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^\infty(\mathbb{T}^3))$; moreover the following estimates hold*

$$\|u\|_{\mathrm{L}^1(0,t_0;\mathrm{L}^\infty(\mathbb{T}^3))} \leq \mathrm{C}_0, \tag{4.6}$$

and

$$\begin{aligned}\sup_{[0,t_0]} &\left\{\|\rho_f(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} + \|j_f(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\right\} \\ &\leq \varphi\left(\|u_0\|_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + M_\alpha f_0 + N_q(f_0) + \mathrm{E}(0) + \nu^{-1}\right), \end{aligned}\tag{4.7}$$

for $\mathrm{C}_0 > 0$ depending on the initial condition and t_0 and some onto continuous nondecreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$.

Proof. In this proof we use the notation $\lesssim_t$ to refer to an inequality of the form $A \leq \psi(t)B$ where ψ is some universal continuous function. For readability we will denote by C_0 any constant like in the right hand side of (4.7) for some continuous nondecreasing and onto $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$.

Since $M_2 f_0 < +\infty$, we have also $M_3 f_0 \lesssim M_2 f_0 + M_\alpha f_0 < +\infty$. We can apply Lemma 3.3 to infer

$$\|\rho_f(t)\|_{\mathrm{L}^2(\mathbb{T}^3)} + \|j_f(t)\|_{\mathrm{L}^{3/2}(\mathbb{T}^3)} \lesssim_t \mathrm{C}_0.$$

In particular recalling the notation $S := F - (u \cdot \nabla)u$ we infer, using Hölder's inequality and another time the Sobolev embedding $\mathrm{H}^1(\mathbb{T}^3) \hookrightarrow \mathrm{L}^6(\mathbb{T}^3)$ and the energy estimate (1.12),

$$\int_0^t \|S(s)\|^2_{\mathrm{L}^{3/2}(\mathbb{T}^3)}\,\mathrm{d}s \lesssim_t \mathrm{C}_0.$$

Now consider w the unique solution of ($\mathbb{P}$ stands for the Leray projector, that is the projection on divergence free vector fields)

$$\begin{aligned}\partial_t w - \Delta w &= \mathbb{P}S,\\ \operatorname{div} w &= 0,\\ w(0) &= 0,\end{aligned}$$

so that $u - w = e^{t\Delta}u_0$. Since $u_0 \in \mathrm{H}^{1/2}(\mathbb{T}^3)$, we infer from [12, Lemma 3.3] that $u - w \in \mathrm{L}^2(\mathbb{R}_+; \mathrm{L}^\infty(\mathbb{T}^3))$ with the estimate

$$\int_0^\infty \|(u-w)(s)\|^2_{\mathrm{L}^\infty(\mathbb{T}^3)}\,\mathrm{d}s \lesssim \|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)}.$$

Thanks to the $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^{3/2}(\mathbb{T}^3))$ estimate on S that we obtained above, we infer from the continuity of $\mathbb{P}$ on $\mathrm{L}^{3/2}(\mathbb{T}^3)$ and the maximal regularity of the Stokes operator that $\mathrm{D}^2 w \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^{3/2}(\mathbb{T}^3))$ and thus $w \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^p(\mathbb{T}^3))$ for all $p < \infty$, by Sobolev's embedding with the bound

$$\int_0^t \|w(s)\|^2_{\mathrm{L}^p(\mathbb{T}^3)}\,\mathrm{d}s \lesssim_t \mathrm{C}_0,$$

and we therefore established $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^p(\mathbb{T}^d))$ with

$$\int_0^t \|u(s)\|^2_{\mathrm{L}^p(\mathbb{T}^3)} \, \mathrm{d}s \lesssim_t \mathrm{C}_0. \tag{4.8}$$

In particular, we get $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^{\alpha+3}(\mathbb{T}^d))$ and we can obtain the following variant of Lemma 3.3: using estimate (3.4) of Lemma 3.2 to propagate the assumption $M_\alpha f_0 < +\infty$ we first have

$$\begin{aligned} M_\alpha f(t) &\lesssim \left(M_\alpha f_0 + e^{\frac{3t}{\alpha+3}} \int_0^t \|u(s)\|_{\alpha+3} \, \mathrm{d}s \right)^{\alpha+3} \\ &\lesssim_t \mathrm{C}_0. \end{aligned}$$

We use the interpolation estimate (3.6) with $k = \alpha$ and $\ell \in \{0, 1\}$ to obtain this time

$$\|\rho_f(t)\|_{\mathrm{L}^{\frac{\alpha+3}{3}}(\mathbb{T}^3)} + \|j_f(t)\|_{\mathrm{L}^{\frac{\alpha+3}{4}}(\mathbb{T}^3)} \lesssim_t \mathrm{C}_0,$$

where the integration exponents are stricly larger than $3/2$. Using (4.8) we can estimate $(u \cdot \nabla)u$ in some $\mathrm{L}^\gamma_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^r(\mathbb{T}^3))$ for $\gamma > 1$ and $r > 3/2$ leading to the following estimate on the source S :

$$\int_0^t \|S(s)\|^\gamma_{\mathrm{L}^r(\mathbb{T}^3)} \, \mathrm{d}s \lesssim_t \mathrm{C}_0.$$

Since $r > 3/2$, using another time the maximal regularity of the Stokes operator we eventually infer

$$\int_0^t \|w(s)\|^\gamma_{\mathrm{L}^\infty(\mathbb{T}^3)} \, \mathrm{d}s \lesssim_t \mathrm{C}_0.$$

All in all, we recovered that $u = (u - w) + w \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^\infty(\mathbb{T}^3))$ and using estimate (3.7) of Lemma 3.4 we infer that both ρ_f, j_f belong to $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^\infty(\mathbb{T}^d))$ with the estimate

$$\|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} + \|j_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \lesssim_t N_q(f_0)\mathrm{C}_0,$$

so that (4.7) is a simple a consequence of the meaning of $\lesssim_t$ and C_0. □

Unfortunately the parabolic regularization that we have exhibited in Proposition 4.4 in the 2D case is in dimension 3 only satisfied locally, or globally at the cost of a smallness assumption. Of course this is related to the fact that we don't know if the Leray solution is unique in dimension $d = 3$.

Proposition 4.6. *If for some $T > 0$ there holds*

$$\|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \frac{\mathrm{C}_\star}{\nu} \int_0^T \|F(s)\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)} \, \mathrm{d}s \leq \frac{\nu^2}{\mathrm{C}_\star^2}, \tag{4.9}$$

where $\mathrm{C}_\star$ is the universal constant given by Proposition 4.3, then one has for all $1 \leq t \leq T$ the estimate

$$\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^3)} + \nu \int_1^t \|\Delta u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)} \, \mathrm{d}s \leq \psi\left(\mathrm{E}(0) + \nu^{-1}\right) \left(1 + \sup_{[0,t]} \|\rho_f(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \right), \tag{4.10}$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is the (universal) onto nondecreasing continuous function given in Proposition 4.3.

Proof. The fact that $F \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{H}^{-1/2}(\mathbb{T}^3))$ is a consequence of Proposition 4.5 and the embedding $\mathrm{L}^{3/2}(\mathbb{T}^3) \hookrightarrow \mathrm{H}^{-1/2}(\mathbb{T}^3)$. For the second estimate, thanks to the smallness condition (4.9), we invoke Proposition 4.3 and Lemma 4.1. □

4.4 Strong existence times and higher order estimates

The assumptions of Theorem 1.3 ensure, thanks to Proposition 4.4 and Proposition 4.5, that ρ_f belongs to $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathrm{L}^\infty(\mathbb{T}^d))$. We can therefore focus on proving that $t \mapsto \|\rho_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}$ is bounded on $[1,+\infty)$. It is convenient to introduce the following notations.

Definition 4.1. *We set for $t_0 > 0$ and $t \geq t_0$*

$$\mathrm{M}_{\rho_f}(t_0,t) := \sup_{[t_0,t]} \|\rho_f(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}, \quad \mathrm{M}_{j_f}(t_0,t) := \sup_{[t_0,t]} \|j_f(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}, \tag{4.11}$$

$$\mathrm{M}_{\rho_f,j_f}(t_0,t) := \mathrm{M}_{\rho_f}(t_0,t) + \mathrm{M}_{j_f}(t_0,t). \tag{4.12}$$

The estimates (4.4) and (4.7) motivate the following notation.

Notation 4.2. *From now on, $A \lesssim_{0,\nu} B$ will mean*

$$A \leq \varphi\left(N_q(f_0) + \mathrm{E}(0) + \nu^{-1}\right) B, \text{ for } d = 2,$$
$$A \leq \varphi\left(\|u_0\|_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + M_\alpha f_0 + N_q(f_0) + \mathrm{E}(0) + \nu^{-1}\right) B, \text{ for } d = 3,$$

where q and α are the exponents given in the statements of Theorem 1.1 and Theorem 1.2. Note that $\lesssim_{0,\nu}$ may depend on the integration exponents appearing in the inequality, but this will always be harmless.

Definition 4.2 (Strong existence times)**.** *In dimension $d = 3$, a real number $t \geq 0$ will be said to be a* strong existence time *whenever* (4.9) *holds. In dimension $d = 2$, we take the convention that all $t \geq 0$ are strong existence times.*

The following lemma asserts that within our set of assumptions, we have a lower bound for strong existence times.

Lemma 4.7. *Under the assumptions of Theorem 1.2,* 1 *is a strong existence time.*

Proof. The proof is based on the affine change of variables of Lemma 3.4. Let us consider the case ν large. By Lemma 4.1, Lemma 3.4 and (4.6), we can bound

$$\begin{aligned}\int_0^1 \|F(s)\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)}\,\mathrm{d}s &\lesssim \int_0^1 \|F(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s \\ &\lesssim \min(\mathrm{E}(0),\mathscr{E}(0)) \sup_{s\in[0,1]} \|\rho_f(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \\ &\lesssim \min(\mathrm{E}(0),\mathscr{E}(0)) \left(1 + \left(\int_0^1 e^s \|u(s) - \langle u(s)\rangle\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\,\mathrm{d}s\right)^3\right) \\ &\lesssim \mathrm{C}_0 \min(\mathrm{E}(0),\mathscr{E}(0)),\end{aligned}$$

so that we ce can find a continuous nondecreasing onto function $\psi_0 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\nu > \psi_0\left(N_q(f_0) + M_\alpha f_0 + \mathrm{E}(0) + \|u_0\|_{\mathrm{H}^{1/2}(\mathbb{T}^3)}\right)$$

implies that (4.9) is satisfied for $T = 1$. We impose $\psi_0 \geq \psi$. The proof is similar in the case $\mathscr{E}(0)$ small and therefore we skip it.

$\square$

We sum up the results (in dimension $d = 2$ and $d = 3$) of this subsection with the following result. From (4.7) and (4.4) we infer the following consequence of Propositions 4.4 and 4.6.

Corollary 4.8. *Let $t_0 > 0$. The following estimate holds for any strong existence time $t \geq t_0$:*

$$\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^d)} + \nu \int_{t_0}^t \|\Delta u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^d)}\,\mathrm{d}s \lesssim_{0,\nu} 1 + \mathrm{M}_{\rho_f}(t_0,t). \tag{4.13}$$

5 Proof of Theorem 1.3 in the case ν large

In this section, we prove Theorem 1.3 when the viscosity $\nu > 0$ is large enough. The general strategy is as follows.

- We rely on the key Lemma 3.4 to obtain the control (3.7) on the L^∞ norm of ρ_f. For this estimate to be meaningful, we have to prove that $\|u(s) - \langle u(s)\rangle\|_{\mathrm{L}^\infty(\mathbb{T}^d)}$ is decaying faster than e^{-s}.
- To this purpose, we rely on the analysis of Section 4, namely we use Corollary 4.8 to obtain a $\mathrm{L}^2_t\mathrm{H}^2_x$ estimate for u.
- We interpolate this $\mathrm{L}^2_t\mathrm{H}^2_x$ estimate with the pointwise L^2_x estimate for $(u - \langle u\rangle)$, which quantity decays exponentially fast, as stated in Lemma 2.5. The required decay for $\|u(s) - \langle u(s)\rangle\|_{\mathrm{L}^\infty(\mathbb{T}^d)}$ is reached if ν is large enough.

In dimension $d = 2$, this procedure results in an *sublinear* inequality that allows to conclude on the spot that $\|\rho_f\|_{\mathrm{L}^\infty(\mathbb{T}^d)}$ is bounded on $\mathbb{R}_+$. In dimension $d = 3$, this inequality becomes *superlinear*, but requiring ν large enough is enough to bound $\|\rho_f\|_{\mathrm{L}^\infty(\mathbb{T}^d)}$ for all strong existence times. We finally rely on an additional bootstrap argument in order to show that the set of *strong existence times* is $\mathbb{R}_+$.

5.1 The 2D case

By Corollary 4.8 applied with $t_0 = 1$, using the straightforward inequality $\|\mathrm{D}^2 g\|_{\mathrm{L}^2(\mathbb{T}^2)} \lesssim \|\Delta g\|_{\mathrm{L}^2(\mathbb{T}^d)}$ on the torus, we infer that for all $t \geq 1$,

$$\int_1^t \|\mathrm{D}^2 u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^2)}\,\mathrm{d}s \lesssim_{0,\nu} 1 + \mathrm{M}_{\rho_f}(1,t). \tag{5.1}$$

We now rely on an interpolation argument. We invoke the Gagliardo-Nirenberg-Sobolev estimate of Theorem 9.4, applied for $g = u(s) - \langle u(s)\rangle$ with $p = \infty$, $q = r = m = 2$, $j = 0$ and, therefore, $\alpha = 1/2$:

$$\|u(s) - \langle u(s)\rangle\|_{\mathrm{L}^\infty(\mathbb{T}^2)} \lesssim \|\mathrm{D}^2 u(s)\|^{1/2}_{\mathrm{L}^2(\mathbb{T}^2)} \|u(s) - \langle u(s)\rangle\|^{1/2}_{\mathrm{L}^2(\mathbb{T}^2)}.$$

By Lemma 2.5, we deduce

$$\|u(s) - \langle u(s)\rangle\|_{\mathrm{L}^\infty(\mathbb{T}^2)} \lesssim \mathscr{E}(0)^{1/4} \|\mathrm{D}^2 u(s)\|^{1/2}_{\mathrm{L}^2(\mathbb{T}^2)} e^{-\nu c_P^2 s/2}. \tag{5.2}$$

and thus by Hölder's inequality

$$\begin{aligned}&\int_1^t e^s \|u(s) - \langle u(s)\rangle\|_\infty\,\mathrm{d}s \\ &\qquad\lesssim \mathscr{E}(0)^{1/4} \left(\int_1^t \|\mathrm{D}^2 u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^2)}\,\mathrm{d}s\right)^{1/4} \left(\int_1^t e^{\frac{4}{3}s(1-\nu c_P^2/2)}\,\mathrm{d}s\right)^{3/4}.\end{aligned}$$

Since by assumption $\nu > 2/c_P^2$ we thus infer, using (5.1)

$$\begin{aligned}\left(\int_1^t e^s \|u(s) - \langle u(s)\rangle\|_{\mathrm{L}^\infty(\mathbb{T}^2)}\,\mathrm{d}s\right)^2 &\lesssim_{0,\nu} \mathscr{E}(0)^{1/2} \left(\int_1^t \|\mathrm{D}^2 u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^2)}\,\mathrm{d}s\right)^{1/2} \\ &\lesssim_{0,\nu} \mathscr{E}(0)^{1/2}(1 + \mathrm{M}_{\rho_f}(1,t)^{1/2}).\end{aligned} \tag{5.3}$$

By Lemma 3.8 with $t_0 = 1$ and (4.3), we have thus proved for all $t \in [1, +\infty)$,

$$\mathrm{M}_{\rho_f}(1,t) \lesssim_{0,\nu} \mathscr{E}(0)^{1/2}(1 + \mathrm{M}_{\rho_f}(1,t)^{1/2}),$$

from which we straightforwardly infer that $\mathrm{M}_{\rho_f}(1,t)$ is bounded on $[1, +\infty)$. Since we also have (4.4) with $t_0 = 1$, we conclude that $\sup_{\mathbb{R}_+} \|\rho_f\|_{\mathrm{L}^\infty(\mathbb{T}^2)} < +\infty$.

We can therefore apply Lemma 2.6 with $T = +\infty$ and this concludes the proof of Theorem 1.3 in this case.

5.2 The 3D case

The beginning of the proof is close to that for the case $d = 2$. Thanks to Corollary 4.8 applied with $t_0 = 1$, for any strong existence time $t \geq 1$, we have,

$$\nu \int_1^t \|\mathrm{D}^2 u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)} \,\mathrm{d}s \lesssim \nu \int_1^t \|\Delta u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)} \,\mathrm{d}s \lesssim_{0,\nu} 1 + \mathrm{M}_{\rho_f}(1,t). \tag{5.4}$$

Once again we use an interpolation argument and invoke the Gagliardo-Nirenberg-Sobolev estimate, the only difference with the 2D case being the interpolation exponent: using $g = u(s) - \langle u(s)\rangle$, $(d,p,q,r,m,j) = (3,\infty,2,2,2,0)$ leads to $\alpha = 3/4$, so that

$$\|u(s) - \langle u(s)\rangle\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \lesssim \|\mathrm{D}^2 u(s)\|^{3/4}_{\mathrm{L}^2(\mathbb{T}^3)} \|u(s) - \langle u(s)\rangle\|^{1/4}_{\mathrm{L}^2(\mathbb{T}^3)}.$$

Using Lemma 2.5 we infer

$$\|u(s) - \langle u(s)\rangle\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \lesssim \mathscr{E}(0)^{1/8} \|\mathrm{D}^2 u(s)\|^{3/4}_{\mathrm{L}^2(\mathbb{T}^3)} e^{-\nu c_P^2 s/4}, \tag{5.5}$$

so that, as soon $\nu > 4/c_P^2$, using Hölder's inequality and (5.4)

$$\begin{aligned}\int_1^t e^s \|u(s) - \langle u(s)\rangle\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \,\mathrm{d}s &\lesssim_{0,\nu} \mathscr{E}(0)^{1/8} \left(\int_1^t \|\mathrm{D}^2 u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)} \,\mathrm{d}s\right)^{3/8} \\ &\lesssim_{0,\nu} \frac{\mathscr{E}(0)^{1/8}}{\nu^{3/8}} \left(1 + \mathrm{M}_{\rho_f}(1,t)\right)^{3/8},\end{aligned}$$

and therefore,

$$\left(\int_1^t e^s \|u(s) - \langle u(s)\rangle\|_{\mathrm{L}^\infty(\mathbb{T}^3)} \,\mathrm{d}s\right)^3 \lesssim_{0,\nu} \frac{\mathscr{E}(0)^{3/8}}{\nu^{9/8}} (1 + \mathrm{M}_{\rho_f}(1,t))^{9/8}.$$

By Lemma 3.8 with $t_0 = 1$ and (4.6), we have thus for any strong existence time $t \geq 1$

$$\mathrm{M}_{\rho_f}(1,t) \lesssim_{0,\nu} 1 + \frac{\mathscr{E}(0)^{3/8}}{\nu^{9/8}} (1 + \mathrm{M}_{\rho_f}(1,t))^{9/8}. \tag{5.6}$$

This is a superlinear estimate from which we cannot derive an *a priori* bound on M_{ρ_f} depending only on the initial data. This is the difference with the bidimensional case: we will have to take ν (a lot) larger than $4/c_P^2$ (or at least big with respect to the initial data) in order to conclude.

Recalling Notation 4.2, since $\nu > 4/c_P^2$, we infer that for some onto continuous nondecreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that the last estimates rewrites, for all strong existence times $t \geq 1$

$$\mathrm{M}_{\rho_f}(1,t) \leq \mathrm{C}_0 \left(1 + \frac{\mathrm{M}_{\rho_f}(1,t)^{9/8}}{\nu^{9/8}}\right),$$

for some constant C_0 of the form

$$\mathrm{C}_0 = \varphi\left(\|u_0\|_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + M_\alpha f_0 + N_q(f_0) + \mathrm{E}(0) + \mathscr{E}(0) + \frac{c_P^2}{4}\right).$$

From now on, C_0 may change from line to line but will always of this form. Thanks to (4.7), we have on the other hand

$$\sup_{s\in[0,1]} \left\{\|\rho_f(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} + \|j_f(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\right\} \leq \mathrm{C}_0. \tag{5.7}$$

We shall now rely on the following continuity result.

Lemma 5.1. *Consider $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous and nondecreasing function and $\mathrm{C} > 0$. For any $\nu > \Phi(2C)$ the following implication holds for any $t^\star \geq 1$ (possibly infinite) and all continuous functions $g : [1, t^\star) \to \mathbb{R}_+$:*

$$\left.\begin{aligned} g(1) \leq \mathrm{C} \\ g \leq \mathrm{C}\Big(1 + \frac{1}{\nu}\Phi(g)\Big) \ \text{ on } [1, t^\star) \end{aligned}\right\} \Rightarrow g < 2C \ \text{ on } [1, t^\star).$$

Proof of Lemma 5.1. Since Φ is nondecreasing we have $\sup_{[0,2C]} \Phi \leq \Phi(2C)$. If g satisfies the two inequalities above, then the inequality $g < 2C$ which is initially true at $t = 1$ propagates all over $[1, t^\star)$. ☐

We apply Lemma 5.1 to (5.6) and therefore we infer that if $\nu > 2\mathrm{C}_0$, one has

$$\mathrm{M}_{\rho_f}(1, t) < 2\mathrm{C}_0 \tag{5.8}$$

for all strong existence times $t \geq 1$.

Remark 5.1 (♠). *Our application of Lemma 5.1 is formal: this result holds only for continuous functions, and ρ_f* a priori *belongs only to $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{L}^\infty(\mathbb{T}^3))$ (thanks to Proposition 4.5). However, the desired estimate $\mathrm{M}_{\rho_f}(t) \leq 2\mathrm{C}_0$ is stable with respect to weak convergence, and one can use the strong stability property of DiPerna-Lions theory to first prove it for a sequence of approximating smooth solutions and then pass to the limit.*

Our last task is to check that the set of strong existence times is indeed $\mathbb{R}_+$. Again, this will be done by assuming ν large enough. Actually, if $\mathrm{C}_1 > 0$ denotes the best constant related to the embeddings $\mathrm{L}^2(\mathbb{T}^3) \hookrightarrow \mathrm{L}^{3/2}(\mathbb{T}^3) \hookrightarrow \mathrm{H}^{-1/2}(\mathbb{T}^3)$, that is the best constant in the inequalities

$$\begin{aligned} \|U\|_{\mathrm{L}^{3/2}(\mathbb{T}^3)} &\leq \mathrm{C}_1 \|U\|_{\mathrm{L}^2(\mathbb{T}^3)}, \qquad \forall U \in (\mathrm{L}^2(\mathbb{T}^3))^3, \\ \|U\|_{\mathrm{H}^{-1/2}(\mathbb{T}^3)} &\leq \mathrm{C}_1 \|U\|_{\mathrm{L}^2(\mathbb{T}^3)}, \qquad \forall U \in (\mathrm{L}^{3/2}(\mathbb{T}^3))^3, \end{aligned} \tag{5.9}$$

the decisive inequality is the following one: if ν is large enough so that

$$\|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \frac{(2\mathrm{C}_0 + \mathrm{C}_0')\mathrm{C}_1^2 \mathrm{E}(0)\mathrm{C}_\star}{\nu} < \frac{\nu^2}{\mathrm{C}_\star^2}, \tag{5.10}$$

with C_0 is the constant in (5.8) and (4.6), then all $t \geq 0$ are strong existence times. Note that this inequality is indeed of the form (1.23) for an appropriate definition of the function ψ, thanks to the definition of C_0. First, by Proposition 4.3, we know the existence of strong existence times $t > 1$. On the other hand, using the aforementioned embeddings and Lemma 4.1, we have for all $t \geq 0$ (whether strong existence time or not)

$$\int_0^t \|F(s)\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)} \, \mathrm{d}s \leq \mathrm{C}_1^2 \mathrm{E}(0) \sup_{[0,t]} \|\rho_f\|_{\mathrm{L}^\infty(\mathbb{T}^3)}.$$

In particular, since we have established $\mathrm{M}_{\rho_f}(1, t) < 2\mathrm{C}_0$ for any strong existence time $t > 1$, we have

$$\|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \frac{\mathrm{C}_\star}{\nu} \int_0^t \|F(s)\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)} \, \mathrm{d}s \leq \|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \frac{2\mathrm{C}_0 \mathrm{C}_1^2 \mathrm{E}(0)\mathrm{C}_\star}{\nu} < \frac{\nu^2}{\mathrm{C}_\star^2}.$$

In particular, for any strong existence time $t > 1$, the equality

$$\|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \frac{\mathrm{C}_\star}{\nu} \int_0^t \|F(s)\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)} \, \mathrm{d}s = \frac{\nu^2}{\mathrm{C}_\star^2} \tag{5.11}$$

is impossible, which implies by continuity that (4.9) holds for all times. This means that the set of existence times is $\mathbb{R}_+$ and allows to conclude the proof.

6 Higher order regularity estimates

In this subsection, the goal is to obtain higher order estimates for u, namely $\mathrm{L}^1_t\mathrm{W}^{1,\infty}_x$ estimates. These estimates will be useful to treat the case $\mathscr{E}(0)$ small.

This will be achieved by using higher order parabolic maximal estimates for the heat equation and in prevision of the proof of Theorem 1.3 (in view of the arguments already developed in Section 5) we will right away interpolate them with the pointwise decay estimates of the fluid part in L^2 norm obtained in Lemma 2.5.

This is the content of the following lemma.

Lemma 6.1. *For any $\alpha \in [1/2, 1)$, $c \in [1, \infty)$, any finite $a, b \geq \max(1, c\alpha)$ and any $t \geq 1$, the following estimate holds (for a possible infinite right hand side)*

$$
\begin{aligned}
&\|\nabla u\|_{\mathrm{L}^c(1,t;\mathrm{L}^p(\mathbb{T}^d))} \\
&\quad \lesssim_{0,\nu} \mathscr{E}(0)^{(1-\alpha)/2} \left(1 + \left(\int_{1/2}^t \|(u\cdot\nabla u)(s)\|^a_{\mathrm{L}^r(\mathbb{T}^d)}\,\mathrm{d}s\right)^{\frac{\alpha}{a}} + \left(\int_{1/2}^t \|F(s)\|^b_{\mathrm{L}^r(\mathbb{T}^d)}\,\mathrm{d}s\right)^{\frac{\alpha}{b}}\right),
\end{aligned}
\tag{6.1}
$$

for any $p \in [1, \infty]$ and $r \in (1, \infty)$ satisfying

$$
\frac{1}{p} = \frac{1}{d} + \alpha\left(\frac{1}{r} - \frac{2}{d}\right) + \frac{1-\alpha}{2}. \tag{6.2}
$$

Proof. Similarly to what we have done in the proof of Proposition 4.5, we introduce w_1 and w_2 as the unique divergence-free solutions on $[1/2, +\infty)$ of

$$
\begin{aligned}
\partial_t w_1 - \Delta w_1 &= \mathbb{P}(u\cdot\nabla)u, \\
\partial_t w_2 - \Delta w_2 &= \mathbb{P}F,
\end{aligned}
$$

with initial conditions $w_1(1/2) = w_2(1/2) = 0$ so that, denoting $u_h := u - (w_1 + w_2)$, we have $u_h(t+1/2) = e^{t\Delta}u(1/2)$. Thanks to the maximal regularity of the heat equation and the continuity of $\mathbb{P}$ on $\mathrm{L}^r(\mathbb{T}^d)$, we infer for $t \geq 1/2$

$$
\left(\int_{1/2}^t \|\mathrm{D}^2 w_1(s)\|^a_{\mathrm{L}^r(\mathbb{T}^d)}\,\mathrm{d}s\right)^{1/a} \lesssim \left(\int_{1/2}^t \|(u\cdot\nabla u)(s)\|^a_{\mathrm{L}^r(\mathbb{T}^d)}\,\mathrm{d}s\right)^{1/a}, \tag{6.3}
$$

$$
\left(\int_{1/2}^t \|\mathrm{D}^2 w_2(s)\|^b_{\mathrm{L}^r(\mathbb{T}^d)}\,\mathrm{d}s\right)^{1/b} \lesssim \left(\int_{1/2}^t \|F(s)\|^b_{\mathrm{L}^r(\mathbb{T}^d)}\,\mathrm{d}s\right)^{1/b}. \tag{6.4}
$$

On the other hand, since $u_h(t+1/2) = e^{t\Delta}u(1/2)$, where we write

$$
u(1/2, x) =: \sum_{k\in\mathbb{Z}^d} c_k e^{2i\pi k\cdot x} \in \mathrm{L}^2(\mathbb{T}^d),
$$

we have for $t \geq 1/2$

$$
u_h(t,x) = \sum_{k\in\mathbb{Z}^d} c_k e^{-(2\pi|k|)^2(t-1/2)} e^{2i\pi k\cdot x},
$$

and in particular for $t \geq 1$ and $\ell \geq 1$

$$
\begin{aligned}
\|u_h(t)\|^2_{\dot{\mathrm{H}}^\ell(\mathbb{T}^d)} &= \sum_{k\in\mathbb{Z}^d} |c_k|^2 |k|^{2\ell} e^{-(2\pi|k|)^2(t-1/2)} \\
&\lesssim_\ell \sum_{k\in\mathbb{Z}^d} |c_k|^2 e^{-|k|^2(t-1/2)} \\
&\leq \|u(1/2)\|^2_{\mathrm{L}^2(\mathbb{T}^d)}\, e^{-(t-1/2)},
\end{aligned}
$$

so that for any $\ell \geq 1$ we obtain

$$\int_1^{+\infty} \|u_h(s)\|^c_{\dot{\mathrm{H}}^\ell(\mathbb{T}^d)} \,\mathrm{d}s \lesssim_\ell \|u(1/2)\|^c_{\mathrm{L}^2(\mathbb{T}^d)} \int_1^{+\infty} e^{-c(s-1/2)/2} \,\mathrm{d}s \lesssim_\ell \|u(1/2)\|^c_{\mathrm{L}^2(\mathbb{T}^d)}. \tag{6.5}$$

The Gagliardo-Nirenberg-Sobolev estimate of Theorem 9.4 for $d \in \{2,3\}$, $(j,m,q) = (1,2,2)$ allows us to write for any $\alpha \in [1/2,1)$ and $s \geq 1$

$$\|\nabla u(s)\|_{\mathrm{L}^p(\mathbb{T}^d)} \lesssim \|\mathrm{D}^2 u(s)\|^\alpha_{\mathrm{L}^r(\mathbb{T}^d)} \|u(s) - \langle u(s)\rangle\|^{1-\alpha}_{\mathrm{L}^2(\mathbb{T}^d)},$$

for p, r satisfying (6.2). Thanks to estimate (2.10), we have therefore, using the decomposition $u = w_1 + w_2 + u_h$

$$\begin{aligned}&\|\nabla u(s)\|_{\mathrm{L}^p(\mathbb{T}^d)} \\ &\lesssim \mathscr{E}(0)^{(1-\alpha)/2} e^{-\nu c_P^2(1-\alpha)s/2}\Big(\|\mathrm{D}^2 w_1(s)\|^\alpha_{\mathrm{L}^r(\mathbb{T}^d)} + \|\mathrm{D}^2 w_2(s)\|^\alpha_{\mathrm{L}^r(\mathbb{T}^d)} + \|\mathrm{D}^2 u_h(s)\|^\alpha_{\mathrm{L}^r(\mathbb{T}^d)}\Big),\end{aligned} \tag{6.6}$$

and thus, by Hölder's inequality in time (using the assumption $a, b \geq c\alpha$)

$$\begin{aligned}&\|\nabla u\|_{\mathrm{L}^c(1,t;\mathrm{L}^p(\mathbb{T}^d))} \\ &\lesssim_{0,\nu} \mathscr{E}(0)^{(1-\alpha)/2}\Big(\|\mathrm{D}^2 w_1\|^\alpha_{\mathrm{L}^a(1,t;\mathrm{L}^r(\mathbb{T}^d))} + \|\mathrm{D}^2 w_2\|^\alpha_{\mathrm{L}^b(1,t;\mathrm{L}^r(\mathbb{T}^d))} + \|\mathrm{D}^2 u_h\|^\alpha_{\mathrm{L}^c(1,t;\mathrm{L}^r(\mathbb{T}^d))}\Big).\end{aligned}$$

By the energy estimate (1.12), we have $\|u(1/2)\|_{\mathrm{L}^2(\mathbb{T}^d)} \lesssim_{0,\nu} 1$, so using (6.5) for ℓ large enough, we obtain by Sobolev embedding

$$\|\nabla u\|_{\mathrm{L}^c(1,t;\mathrm{L}^p(\mathbb{T}^d))} \lesssim_{0,\nu} \mathscr{E}(0)^{(1-\alpha)/2}\Big(\|\mathrm{D}^2 w_1\|^\alpha_{\mathrm{L}^a(1,t;\mathrm{L}^r(\mathbb{T}^d))} + \|\mathrm{D}^2 w_2\|^\alpha_{\mathrm{L}^b(1,t;\mathrm{L}^r(\mathbb{T}^d))} + 1\Big),$$

and using eventually (6.3)–(6.4) we obtain (6.1). □

Lemma 6.1 invites us to study certain $\|\cdot\|_{\mathrm{L}^a(1/2,t;\mathrm{L}^b(\mathbb{T}^d))}$ norms of $u\cdot\nabla u$ and F. This is the object of the next two lemmas.

Lemma 6.2. *There exists $a \in (2,4)$ and $r_a > 2$ such that the following interpolation estimate holds for $t \geq 1$:*

$$\|(u\cdot\nabla)u\|_{\mathrm{L}^a(1/2,t;\mathrm{L}^{r_a}(\mathbb{T}^d))} \leq \|u\|_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^6(\mathbb{T}^d))} \|\nabla u\|^{\frac{2}{a}}_{\mathrm{L}^2(1/2,t;\mathrm{L}^6(\mathbb{T}^d))} \|\nabla u\|^{1-\frac{2}{a}}_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^2(\mathbb{T}^d))}. \tag{6.7}$$

Proof. For any $a > 2$, we have by Hölder inequality and interpolation $[(2,6),(\infty,2)]_\theta$,

$$\|u\cdot\nabla u\|_{\mathrm{L}^a(1/2,t;\mathrm{L}^{r_a}(\mathbb{T}^d))} \leq \|u\|_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^6(\mathbb{T}^d))} \|\nabla u\|^\theta_{\mathrm{L}^2(1/2,t;\mathrm{L}^6(\mathbb{T}^d))} \|\nabla u\|^{1-\theta}_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^2(\mathbb{T}^d))},$$

with the following identity

$$\left(\frac{1}{a}, \frac{1}{r_a}\right) = \left(0, \frac{1}{6}\right) + \theta\left(\frac{1}{2}, \frac{1}{6}\right) + (1-\theta)\left(0, \frac{1}{2}\right).$$

In particular, $\theta = 2/a$ and we recover the exponents in estimate (6.7). From the previous identity we also deduce the value of r_a, because $\frac{1}{r_a} = \frac{1}{6}(1+\frac{2}{a}) + \frac{1}{2}(1-\frac{2}{a})$. In the limit case $a = 2$ we get $r_a = 3$, so that taking $|a-2|$ small enough we have indeed $r_a > 2$ and $a \in (2,4)$. □

Lemma 6.3. *For any finite $b > 4$, the following estimate holds for some $r_b > 3$ and all strong existence times $t \geq 1$:*

$$\|F\|_{\mathrm{L}^b(1/2,t;\mathrm{L}^{r_b}(\mathbb{T}^d))} \lesssim_{0,\nu} 1 + \mathrm{M}_{\rho_f,j_f}(1,t)^{\frac{3}{2}-\frac{2}{b}}. \tag{6.8}$$

Proof. Thanks to Lemma 4.1 and (4.4) – (4.7) we have

$$\|F\|_{\mathrm{L}^2(1/2,t;\mathrm{L}^2(\mathbb{T}^d))}2,2;t \lesssim 1+\mathrm{M}_{\rho_f}(1,t)^{1/2} \leq 1+\mathrm{M}_{\rho_f,j_f}(1,t)^{1/2}. \tag{6.9}$$

By interpolation $[(2,2);(\infty,6)]_\theta$, we have

$$\|F\|_{\mathrm{L}^b(1/2,t;\mathrm{L}^{r_b}(\mathbb{T}^d))} \leq \|F\|^{\theta}_{\mathrm{L}^2(1/2,t;\mathrm{L}^2(\mathbb{T}^d))}\|F\|^{1-\theta}_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^6(\mathbb{T}^d))}, \tag{6.10}$$

where θ and r_b are defined by the equality $(\frac{1}{b},\frac{1}{r_b})=\theta(\frac12,\frac12)+(1-\theta)(0,\frac16)$ from which we get $\theta=2/b$ and $\frac{1}{r_b}=\frac{2}{3b}+\frac16$; we notice that $b>4$ implies $r_b>3$. Using the triangle inequality, we have

$$\begin{aligned}\|F\|_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^6(\mathbb{T}^d))} &= \|j_f-\rho_f u\|_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^6(\mathbb{T}^d))}\\ &\lesssim \mathrm{M}_{\rho_f,j_f}(1/2,t)(1+\|u\|_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^6(\mathbb{T}^d))}),\end{aligned}$$

which in turn, using the Sobolev embedding $\mathrm{H}^1(\mathbb{T}^d)\hookrightarrow\mathrm{L}^6(\mathbb{T}^d)$ together with (4.5) and the energy estimate (1.12), implies

$$\|F\|_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^6(\mathbb{T}^d))} \lesssim \mathrm{M}_{\rho_f,j_f}(1,t)(1+\mathrm{M}_{\rho_f,j_f}(1,t)^{1/2}).$$

Combining the previous with (6.9) in (6.10) we therefore get

$$\|F\|_{\mathrm{L}^b(1/2,t;\mathrm{L}^{r_b}(\mathbb{T}^d))} \lesssim 1+\mathrm{M}_{\rho_f,j_f}(1,t)^{3/2-\theta},$$

which is exactly (6.8) because $b=2/\theta$. □

Gathering all pieces together, we obtain the following regularity result.

Proposition 6.4. *Assume $\mathscr{E}(0)\leq 1$. There exist two positive exponents $\gamma_{d,1},\gamma_{d,2}$ such that, for any strong existence time $t\geq 1$, one has $\nabla u\in\mathrm{L}^1(1,t;\mathrm{L}^\infty(\mathbb{T}^d))$ with the estimate*

$$\|\nabla u\|_{\mathrm{L}^1(1,t;\mathrm{L}^\infty(\mathbb{T}^d))} \lesssim_{0,\nu} \mathscr{E}(0)^{\gamma_{d,1}}(1+\mathrm{M}_{\rho_f,j_f}(1,t)^{\gamma_{d,2}}). \tag{6.11}$$

Proof. We start by using Lemma 6.2 and Lemma 6.3 together with Corollary 4.8to infer for $t_0=1/2$ and any strong existence time $t\geq 1/2$,

$$\left(\int_{1/2}^t \|(u\cdot\nabla)u(s)\|^a_{r_a}\,\mathrm{d}s\right)^{1/a} \lesssim_{0,\nu} 1+\mathrm{M}_{\rho_f,j_f}(1,t), \tag{6.12}$$

$$\left(\int_{1/2}^t \|F(s)\|^b_{r_b}\,\mathrm{d}s\right)^{1/b} \lesssim_{0,\nu} 1+\mathrm{M}_{\rho_f,j_f}(1,t)^{\frac32-\frac2b}, \tag{6.13}$$

for some $b>4>a>2$ and $r_a>2$, $r_b>3$. Let us now apply estimate (6.1) of Lemma 6.1, with $r:=\min(r_a,r_b)$. Combining with (6.12)–(6.13) for all strong existence times $t\geq 1$, we infer

$$\|\nabla u\|_{\mathrm{L}^c(1,t;\mathrm{L}^p(\mathbb{T}^d))} \lesssim_{0,\nu} \mathscr{E}(0)^{(1-\alpha)/2}\left(1+\mathrm{M}_{\rho_f,j_f}(1,t)^{\alpha(\frac32-\frac2b)}\right), \tag{6.14}$$

which holds for any $\alpha\in[1/2,1)$ and p defined by (6.2), provided that that $\alpha c<\min(a,b)$.

We may now hope that from (6.14), we could conclude by taking $c=1$ and adjusting the values of α so as $p=\infty$. It turns out that this is possible only in the case $d=2$. Indeed, taking $p=\infty$ in (6.2) leads to

$$\frac1d+\frac12=\alpha\left(\frac2d+\frac12-\frac1r\right),$$

which, since $\alpha < 1$, is possible only if $\frac{1}{d} > \frac{1}{r}$, that is $r > d$. We have chosen $r := \min(r_a, r_b)$, and $r_a > 2$, $r_b > 3$ (see Lemma 6.2 and Lemma 6.3), so we do have $r > 2$. In the bidimensional case we can therefore choose $\alpha = (\frac{3}{2} - \frac{1}{r})^{-1}$, $c = 1$ in (6.14), and deduce (6.11) for

$$\gamma_{2,1} := \frac{1-\alpha}{2}, \qquad \gamma_{2,2} := \alpha\left(\frac{3}{2} - \frac{2}{b}\right).$$

The three dimensional case is more involved. Since $a < b$, we can take $c = a$ in (6.14). Going back to (6.2), we see that the limit case $\alpha = 1$ leads to the equality

$$\frac{1}{p} = \frac{1}{r} - \frac{1}{3},$$

which, since $r > 2$ implies $\frac{1}{p} < \frac{1}{6}$, that is $p > 6$. Taking $\alpha \in [1/2, 1)$ close enough to 1, we therefore infer the existence of $p > 6$ such that, for all strong existence times $t \geq 1$,

$$\|\nabla u\|_{\mathrm{L}^a(1,t;\mathrm{L}^p(\mathbb{T}^3))} \lesssim_{0,\nu} \mathscr{E}(0)^{(1-\alpha)/2}\left(1 + \mathrm{M}_{\rho_f,j_f}(1,t)^{\alpha(\frac{3}{2}-\frac{1}{b})}\right).$$

Since $p > 6$, we infer from Hölder's inequality, for some $\widetilde{r}_a > 3$ that for all strong existence times $t \geq 1$

$$\begin{aligned}\Big(\int_{1/2}^t \|(u\cdot\nabla)u(s)\|^a_{\mathrm{L}^{\tilde{r}_a}(\mathbb{T}^3)}\,\mathrm{d}s\Big)^{1/a} &\\ &\leq \|u\|_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^6(\mathbb{T}^3))}\|\nabla u\|_{\mathrm{L}^a(1/2,t;\mathrm{L}^p(\mathbb{T}^3))}\\ &\lesssim_{0,\nu} \|u\|_{\mathrm{L}^\infty(1/2,t;\mathrm{L}^6(\mathbb{T}^3))}\mathscr{E}(0)^{(1-\alpha)/2}\left(1 + \mathrm{M}_{\rho_f,j_f}(1,t)^{\alpha(\frac{3}{2}-\frac{2}{b})}\right).\end{aligned}$$

Using the previous inequality together with (4.13) and the energy estimate (1.12), we infer

$$\left(\int_{1/2}^t \|(u\cdot\nabla)u(s)\|^a_{\mathrm{L}^{\tilde{r}_a}(\mathbb{T}^3)}\,\mathrm{d}s\right)^{1/a} \lesssim_{0,\nu} \mathscr{E}(0)^{(1-\alpha)/2}\left(1 + \mathrm{M}_{\rho_f,j_f}(1,t)^{\frac{1}{2}+\alpha(\frac{3}{2}-\frac{2}{b})}\right). \tag{6.15}$$

The point is that (6.15) can now replace (6.12) in the analysis that we have performed earlier with the advantage that, now $\widetilde{r}_a > 3$. This yields that $\widetilde{r} := \min(r_b, \widetilde{r}_a) > 3$ and hence taking

$$\widetilde{\alpha} = 5\left(7 - \frac{6}{\widetilde{r}}\right)^{-1} < 1,$$

we can check that $\widetilde{\alpha} \in [1/2, 1)$ and satisfies

$$0 = \frac{1}{3} + \widetilde{\alpha}\left(\frac{1}{\widetilde{r}} - \frac{2}{3}\right) + \frac{1-\widetilde{\alpha}}{2}.$$

So we invoke estimate (6.1) with $r = \widetilde{r} > 3$, $c = 1$ and $\widetilde{\alpha}$ as above to infer for strong existence times $t \geq 1$

$$\begin{aligned}&\|\nabla u\|_{\mathrm{L}^1(1,t;\mathrm{L}^\infty(\mathbb{T}^3))}\\ &\lesssim_{0,\nu} \mathscr{E}(0)^{(1-\tilde{\alpha})/2}\left(1 + \mathscr{E}(0)^{(1-\alpha)/2}\mathrm{M}_{\rho_f,j_f}(1,t)^{\tilde{\alpha}(\frac{1}{2}+\alpha(\frac{3}{2}-\frac{2}{b}))} + \mathrm{M}_{\rho_f,j_f}(1,t)^{\tilde{\alpha}(\frac{3}{2}-\frac{1}{b})}\right),\end{aligned}$$

and deduce therefore (6.11) for

$$\gamma_{3,1} := \frac{1-\widetilde{\alpha}}{2}, \qquad \gamma_{3,2} := \max\left\{\tilde{\alpha}\left(\frac{1}{2} + \alpha\left(\frac{3}{2} - \frac{2}{b}\right)\right), \tilde{\alpha}\left(\frac{3}{2} - \frac{2}{b}\right)\right\}.$$

□

7 Proof of Theorem 1.3 in the case $\mathscr{E}(0)$ small

In this section, we prove Theorem 1.3 when the initial modulated energy $\mathscr{E}(0)$ is small enough. The general strategy is as follows:

- We rely on the key Lemmas 3.5 and 3.6 (in the form of Proposition 7.1 stated below) to obtain the controls (3.21) and (3.22) on the L^∞ norm of ρ_f and j_f. In order to apply this result, we need to ensure that $\int_1^t \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\,\mathrm{d}s$ is small enough.
- To this purpose, we rely on the higher order estimates obtained in Section 6, namely we use Proposition 6.4 to obtain a $\mathrm{L}^2_t\mathrm{W}^{1,\infty}_x$ estimate for u (note that here we also need an interpolation argument based on Lemma 2.5 but this is already included in the statement of Proposition 6.4).

We gather in the following proposition the results of Lemma 3.6 and Corollary 4.8.

Proposition 7.1. *Assume the existence of a strong existence time strictly greater than 1. Define*

$$t^\star := \sup\left\{\textit{strong existence time } t \textit{ such that } \int_1^t \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\,\mathrm{d}s < \mathrm{c}(d)\right\}. \tag{7.1}$$

where $\mathrm{c}(d)$ *is given in Lemma 3.5. Then, on* $[1,t^\star)$*, one has* $\mathrm{M}_{\rho_f,j_f}(1,t) \lesssim_{0,\nu} 1$.

Proof. The existence of a strong existence time larger than 1 together with Proposition 6.4 implies that $t^\star > 1$.

For $t \in [1,t^\star)$ we can invoke Lemma 3.8 with $t_0 = 1$ and (4.3) or (4.6), to ensure that $\mathrm{M}_{\rho_f}(1,t) \lesssim_{0,\nu} 1$ and

$$\|j_f(t)\|_{\mathrm{L}^\infty(\mathbb{T}^d)} \lesssim e^{-t}\int_1^t e^s\|u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\;\mathrm{d}s.$$

Thanks to Sobolev's embedding $\mathrm{H}^2(\mathbb{T}^d) \hookrightarrow \mathrm{L}^\infty(\mathbb{T}^d)$ we infer

$$\int_1^t e^s\|u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\,\mathrm{d}s \lesssim \int_1^t e^s\|u(s)\|_{\mathrm{L}^2(\mathbb{T}^d)}\,\mathrm{d}s + \int_1^t e^s\|\mathrm{D}^2u(s)\|_{\mathrm{L}^2(\mathbb{T}^d)}\,\mathrm{d}s,$$

and therefore (using Cauchy-Schwarz's inequality)

$$\int_1^t e^s\|u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^2)}\,\mathrm{d}s \lesssim (e^t-1)\sup_{[1,t]}\|u(s)\|_{\mathrm{L}^2(\mathbb{T}^2)} + \left(\int_1^t e^{2s}\,\mathrm{d}s\right)^{1/2}\left(\int_1^t \|\mathrm{D}^2u(s)\|_2^2\,\mathrm{d}s\right)^{1/2}.$$

Thanks to (4.13) and the energy estimate (1.12) we eventually infer

$$e^{-t}\int_1^t e^s\|u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\,\mathrm{d}s \lesssim_{0,\nu} 1 + \mathrm{M}_{\rho_f}(1,t)$$

and we have already proved that $\mathrm{M}_{\rho_f}(1,t) \lesssim_{0,\nu} 1$. $\square$

7.1 The 2D case

We recall that for $d = 2$, all times are strong existence times. Applying Proposition 7.1, the question thus reduces to ensure $t^\star = +\infty$, that is to say to show that for all $t \geq 1$,

$$\int_1^t \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^2)}\,\mathrm{d}s < \mathrm{c}(2). \tag{7.2}$$

Recalling the meaning of the symbol $\lesssim_{0,\nu}$ (see Notation 4.2), we have the existence of an universal onto nondecreasing continuous function $\varphi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that, on $[1, t^\star)$

$$\mathrm{M}_{\rho_f, j_f}(1,t) \leq \varphi_1\left(N_q(f_0) + \mathrm{E}(0) + \nu^{-1}\right). \tag{7.3}$$

Thanks to Proposition 6.4, there exist two positive exponents $\gamma_{2,1}, \gamma_{2,2}$ and an onto nondecreasing continuous function φ_2 such that

$$\|\nabla u\|_{\mathrm{L}^1(1,t;\mathrm{L}^\infty(\mathbb{T}^2))} \leq \varphi_2\left(N_q(f_0) + \mathrm{E}(0) + \nu^{-1}\right) \mathscr{E}(0)^{\gamma_{2,1}}(1 + \mathrm{M}_{\rho_f, j_f}(1,t)^{\gamma_{2,2}}),$$

and therefore, combining with (7.3), on $[0, t^\star)$, we get that

$$\|\nabla u\|_{\mathrm{L}^1(1,t;\mathrm{L}^\infty(\mathbb{T}^2))} \leq \mathscr{E}(0)^{\gamma_{2,1}} \varphi\left(N_q(f_0) + \mathrm{E}(0) + \nu^{-1}\right),$$

where $\varphi := \varphi_2(1 + \varphi_1^{\gamma_{2,2}})$. Thus setting

$$\psi = \left(\frac{\varphi\left(N_q(f_0) + \mathrm{E}(0) + \nu^{-1}\right)}{\mathrm{c}(2)}\right)^{1/\gamma_{2,1}},$$

the smallness assumption (1.20) reads

$$\mathscr{E}(0) < \left(\frac{\mathrm{c}(2)}{\varphi\left(N_q(f_0) + \mathrm{E}(0) + \nu^{-1}\right)}\right)^{1/\gamma_{2,1}},$$

so that for all $t \in (1, t^\star)$, (7.2) holds and thus by continuity we must have $t^\star = +\infty$. This concludes the proof in this case.

7.2 The 3D case

As in the bidimensional case, the proof relies on Proposition 7.1, and starts exactly like we have argued in Subsection 7.1: we obtain the existence of an onto nondecreasing continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that, if

$$\psi\left(\|u_0\|_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + M_\alpha f_0 + \mathrm{E}(0) + N_q(f_0) + \nu^{-1}\right)\mathscr{E}(0) < 1,$$

then, for any strong existence time $t \geq 1$, one has

$$\int_1^t \|\nabla u(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\,\mathrm{d}s < c(3).$$

The additional task is to show that the set of strong existence times is $\mathbb{R}_+$. Thanks to Proposition 7.1 and Proposition 4.5 (see (4.7)), we get that for any strong existence time $t \geq 1$, there exists another nondecreasing continuous function φ_2 such that

$$\begin{aligned}\sup_{s\in[0,1]} &\|\rho_f(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)} + \mathrm{M}_{\rho_f, j_f}(1,t) \\ &\leq \varphi_2\left(\|u_0\|_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + M_\alpha f_0 + \mathrm{E}(0) + N_q(f_0) + \nu^{-1}\right).\end{aligned} \tag{7.4}$$

Recall that by assumption, we have $\|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} < \nu^2/\mathrm{C}_\star^2$. Using Lemma 4.1, we thus infer that for all strong existence times

$$\begin{aligned}
\|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} &+ \frac{\mathrm{C}_\star}{\nu}\int_0^t \|F(s)\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)}\,\mathrm{d}s \\
&\leq \|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \frac{\mathrm{C}_1^2\mathrm{C}_\star}{\nu}\int_0^t \|F(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s \\
&\leq \|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \mathscr{E}(0)\frac{\mathrm{C}_1^2\mathrm{C}_\star}{\nu}\left(\mathrm{M}_{\rho_f,j_f}(1,t) + \sup_{s\in[0,1]}\|\rho(s)\|_{\mathrm{L}^\infty(\mathbb{T}^3)}\right),
\end{aligned}$$

where C_1 is the constant introduced in (5.9). Combining with (7.4), we therefore obtain

$$\begin{aligned}
\|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} &+ \frac{\mathrm{C}_\star}{\nu}\int_0^t \|F(s)\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)}\,\mathrm{d}s \\
&\leq \|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \mathscr{E}(0)\varphi_3\left(\|u_0\|_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + M_\alpha f_0 + \mathrm{E}(0) + N_q(f_0) + \nu^{-1}\right),
\end{aligned}$$

where $\varphi_3(z) := \mathrm{C}_1^2\mathrm{C}_\star\varphi_3(z)z$. Therefore, setting $\psi := \max(\varphi_1,\varphi_3)$, the smallness assumption (1.24) reads:

$$\begin{aligned}
\psi\left(N_q(f_0) + M_\alpha f_0 + \mathrm{E}(0) + \|u_0\|_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \nu^{-1}\right)\mathscr{E}(0) \\
< \min\left(1, \frac{\nu^2}{\mathrm{C}_\star^2} - \|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)}\right),
\end{aligned}$$

and we deduce that for for any strong existence time,

$$\|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \frac{\mathrm{C}_\star}{\nu}\int_0^t \|F(s)\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)}\,\mathrm{d}s \leq \frac{\nu^2}{\mathrm{C}_\star^2}.$$

By continuity, we deduce that the set of strong existence times is indeed $\mathbb{R}_+$ and this concludes the proof.

8 Further description of the asymptotic state

The goal of this section is to provide a sharper description of the asymptotic dynamics reached as $t \to +\infty$. We first provide the proof of Corollary 1.4. Then the main focus will be to prove the following modified scattering result, obtained at the expense of imposing more stringent conditions on ν (that must be large enough) and $\mathscr{E}(0)$ (that must be small enough).

Corollary 8.1. *Assume that ν is large enough and that $\mathscr{E}(0)$ is small enough (depending on the dimension d). In particular assume that the assumptions of* **all sub-cases** *of Theorem 1.1 if $d = 2$ or Theorem 1.2 if $d = 3$ are satisfied. There exists a vector field $(s,x,v) \mapsto (\widetilde{\mathrm{X}}_{s,\infty}(x,v), \widetilde{\mathrm{V}}_{s,\infty}(x,v)) \in \mathscr{C}^1(\mathbb{R}_+ \times \mathbb{T}^d \times \mathbb{R}^d)$ such that, setting*

$$\widetilde{\rho}(t,x) := \int_{\mathbb{R}^d} f_0\left(\widetilde{\mathrm{X}}_{0,\infty}(x,v) - t\frac{\langle u_0 + j_{f_0}\rangle}{2}, \widetilde{\mathrm{V}}_{0,\infty}(x,v)\right)\mathrm{d}v, \tag{8.1}$$

with

$$\begin{aligned}
\widetilde{\mathrm{X}}_{0,\infty}(x,v) &:= x - v + \frac{\langle u_0 + j_{f_0}\rangle}{2} - \int_0^{+\infty}\left(\langle u(\tau)\rangle - \frac{\langle u_0 + j_{f_0}\rangle}{2}\right)\mathrm{d}\tau \\
&\quad + \int_0^{+\infty}(e^\tau - 1)\left(u(\tau, \widetilde{\mathrm{X}}_{\tau,\infty}(x,v)) - \langle u(\tau)\rangle\right)\mathrm{d}\tau, \qquad (8.2)\\
\widetilde{\mathrm{V}}_{0,\infty}(x,v) &:= v - \int_0^{+\infty} e^\tau\left(u(\tau, \widetilde{\mathrm{X}}_{\tau,\infty}(x,v)) - \langle u(\tau)\rangle\right)\mathrm{d}\tau,
\end{aligned}$$

we have the following asymptotic behavior. For all $0 < \lambda < 1$*, for all* $t \geq 0$*,*

$$\mathrm{W}_1\left(f(t), \widetilde{\rho}(t,x) \otimes \delta_{\frac{\langle u_0 + j_{f_0}\rangle}{2}}\right) \lesssim e^{-\lambda t}. \tag{8.3}$$

Remark 8.1. *We recover the result of Corollary 1.4 and we have* $\overline{\rho} = \widetilde{\rho}$*.*

It is interesting to compare the statement of Corollary 8.1 with the explicit asymptotic behavior of solutions to the linearized equation

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v\left(\left(\frac{\langle u_0 + j_{f_0}\rangle}{2} - v\right) f\right) = 0,$$

for which we recall we have

$$\mathrm{W}_1\left(f(t,x,v), \widetilde{\rho}_0(t,x) \otimes \delta_{\frac{\langle u_0 + j_{f_0}\rangle}{2}}\right) \xrightarrow[t\to\infty]{} 0,$$

with

$$\widetilde{\rho}_0(t,x) := \int_{\mathbb{R}^d} f_0\left(x - v - (t-1)\frac{\langle u_0 + j_{f_0}\rangle}{2}, v\right) \mathrm{d}v.$$

As will be clear from the estimates we will obtain in the proof of Corollary 8.1, we have

$$\begin{aligned}
&\left\| \int_0^{+\infty} (e^\tau - 1)\left(u(\tau, \widetilde{\mathrm{X}}_{\tau,\infty}) - \langle u(\tau)\rangle\right) \mathrm{d}\tau \right\|_{\mathrm{L}^\infty(\mathbb{T}^d\times\mathbb{R}^d)} \\
&+ \left| \int_0^{+\infty} \left(\langle u(\tau)\rangle - \frac{\langle u_0 + j_{f_0}\rangle}{2}\right) \mathrm{d}\tau \right| + \left\| \int_0^{+\infty} e^\tau \left(u(\tau, \widetilde{\mathrm{X}}_{\tau,\infty}) - \langle u(\tau)\rangle\right) \mathrm{d}\tau \right\|_{\mathrm{L}^\infty(\mathbb{T}^d\times\mathbb{R}^d)} \\
&\lesssim \varphi(\mathscr{E}(0)),
\end{aligned}$$

for some continuous non-negative function φ cancelling at 0 (see also (8.20)). We therefore see that the deviation from the linearized behavior is small, as we have

$$\|\widetilde{\rho} - \widetilde{\rho}_0\|_{\mathrm{L}^\infty(0,+\infty;\mathrm{L}^\infty(\mathbb{T}^d))} \ll 1,$$

for $\mathscr{E}(0) \ll 1$.

8.1 Convergence when $t \to +\infty$: proof of Corollary 1.4

We give in this section the proof of Corollary 1.4. We first prove that there exists a function $\overline{\rho}(x) \in \mathrm{L}^\infty(\mathbb{T}^d)$ such that

$$\mathrm{W}_1\left(\rho_f(t,x), \overline{\rho}\left(x + t\frac{\langle u_0 + j_{f_0}\rangle}{2}\right)\right) \to_{t\to+\infty} 0,$$

with (sharp) exponential decay.

To this end, we rely on an argument of Jabin [20] used in the context of the large time behavior of the Vlasov-Stokes system. The proof proceeds by weak compactness, and heavily relies on Theorem 1.3. By (1.28), we know that

$$\|\rho_f\|_{\mathrm{L}^\infty(0,+\infty;\mathrm{L}^\infty(\mathbb{T}^d))} < +\infty,$$

Therefore, setting

$$\overline{\rho}_f(t,x) := \rho_f\left(t, x + t\frac{\langle u_0 + j_{f_0}\rangle}{2}\right), \tag{8.4}$$

by weak-$\star$ compactness, there is an increasing sequence of positive times $(t_n)_{n\in\mathbb{N}}$ and a function $\overline{\rho} \in \mathrm{L}^\infty(\mathbb{T}^d)$ such that

$$\mathrm{W}_1(\overline{\rho}_f(t_n, x), \overline{\rho}(x)) \to_{n\to+\infty} 0.$$

Consider two such sequences $(t_n^1)_{n\in\mathbb{N}}$ and $(t_n^2)_{n\in\mathbb{N}}$ and two corresponding limit functions $\overline{\rho}^1, \overline{\rho}^2 \in \mathrm{L}^\infty(\mathbb{T}^d)$ such that

$$\mathrm{W}_1(\overline{\rho}_f(t_n^i, x), \overline{\rho}^i(x)) \to_{n\to+\infty} 0, \ \text{for } i = 1, 2.$$

Let ψ be a smooth test function, such that $\|\nabla_x \psi\|_{\mathrm{L}^\infty(\mathbb{T}^d)} \leq 1$. Recall the local conservation of mass

$$\partial_t \rho_f(t,x) + \nabla_x \cdot j_f(t,x) = 0,$$

so that setting as well $\overline{j}_f(t,x) := j_f\left(t, x + t\frac{\langle u_0 + j_{f_0}\rangle}{2}\right)$, $\overline{\rho}_f$ satisfies

$$\partial_t \overline{\rho}_f + \nabla_x \cdot \overline{j}_f = \frac{\langle u_0 + j_{f_0}\rangle}{2} \cdot \nabla_x \overline{\rho}_f,$$

from which we obtain that for all $0 \leq s \leq t$,

$$\begin{aligned}
&\left| \int_{\mathbb{T}^d} \overline{\rho}_f(t,x)\psi(x)\,\mathrm{d}x - \int_{\mathbb{T}^d} \overline{\rho}_f(s,x)\psi(x)\,\mathrm{d}x \right| \\
&\quad \leq \int_s^t \left| \int_{\mathbb{T}^d} \left(\nabla_x \cdot \overline{j}_f - \nabla_x \overline{\rho}_f \cdot \frac{\langle u_0 + j_{f_0}\rangle}{2} \right) \psi\,\mathrm{d}x \right| \mathrm{d}\tau \\
&\quad \leq \int_s^t \left| \int_{\mathbb{T}^d} \left(\nabla_x \cdot \overline{j}_f - \nabla_x \cdot \left(\overline{\rho}_f \frac{\langle u_0 + j_{f_0}\rangle}{2} \right) \right) \psi\,\mathrm{d}x \right| \mathrm{d}\tau \\
&\quad \leq \int_s^t \left| \int_{\mathbb{T}^d} \left(j_f(\tau,x) - \rho_f(\tau,x) \frac{\langle u_0 + j_{f_0}\rangle}{2} \right) \cdot (\nabla_x \psi)\left(x - \tau \frac{\langle u_0 + j_{f_0}\rangle}{2} \right) \mathrm{d}x \right| \mathrm{d}\tau \\
&\quad = \int_s^t \left| \int_{\mathbb{T}^d\times\mathbb{R}^d} f(\tau,x,v) \left(v - \frac{\langle u_0 + j_{f_0}\rangle}{2} \right) \cdot (\nabla_x \psi)\left(x - \tau \frac{\langle u_0 + j_{f_0}\rangle}{2} \right) \mathrm{d}v\mathrm{d}x \right| \mathrm{d}\tau.
\end{aligned} \tag{8.5}$$

By Cauchy-Scharwz inequality and (i)-(ii) in Lemma 2.1, this yields

$$\begin{aligned}
&\left| \int_{\mathbb{T}^d} \overline{\rho}_f(t,x)\psi(x)\,\mathrm{d}x - \int_{\mathbb{T}^d} \overline{\rho}_f(s,x)\psi(x)\,\mathrm{d}x \right| \\
&\quad \leq \int_s^t \left(\int_{\mathbb{T}^d\times\mathbb{R}^d} \left| v - \frac{\langle u_0 + j_{f_0}\rangle}{2} \right|^2 f(\tau,x,v)\,\mathrm{d}v\mathrm{d}x \right)^{1/2} \\
&\qquad\qquad \times \left(\int_{\mathbb{T}^d} \rho_f(\tau,x) \|\nabla_x \psi\|^2_{\mathrm{L}^\infty(\mathbb{T}^d)}\,\mathrm{d}x \right)^{1/2} \mathrm{d}\tau \\
&\quad \lesssim \int_s^t \left(\int_{\mathbb{T}^d\times\mathbb{R}^d} |v - \langle j_f\rangle|^2 f(\tau,x,v)\,\mathrm{d}v\mathrm{d}x \right)^{1/2} \mathrm{d}\tau \\
&\qquad\qquad + \int_s^t \left| \langle j_f\rangle - \frac{\langle u_0 + j_{f_0}\rangle}{2} \right| \left(\int_{\mathbb{T}^d} \rho(\tau)\,\mathrm{d}x \right)^{1/2} \mathrm{d}\tau.
\end{aligned}$$

Therefore, by Lemma 2.2, we end up with

$$\left| \int_{\mathbb{T}^d} \overline{\rho}_f(t,x)\psi(x)\,\mathrm{d}x - \int_{\mathbb{T}^d} \overline{\rho}_f(s,x)\psi(x)\,\mathrm{d}x \right| \lesssim \int_s^t \mathscr{E}(\tau)^{1/2}\,\mathrm{d}\tau. \tag{8.6}$$

We deduce

$$\left| \int_{\mathbb{T}^d} \overline{\rho}_f(t_n^1,x)\psi(x)\,\mathrm{d}x - \int_{\mathbb{T}^d} \overline{\rho}_f(t_n^2,x)\psi(x)\,\mathrm{d}x \right| \lesssim \int_{\min(t_n^1,t_n^2)}^{+\infty} \widetilde{\mathscr{E}}(\tau)^{1/2}\,\mathrm{d}\tau$$

which converges to 0 as $n \to +\infty$ since by by (1.27), $\widetilde{\mathscr{E}}^{1/2}$ is integrable at $+\infty$. This yields that $\overline{\rho}^1 = \overline{\rho}^2$, which already proves that

$$\mathrm{W}_1(\overline{\rho}_f(t,x), \overline{\rho}(x)) \to_{t\to+\infty} 0.$$

This yields

$$\mathrm{W}_1\left(\rho_f(t,x),\overline{\rho}\left(x-t\frac{\langle u_0+j_{f_0}\rangle}{2}\right)\right)\to_{t\to+\infty}0.$$

To obtain exponential decay, observe that by (8.6) for any $0\le t<s<+\infty$ and (1.27), we have, for all $0<\lambda<1$,

$$\left|\int_{\mathbb{T}^d}\overline{\rho}_f(t,x)\psi(x)\,\mathrm{d}x-\int_{\mathbb{T}^d}\overline{\rho}_f(s,x)\psi(x)\,\mathrm{d}x\right|\le\int_t^{+\infty}\widetilde{\mathscr{E}}(\tau)^{1/2}\,\mathrm{d}\tau\lesssim e^{-\lambda t}.$$

Letting $s\to+\infty$, by Monge-Kantorovitch duality, we obtain

$$\mathrm{W}_1\left(\rho_f(t,x),\overline{\rho}\left(x-t\frac{\langle u_0+j_{f_0}\rangle}{2}\right)\right)\lesssim e^{-\lambda t}.$$

We finally conclude, combining with Theorem 1.1 or 1.2, that

$$\mathrm{W}_1\left(f,\overline{\rho}\left(x-t\frac{\langle u_0+j_{f_0}\rangle}{2}\right)\otimes\delta_{\frac{\langle u_0+j_{f_0}\rangle}{2}}\right)\lesssim e^{-\lambda t},$$

hence concluding the proof.

8.2 Further description of the asymptotic state: the case $\langle u_0+j_{f_0}\rangle=0$

We provide in this section a proof of Corollary 8.1 in the case $\langle u_0+j_{f_0}\rangle=0$.

Consider $(\mathrm{X}(s;t,x,v),\mathrm{V}(s,t;x,v))$ the characteristics as defined in (3.1) and introduce the *renormalized characteristics* that are defined as follows:

Definition 8.1. *For any $0\le s,t,<+\infty$, we set*

$$\begin{aligned}
\widetilde{\mathrm{X}}_{s,t}(x,v)&:=\mathrm{X}\left(s;t,x,e^{-t}\left(v+\int_0^t e^{\tau}\langle u(\tau)\rangle\,\mathrm{d}\tau\right)\right),\\
\widetilde{\mathrm{V}}_{s,t}(x,v)&:=e^{s}\mathrm{V}\left(s;t,x,e^{-t}\left(v+\int_0^t e^{\tau}\langle u(\tau)\rangle\,\mathrm{d}\tau\right)\right)-\int_0^s e^{\tau}\langle u(\tau)\rangle\,\mathrm{d}\tau.
\end{aligned}\tag{8.7}$$

Observe that by construction, $(\widetilde{\mathrm{X}}_{s,t}(x,v),\widetilde{\mathrm{V}}_{s,t}(x,v))$ satisfies the equation

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\widetilde{\mathrm{X}}_{s,t}(x,v)&=e^{-s}\widetilde{\mathrm{V}}_{s,t}(x,v)+e^{-s}\int_0^s e^{\tau}\langle u(\tau)\rangle\,\mathrm{d}\tau,\\
\frac{\mathrm{d}}{\mathrm{d}s}\widetilde{\mathrm{V}}_{s,t}(x,v)&=e^{s}\left(u(s,\widetilde{\mathrm{X}}_{s,t}(x,v))-\langle u(s)\rangle\right),
\end{aligned}\tag{8.8}$$

with $(\widetilde{\mathrm{X}}_{t,t}(x,v),\widetilde{\mathrm{V}}_{t,t}(x,v))=(x,v)$. We also note that for all $0\le s,t<+\infty$ and all $(x,v)\in\mathbb{T}^d\times\mathbb{R}^d$, there holds

$$\begin{aligned}
(\widetilde{\mathrm{X}}_{s,t},\widetilde{\mathrm{V}}_{s,t})(\widetilde{\mathrm{X}}_{t,s}(x,v),\widetilde{\mathrm{V}}_{t,s}(x,v))&=(x,v),\\
(\widetilde{\mathrm{X}}_{t,s},\widetilde{\mathrm{V}}_{t,s})(\widetilde{\mathrm{X}}_{s,t}(x,v),\widetilde{\mathrm{V}}_{s,t}(x,v))&=(x,v).
\end{aligned}\tag{8.9}$$

We write the representation formula

$$\rho_f(t,x)=e^{dt}\int_{\mathbb{R}^d}f_0(\mathrm{X}(0;t,x,v),\mathrm{V}(0;t,x,v))\,\mathrm{d}v$$

and use the change of variables $v\mapsto e^t v-\int_0^t e^{\tau}\langle u(\tau)\rangle\,\mathrm{d}\tau$. This yields

$$\rho_f(t,x)=\int_{\mathbb{R}^2}f_0(\widetilde{\mathrm{X}}_{0,t}(x,v),\widetilde{\mathrm{V}}_{0,t}(x,v))\,\mathrm{d}v.$$

Now the goal is to show the following long time behavior behavior for the renormalized characteristics. We shall from now on assume that the viscosity ν is taken large enough.

Lemma 8.2. *Assume ν is large enough (depending on the dimension d). For all $s \geq 0, (x,v) \in \mathbb{T}^d \times \mathbb{R}^d$, there exist limit characteristics $(\widetilde{\mathrm{X}}_{s,\infty}(x,v), \widetilde{\mathrm{V}}_{0,\infty}(x,v))$ and $(\widetilde{\mathrm{X}}_{\infty,s}(x,v), \widetilde{\mathrm{V}}_{\infty,s}(x,v))$ such that for all $s \geq 0$ and all $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$,*

$$\lim_{t \to +\infty} |(\widetilde{\mathrm{X}}_{s,t}(x,v), \widetilde{\mathrm{V}}_{s,t}(x,v)) - (\widetilde{\mathrm{X}}_{s,\infty}(x,v), \widetilde{\mathrm{V}}_{s,\infty}(x,v))| = 0, \tag{8.10}$$

and

$$\lim_{t \to +\infty} |(\widetilde{\mathrm{X}}_{t,s}(x,v), \widetilde{\mathrm{V}}_{t,s}(x,v)) - (\widetilde{\mathrm{X}}_{\infty,s}(x,v), \widetilde{\mathrm{V}}_{\infty,s}(x,v))| = 0. \tag{8.11}$$

Moreover, for all $s \geq 0$, $(\widetilde{\mathrm{X}}_{s,\infty}(x,v), \widetilde{\mathrm{V}}_{s,\infty}(x,v))$ satisfies the system

$$\begin{aligned}
\widetilde{\mathrm{V}}_{s,\infty}(x,v) &= v - \int_s^{+\infty} e^{\tau} \Big(u(\tau, \widetilde{\mathrm{X}}_{\tau,\infty}(x,v)) - \langle u(\tau) \rangle \Big)\, \mathrm{d}\tau, \\
\widetilde{\mathrm{X}}_{s,\infty}(x,v) &= x - e^{-s} v - \int_s^{+\infty} \langle u(\tau) \rangle \, \mathrm{d}\tau - \int_0^s e^{\tau - s} \langle u(\tau) \rangle \, \mathrm{d}\tau \\
&\qquad + \int_s^{+\infty} (e^{\tau - s} - 1) \Big(u(\tau, \widetilde{\mathrm{X}}_{\tau,\infty}(x,v)) - \langle u(\tau) \rangle \Big)\, \mathrm{d}\tau.
\end{aligned} \tag{8.12}$$

and we have

$$\begin{aligned}
(\widetilde{\mathrm{X}}_{s,\infty}, \widetilde{\mathrm{V}}_{s,\infty})(\widetilde{\mathrm{X}}_{\infty,s}(x,v), \widetilde{\mathrm{V}}_{\infty,s}(x,v)) &= (x,v), \\
(\widetilde{\mathrm{X}}_{\infty,s}, \widetilde{\mathrm{V}}_{\infty,s})(\widetilde{\mathrm{X}}_{s,\infty}(x,v), \widetilde{\mathrm{V}}_{s,\infty}(x,v)) &= (x,v).
\end{aligned} \tag{8.13}$$

Proof. We focus on (8.10) (the other limit (8.11) being proved similarly). Let $0 \leq s \leq t \leq t' < +\infty$. Fix $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$. Let us forget about the dependence in x, v for the sake of readability. By integration of the characteristics equation (8.8) between s and t we first have

$$\begin{aligned}
\widetilde{\mathrm{X}}_{s,t} - \widetilde{\mathrm{X}}_{s,t'} &= x - \widetilde{\mathrm{X}}_{t,t'} - \int_s^t e^{-\tau} [\widetilde{\mathrm{V}}_{\tau,t} - \widetilde{\mathrm{V}}_{\tau,t'}]\, \mathrm{d}\tau, \\
\widetilde{\mathrm{V}}_{s,t} - \widetilde{\mathrm{V}}_{s,t'} &= v - \widetilde{\mathrm{V}}_{t,t'} - \int_s^t e^{\tau} [u(\tau, \widetilde{\mathrm{X}}_{\tau,t}) - u(\tau, \widetilde{\mathrm{X}}_{\tau,t'})]\, \mathrm{d}\tau.
\end{aligned} \tag{8.14}$$

On the other hand, still using (8.8), we can rewrite

$$\begin{aligned}
x - \widetilde{\mathrm{X}}_{t,t'} &= \widetilde{\mathrm{X}}_{t',t'} - \widetilde{\mathrm{X}}_{t,t'} \\
&= \int_t^{t'} e^{-\tau} \widetilde{\mathrm{V}}_{\tau,t'}\, \mathrm{d}\tau + \int_t^{t'} (1 - e^{\tau - t'}) \langle u(\tau) \rangle \, \mathrm{d}\tau + [e^{-t} - e^{-t'}] \int_0^t e^{\tau} \langle u(\tau) \rangle \, \mathrm{d}\tau, \\
v - \widetilde{\mathrm{V}}_{t,t'} &= \widetilde{\mathrm{V}}_{t',t'} - \widetilde{\mathrm{V}}_{t,t'} = \int_t^{t'} e^{\tau} \Big(u(\tau, \widetilde{\mathrm{X}}_{\tau,t'}) - \langle u(\tau) \rangle \Big)\, \mathrm{d}\tau.
\end{aligned} \tag{8.15}$$

The goal is now to prove that $t \mapsto (\widetilde{\mathrm{X}}_{s,t}, \widetilde{\mathrm{V}}_{s,t})$ satisfies Cauchy's criterion as $t \to +\infty$, from which we will deduce the existence of (and convergence to) the limit characteristics $(\widetilde{\mathrm{X}}_{\infty,s}(x,v), \widetilde{\mathrm{V}}_{\infty,s}(x,v))$. In order to estimate the contribution of the terms in (8.14) and (8.15), we use the following improved decay estimates for u, relying on the analysis already led in Section 5.1 for dimension 2 and Section 5.2 for dimension 3. The improvement comes at the expense of taking ν large enough.

Lemma 8.3. *Assume ν is large enough (depending on the dimension d). Then there are $\gamma > 2$, $p > 1$ and a non-decreasing function φ cancelling at 0, such that*

$$\| e^{\gamma t} (u - \langle u \rangle) \|_{\mathrm{L}^p(0,\infty; \mathrm{L}^\infty(\mathbb{T}^d))} \leq \varphi(\mathscr{E}(0)),$$

for some non-negative continuous function φ cancelling at 0.

Proof. Let us start with dimension 2. By (5.2), we recall that

$$\|u(s)-\langle u(s)\rangle\|_{\mathrm{L}^\infty(\mathbb{T}^2)} \lesssim \mathscr{E}(0)^{1/4}\|\mathrm{D}_x^2 u(s)\|^{1/2}_{\mathrm{L}^2(\mathbb{T}^2)} e^{-\nu c_P^2 s/2}.$$

Thus, for $p=4$, combining with (5.1), taking ν large enough, we can ensure for some $\gamma>1$ that the following holds

$$\|e^{\gamma t}(u(s)-\langle u(s)\rangle)\|_{\mathrm{L}^4(0,\infty;\mathrm{L}^\infty(\mathbb{T}^2))} \lesssim \mathscr{E}(0).$$

In dimension 3, we may argue as above, using this time (5.5), (5.4) and the fact that $\|\rho\|_{\mathrm{L}^\infty(\mathbb{R}_+\times\mathbb{T}^3)}<+\infty$, (in this case, take $p=8/3$), we reach the same conclusion. □

Thanks to Lemma 8.3, we thus obtain, denoting by p' the dual exponent of p,

$$\begin{aligned}\left|\int_t^{t'} e^\tau\left(u(\tau,\mathrm{X}_{\tau,t'})-\langle u(\tau)\rangle\right)\mathrm{d}\tau\right| &\le \int_t^{t'} e^\tau\|u(\tau)-\langle u(\tau)\rangle\|_\infty\, d\tau\\ &\lesssim \left(\int_t^{t'} e^{p'(1-\gamma)\tau}\,d\tau\right)^{1/p'}\\ &\lesssim \left(e^{p'(1-\gamma)t}-e^{p'(1-\gamma)t'}\right)^{1/p'}.\end{aligned}\tag{8.16}$$

We therefore deduce

$$\begin{aligned}\left|\int_t^{t'} e^{-\tau}\widetilde{\mathrm{V}}_{\tau,t'}\,d\tau\right| &\lesssim \int_t^{t'} e^{-\tau}\left[|v|+\left(e^{p'(1-\gamma)\tau}-e^{p'(1-\gamma)t'}\right)^{1/p'}\right]\mathrm{d}\tau\\ &\lesssim |v|[e^{-t}-e^{-t'}]+\left(e^{-pt}-e^{-pt'}\right)^{1/p}\left(e^{p'(1-\gamma)t}-e^{p'(1-\gamma)t'}\right)^{1/p'}.\end{aligned}\tag{8.17}$$

On the other hand, recalling that $\langle u_0+j_{f_0}\rangle=0$, we can use Theorem 1.3 for $\lambda=1/2$ (combined with (2.5)) to obtain

$$\begin{aligned}\left|\int_t^{t'}(1-e^{\tau-t'})\langle u(\tau)\rangle\,\mathrm{d}\tau\right| &\lesssim \left(e^{-t/2}-e^{-t'/2}\right)+\left(e^{t/2}-e^{t'/2}\right)e^{-t'},\\ \left|[e^{-t}-e^{-t'}]\int_0^t e^\tau\langle u(\tau)\rangle\,\mathrm{d}\tau\right| &\lesssim \left|e^{-t}-e^{-t'}\right|[e^{t/2}-1].\end{aligned}\tag{8.18}$$

Eventually we have the bound

$$\left|\int_s^t e^\tau[u(\tau,\widetilde{\mathrm{X}}_{\tau,t})-u(\tau,\widetilde{\mathrm{X}}_{\tau,t'})]\,\mathrm{d}\tau\right| \lesssim \int_s^t e^\tau\|\nabla_x u\|_{\mathrm{L}^\infty(\mathbb{T}^d)}|\widetilde{\mathrm{X}}_{\tau,t}-\widetilde{\mathrm{X}}_{\tau,t'}|\,\mathrm{d}\tau.\tag{8.19}$$

The next lemma concerns improved decay estimates for $\nabla_x u$, relying this time on the analysis of Section 6.

Lemma 8.4. *Assume ν is large enough (depending on the dimension d). Then there is a non-decreasing function φ cancelling at 0, such that*

$$\|e^t\nabla_x u\|_{\mathrm{L}^1(0,\infty;\mathrm{L}^\infty(\mathbb{T}^d))} \le \varphi(\mathscr{E}(0)),$$

for some non-negative continuous function φ cancelling at 0.

Proof. The proof is analogous to that of of Lemma 8.3 and therefore we skip the details. We rely on Lemma 6.1, Proposition 6.4 and the fact that $\sup_{\mathbb{R}_+}\mathrm{M}_{\rho_f,j_f}(1,t)<+\infty$. Note in particular that from (6.6), we can take ν large enough in order to include the weight e^t in all estimates.

□

Using (8.16), (8.17), (8.18) and (8.19), we obtain

$$\begin{aligned}&|\widetilde{\mathrm{X}}_{s,t}-\widetilde{\mathrm{X}}_{s,t'}|+|\widetilde{\mathrm{V}}_{s,t}-\widetilde{\mathrm{V}}_{s,t'}|\\&\qquad\lesssim\varphi(t,t')+\int_s^t\left(e^{\tau}\|\nabla_x u\|_{\mathrm{L}^\infty(\mathbb{T}^d)}|\widetilde{\mathrm{X}}_{\tau,t}-\widetilde{\mathrm{X}}_{\tau,t'}|+e^{-\tau}|\widetilde{\mathrm{V}}_{\tau,t}-\widetilde{\mathrm{V}}_{\tau,t'}|\right)\mathrm{d}\tau,\end{aligned}$$

for some explicit non-negative function φ satisfying the following property: for all $\varepsilon>0$, there exists $T>0$ such that for all $t'\geq t\geq T$, $\varphi(t,t')\leq\varepsilon$.

Using Gronwall inequality, using Lemma 8.4, this finally leads to

$$\begin{aligned}|\widetilde{\mathrm{X}}_{s,t}-\widetilde{\mathrm{X}}_{s,t'}|+|\widetilde{\mathrm{V}}_{s,t}-\widetilde{\mathrm{V}}_{s,t'}|&\lesssim\varphi(t,t')\exp\left(\|e^t\nabla_x u\|_{\mathrm{L}^1(0,\infty;\mathrm{L}^\infty(\mathbb{T}^d))}+1\right)\\&\lesssim\varphi(t,t').\end{aligned}$$

By Cauchy's criterion, we therefore deduce the existence of $(\widetilde{\mathrm{X}}_{s,\infty}(x,v),\widetilde{\mathrm{V}}_{s,\infty}(x,v))$ such that (8.10) holds.

We can moreover pass to the limit in (8.15) for all $(x,v)\in\mathbb{T}^d\times\mathbb{R}^d$ (using the Lebesgue's dominated convergence theorem to justify the limits in the integrals in time) and we get for all $s\geq 0$, the equation (8.12).

Finally, the identities (8.13) follow from (8.9), that is

$$(\widetilde{\mathrm{X}}_{s,t},\widetilde{\mathrm{V}}_{s,t})(\widetilde{\mathrm{X}}_{t,s}(x,v),\widetilde{\mathrm{V}}_{t,s}(x,v))=(x,v),\quad(\widetilde{\mathrm{X}}_{t,s},\widetilde{\mathrm{V}}_{t,s})(\widetilde{\mathrm{X}}_{s,t}(x,v),\widetilde{\mathrm{V}}_{s,t}(x,v))=(x,v),$$

in which we take the limit $t\to+\infty$.

□

The next lemma concerns the regularity of the limit vector fields.

Lemma 8.5. *Assume furthermore that $\mathscr{E}(0)$ is small enough. The limit characteristics $(s,x,v)\mapsto(\widetilde{\mathrm{X}}_{s,\infty}(x,v),\widetilde{\mathrm{V}}_{s,\infty}(x,v))$ and $(s,x,v)\mapsto(\widetilde{\mathrm{X}}_{\infty,s}(x,v),\widetilde{\mathrm{V}}_{\infty,s}(x,v))$ belong to the space $\mathscr{C}^1(\mathbb{R}_+\times\mathbb{T}^d\times\mathbb{R}^d)$, with the estimate*

$$\left\|\mathrm{D}_{x,v}(\widetilde{\mathrm{X}}_{s,\infty}(x,v),\widetilde{\mathrm{V}}_{s,\infty}(x,v))-\begin{pmatrix}\mathrm{I}_d&-e^{-s}\mathrm{I}_d\\0&\mathrm{I}_d\end{pmatrix}\right\|_{\mathrm{L}^\infty(0,+\infty;\mathrm{L}^\infty(\mathbb{T}^d\times\mathbb{R}^d))}\leq\varphi(\mathscr{E}(0)),\tag{8.20}$$

$$\|\mathrm{D}_{x,v}(\widetilde{\mathrm{X}}_{\infty,s}(x,v),\widetilde{\mathrm{V}}_{\infty,s}(x,v))\|_{\mathrm{L}^\infty(0,+\infty;\mathrm{L}^\infty(\mathbb{T}^d\times\mathbb{R}^d))}\lesssim 1,\tag{8.21}$$

for some non-negative continuous function φ cancelling at 0.

Proof. The fact that $(s,x,v)\mapsto(\widetilde{\mathrm{X}}_{s,\infty}(x,v),\widetilde{\mathrm{V}}_{s,\infty}(x,v))$ is $\mathscr{C}^1$ and estimate (8.20) follow from the integral equation (8.12) and an application of Gronwall inequality, relying again on Lemmas 8.3 and 8.4, as we have already argued in the previous proof.

Remark that $\begin{pmatrix}\mathrm{I}_d&-e^{-s}\mathrm{I}_d\\0&\mathrm{I}_d\end{pmatrix}$ is invertible. From (8.20), taking $\mathscr{E}(0)$ small enough, we deduce that $\mathrm{D}_{x,v}(\widetilde{\mathrm{X}}_{s,\infty}(x,v),\widetilde{\mathrm{V}}_{s,\infty}(x,v))$ is also invertible, with

$$\left\|\left[\mathrm{D}_{x,v}(\widetilde{\mathrm{X}}_{s,\infty}(x,v),\widetilde{\mathrm{V}}_{s,\infty}(x,v))\right]^{-1}\right\|_{\mathrm{L}^\infty(0,+\infty;\mathrm{L}^\infty(\mathbb{T}^d\times\mathbb{R}^d)}\lesssim 1.\tag{8.22}$$

Then (8.21) is a consequence of (8.22), using (8.13). We infer indeed from the identity

$$(\widetilde{\mathrm{X}}_{s,\infty},\widetilde{\mathrm{V}}_{s,\infty})(\widetilde{\mathrm{X}}_{\infty,s}(x,v),\widetilde{\mathrm{V}}_{\infty,s}(x,v))=(x,v)$$

and the implicit function theorem that $(s,x,v)\mapsto(\widetilde{\mathrm{X}}_{\infty,s}(x,v),\widetilde{\mathrm{V}}_{\infty,s}(x,v))$ is $\mathscr{C}^1$ and that

$$\mathrm{D}_{x,v}(\widetilde{\mathrm{X}}_{s,\infty},\widetilde{\mathrm{V}}_{s,\infty})\mathrm{D}_{x,v}(\widetilde{\mathrm{X}}_{\infty,s},\widetilde{\mathrm{V}}_{\infty,s})=\mathrm{I}_{2d},$$

from which, combining with (8.22), we deduce (8.21).

□

It is possible to obtain a sharp rate of convergence in Lemma 8.2.

Lemma 8.6. *For all $0<\lambda<1$, for all $0\le s\le t<+\infty$, and all $(x,v)\in\mathbb{T}^d\times\mathbb{R}^d$,*

$$\left|(\widetilde{\mathrm{X}}_{s,t}(x,v),\widetilde{\mathrm{V}}_{s,t}(x,v))-(\widetilde{\mathrm{X}}_{s,\infty}(x,v),\widetilde{\mathrm{V}}_{s,\infty}(x,v))\right|\lesssim(1+|v|)e^{-\lambda t},\tag{8.23}$$

and

$$\left|(\widetilde{\mathrm{X}}_{t,s}(x,v),\widetilde{\mathrm{V}}_{t,s}(x,v))-(\widetilde{\mathrm{X}}_{\infty,s}(x,v),\widetilde{\mathrm{V}}_{\infty,s}(x,v))\right|\lesssim(1+|v|)e^{-\lambda t}.\tag{8.24}$$

Proof. We first focus on (8.23). Using the equations (8.8) and (8.12) we write

$$\begin{aligned}\widetilde{\mathrm{X}}_{s,t}(x,v)-\widetilde{\mathrm{X}}_{s,\infty}(x,v)&=-e^{-t}v-\int_t^{+\infty}\langle u\rangle\,\mathrm{d}\tau-e^{-t}\int_0^t e^{\tau}\langle u\rangle\,\mathrm{d}\tau\\&\quad-\int_s^t(e^{\tau-t}-1)[u(\tau,\widetilde{\mathrm{X}}_{\tau,t})-u(\tau,\widetilde{\mathrm{X}}_{\tau,\infty})]\,\mathrm{d}\tau\\&\qquad+\int_t^{+\infty}(e^{\tau-t}-1)\Big(u(\tau,\widetilde{\mathrm{X}}_{\tau,\infty})-\langle u(\tau)\rangle\Big)\,\mathrm{d}\tau\end{aligned}$$

and

$$\begin{aligned}&\widetilde{\mathrm{V}}_{s,t}(x,v)-\widetilde{\mathrm{V}}_{s,\infty}(x,v)=\\&\qquad-\int_s^t e^{\tau}[u(\tau,\widetilde{\mathrm{X}}_{\tau,t})-u(\tau,\widetilde{\mathrm{X}}_{\tau,\infty})]\,\mathrm{d}\tau+\int_t^{+\infty}e^{\tau}\Big(u(\tau,\widetilde{\mathrm{X}}_{\tau,\infty})-\langle u(\tau)\rangle\Big)\,\mathrm{d}\tau.\end{aligned}$$

We deduce that

$$\begin{aligned}&|\widetilde{\mathrm{X}}_{s,t}-\widetilde{\mathrm{X}}_{s,\infty}|+|\widetilde{\mathrm{V}}_{s,t}-\widetilde{\mathrm{V}}_{s,\infty}|\\&\quad\lesssim|v|e^{-t}-\int_t^{+\infty}\langle u\rangle\,\mathrm{d}\tau-e^{-t}\int_0^t e^{\tau}\langle u\rangle\,\mathrm{d}\tau+\int_t^{+\infty}e^{\tau}\|u(\tau)-\langle u(\tau)\rangle\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\,\mathrm{d}\tau\\&\qquad\qquad+\int_s^t e^{\tau}\|\nabla_x u\|_{\mathrm{L}^\infty(\mathbb{T}^d)}|\widetilde{\mathrm{X}}_{\tau,t}-\widetilde{\mathrm{X}}_{\tau,\infty}|\,\mathrm{d}\tau\end{aligned}$$

By Theorem 1.3 and (8.16), we have for all $0<\lambda<1$, the estimate

$$\begin{aligned}&|v|e^{-t}-\int_t^{+\infty}\langle u\rangle\,\mathrm{d}\tau-e^{-t}\int_0^t e^{\tau}\langle u\rangle\,\mathrm{d}\tau+\int_t^{+\infty}e^{\tau}\|u(\tau)-\langle u(\tau)\rangle\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\,\mathrm{d}\tau\\&\qquad\lesssim(1+|v|)e^{-\lambda t}.\end{aligned}$$

The claimed estimate (8.23) thus follows from Gronwall inequality, using the decay estimates of Lemma 8.3.

For what concerns (8.24), we rely on (8.13). Let $(x,v)\in\mathbb{T}^d\times\mathbb{R}^d$. Let $(x',v')\in\mathbb{T}^d\times\mathbb{R}^d$ such that $(x,v)=(\widetilde{\mathrm{X}}_{s,t}(x',v'),\widetilde{\mathrm{V}}_{s,t}(x',v'))$. We compute

$$\begin{aligned}&(\widetilde{\mathrm{X}}_{t,s}(x,v),\widetilde{\mathrm{V}}_{t,s}(x,v))-(\widetilde{\mathrm{X}}_{\infty,s}(x,v),\widetilde{\mathrm{V}}_{\infty,s}(x,v))\\&=(\widetilde{\mathrm{X}}_{t,s},\widetilde{\mathrm{V}}_{t,s})\circ(\widetilde{\mathrm{X}}_{s,t}(x',v'),\widetilde{\mathrm{V}}_{s,t}(x',v'))-(\widetilde{\mathrm{X}}_{\infty,s},\widetilde{\mathrm{V}}_{\infty,s})\circ(\widetilde{\mathrm{X}}_{s,t}(x',v'),\widetilde{\mathrm{V}}_{s,t}(x',v'))\\&=(x',v')-(\widetilde{\mathrm{X}}_{\infty,s},\widetilde{\mathrm{V}}_{\infty,s})\circ(\widetilde{\mathrm{X}}_{t,s}(x',v'),\widetilde{\mathrm{V}}_{t,s}(x',v'))\\&=(\widetilde{\mathrm{X}}_{\infty,s},\widetilde{\mathrm{V}}_{\infty,s})\circ(\widetilde{\mathrm{X}}_{s,\infty}(x',v'),\widetilde{\mathrm{V}}_{s,\infty}(x',v'))-(\widetilde{\mathrm{X}}_{\infty,s},\widetilde{\mathrm{V}}_{\infty,s})\circ(\widetilde{\mathrm{X}}_{s,t}(x',v'),\widetilde{\mathrm{V}}_{s,t}(x',v')).\end{aligned}$$

Consequently, by Lemma 8.5 and the estimate (8.23), we deduce

$$\begin{aligned}&|(\widetilde{\mathrm{X}}_{t,s}(x,v),\widetilde{\mathrm{V}}_{t,s}(x,v))-(\widetilde{\mathrm{X}}_{\infty,s}(x,v),\widetilde{\mathrm{V}}_{\infty,s}(x,v))|\\&\lesssim\|\nabla_{x,v}(\widetilde{\mathrm{X}}_{\infty,s},\widetilde{\mathrm{V}}_{\infty,s})\|_{\mathrm{L}^\infty(\mathbb{T}^d\times\mathbb{R}^d)}|(\widetilde{\mathrm{X}}_{s,\infty}(x',v'),\widetilde{\mathrm{V}}_{s,\infty}(x',v'))-(\widetilde{\mathrm{X}}_{s,t}(x',v'),\widetilde{\mathrm{V}}_{s,t}(x',v'))|\\&\lesssim(1+|v'|)e^{-\lambda s}.\end{aligned}$$

There remains to bound $|v'| = \widetilde{\mathrm{V}}_{t,s}(x,v)$, which is done from (8.15), using Lemma 8.3, which yields

$$|v'| \lesssim (1+|v|),$$

from which we deduce (8.24). ☐

We are finally in position to conclude the proof of Corollary 8.1. Define

$$\widetilde{\rho}(x) = \int_{\mathbb{R}^d} f_0\left(\widetilde{\mathrm{X}}_{0,\infty}(x,v), \widetilde{\mathrm{V}}_{0,\infty}(x,v)\right) \mathrm{d}v. \tag{8.25}$$

We deduce from (8.12) the formula

$$\begin{aligned}\widetilde{\rho}(x) = \int_{\mathbb{R}^d} f_0\Big(x - v + \int_0^{+\infty} \langle u(\tau)\rangle \,\mathrm{d}\tau - \int_0^{+\infty} (e^{\tau}-1)\left(u(\tau, \widetilde{\mathrm{X}}_{\tau,\infty}) - \langle u(\tau)\rangle\right) \mathrm{d}\tau,\\ v - \int_0^{+\infty} e^{\tau}\left(u(\tau, \widetilde{\mathrm{X}}_{\tau,\infty}) - \langle u(\tau)\rangle\right) \mathrm{d}\tau\Big) \,\mathrm{d}v.\end{aligned} \tag{8.26}$$

We claim that

$$\mathrm{W}_1(\rho_f(t), \widetilde{\rho}) \to_{t\to+\infty} 0,$$

with sharp exponential decay. Indeed, let $\psi(x)$ be a smooth test function, such that $\|\nabla_x \psi\|_{\mathrm{L}^\infty(\mathbb{T}^d)} \leq 1$. We consider the changes of variables $(y,w) := (\widetilde{\mathrm{X}}_{t,0}(x,v), \widetilde{\mathrm{V}}_{0,t}(x,v))$ and $(y,w) := (\widetilde{\mathrm{X}}_{t,0}(x,v), \widetilde{\mathrm{V}}_{0,\infty}(x,v)))$ which yields

$$\begin{aligned}&\left|\int_{\mathbb{T}^d} \rho_f(t,x)\psi(x)\,\mathrm{d}x - \int_{\mathbb{T}^d} \widetilde{\rho}(x)\psi(x)\,\mathrm{d}x\right| \\ &\leq \left|\int_{\mathbb{T}^d\times\mathbb{R}^d} f_0(x,v)\psi(\widetilde{\mathrm{X}}_{t,0}(x,v))\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{T}^d\times\mathbb{R}^d} f_0(x,v)\psi(\widetilde{\mathrm{X}}_{\infty,0}(x,v))\,\mathrm{d}x\right| \\ &\leq \int_{\mathbb{T}^d\times\mathbb{R}^d} f_0(x,v)\|\nabla_x\psi\|_{\mathrm{L}^\infty(\mathbb{T}^d)}|\widetilde{\mathrm{X}}_{t,0}(x,v) - \widetilde{\mathrm{X}}_{\infty,0}(x,v)|\,\mathrm{d}v\,\mathrm{d}x.\end{aligned}$$

Eventually, we use Lemma 8.6 to pass to the limit, which yields that this quantity converges exponentially fast to 0 as $t \to +\infty$. This proves our claim. It follows by (1.22) or (1.26) that for all $0 < \lambda < 1$, for all $t \geq 0$,

$$\mathrm{W}_1(f(t), \widetilde{\rho}(x)\otimes\delta_0) \lesssim e^{-\lambda t},$$

hence concluding the proof.

8.3 Further description of the asymptotic state: the case $\langle u_0 + j_{f_0}\rangle \neq 0$

Assume now that $\langle u_0 + j_{f_0}\rangle \neq 0$. We explain in this section how to adapt the analysis of the previous section to handle this case. The only procedure that is required is another renormalization to take out the drift $t\frac{\langle u_0 + j_{f_0}\rangle}{2}$ in the dynamics in x.

Consider $(\mathrm{X}(s;t,x,v), \mathrm{V}(s,t;x,v))$ the characteristics as defined in (3.1) and introduce now the following renormalized characteristics.

Definition 8.2. *For any $0 \leq s,t, < +\infty$, we set*

$$\begin{aligned}\widetilde{\mathrm{X}}_{s,t}(x,v) &:= \mathrm{X}\left(s;t,x,e^{-t}\left(v + \int_0^t e^{\tau}\langle u(\tau)\rangle\,\mathrm{d}\tau\right)\right) + (t-s)\frac{\langle u_0 + j_{f_0}\rangle}{2},\\ \widetilde{\mathrm{V}}_{s,t}(x,v) &:= e^{s}\mathrm{V}\left(s;t,x,e^{-t}\left(v + \int_0^t e^{\tau}\langle u(\tau)\rangle\,\mathrm{d}\tau\right)\right) - \int_0^s e^{\tau}\langle u(\tau)\rangle\,\mathrm{d}\tau.\end{aligned} \tag{8.27}$$

Then $\widetilde{\mathrm{X}}_{s,t}(x,v), \widetilde{\mathrm{V}}_{s,t}(x,v)$ satisfies the equation

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\widetilde{\mathrm{X}}_{s,t}(x,v) &= e^{-s}\widetilde{\mathrm{V}}_{s,t}(x,v) + e^{-s}\int_0^s e^{\tau}\langle u(\tau)\rangle\,\mathrm{d}\tau - \frac{\langle u_0 + j_{f_0}\rangle}{2},\\
\frac{\mathrm{d}}{\mathrm{d}s}\widetilde{\mathrm{V}}_{s,t}(x,v) &= e^{s}\left(u(s,\widetilde{\mathrm{X}}_{s,t}(x,v)) - \langle u(s)\rangle\right),
\end{aligned} \tag{8.28}$$

with $(\widetilde{\mathrm{X}}_{t,t}(x,v), \widetilde{\mathrm{V}}_{t,t}(x,v)) = (x,v)$. Observe that we can write

$$\int_0^s e^{\tau}\langle u(\tau)\rangle\,\mathrm{d}\tau - \frac{\langle u_0 + j_{f_0}\rangle}{2} = \int_0^s e^{\tau}\left(\langle u(\tau)\rangle - \frac{\langle u_0 + j_{f_0}\rangle}{2}\right)\mathrm{d}\tau - e^{-s}\frac{\langle u_0 + j_{f_0}\rangle}{2}.$$

The analysis of the previous section can be led again, mutatis mutandis. The only difference comes from the drift term in the new definition of $\widetilde{\mathrm{X}}$. We obtain this time an analogue of Lemma 8.2, with limit characteristics $(\widetilde{\mathrm{X}}_{s,\infty}, \widetilde{\mathrm{V}}_{s,\infty})$ satisfying

$$\begin{aligned}
\widetilde{\mathrm{V}}_{s,\infty}(x,v) &= v - \int_s^{+\infty} e^{\tau}\left(u(\tau,\widetilde{\mathrm{X}}_{\tau,\infty}(x,v)) - \langle u(\tau)\rangle\right)\mathrm{d}\tau,\\
\widetilde{\mathrm{X}}_{s,\infty}(x,v) &= x - e^{-s}v + e^{-s}\frac{\langle u_0 + j_{f_0}\rangle}{2} - \int_s^{+\infty}\left(\langle u(\tau)\rangle - \frac{\langle u_0 + j_{f_0}\rangle}{2}\right)\mathrm{d}\tau\\
&\quad - \int_0^s e^{\tau - s}\left(\langle u(\tau)\rangle - \frac{\langle u_0 + j_{f_0}\rangle}{2}\right)\mathrm{d}\tau\\
&\quad + \int_s^{+\infty}(e^{\tau - s} - 1)\left(u(\tau,\widetilde{\mathrm{X}}_{\tau,\infty}(x,v)) - \langle u(\tau)\rangle\right)\mathrm{d}\tau.
\end{aligned} \tag{8.29}$$

Analogues of Lemmas 8.5 and 8.6 hold as well. We now have the formula

$$\rho_f(t,x) = \int_{\mathbb{R}^d} f_0\left(\widetilde{\mathrm{X}}_{0,t}(x,v) - t\frac{\langle u_0 + j_{f_0}\rangle}{2}, \widetilde{\mathrm{V}}_{0,t}(x,v)\right)\mathrm{d}v.$$

Setting

$$\widetilde{\rho}(t,x) := \int_{\mathbb{R}^d} f_0\left(\widetilde{\mathrm{X}}_{0,\infty}(x,v) - t\frac{\langle u_0 + j_{f_0}\rangle}{2}, \widetilde{\mathrm{V}}_{0,\infty}(x,v)\right)\mathrm{d}v, \tag{8.30}$$

the outcome is, as in the previous case,

Lemma 8.7. *For all $0 < \lambda < 1$, we have, for all $t \geq 0$,*

$$\mathrm{W}_1(\rho_f(t), \widetilde{\rho}(t)) \lesssim e^{-\lambda t}.$$

It follows that

$$\mathrm{W}_1\left(f(t), \widetilde{\rho}(t,x) \otimes \delta_{\frac{\langle u_0 + j_{f_0}\rangle}{2}}\right) \lesssim e^{-\lambda t},$$

concluding the proof of Corollary 8.1.

9 Appendix

9.1 Wasserstein distance

To simplify the presentation, X here will denote either $\mathbb{T}^d$ or $\mathbb{T}^d \times \mathbb{R}^d$.

Definition 9.1. *For $m > 0$ we denote by $\mathcal{M}_{1,m}(X)$ the set of all measures μ such that*

$$\int_X \mathrm{d}_X(z,0)\,\mathrm{d}\mu(z) < +\infty, \qquad \mu(X) = m.$$

where d_X stands for the canonical distance on X.

Definition 9.2. *Fix $m > 0$ and consider μ and ν in $\mathcal{M}_{1,m}(X)$. The* Wassertein distance *between μ and ν is*

$$\mathrm{W}_1(\mu, \nu) := \inf_{\gamma \in \Gamma(\mu,\nu)} \int_{X^2} \mathrm{d}_X(z, z')\, \mathrm{d}\gamma(z, z'),$$

where $\Gamma(\mu, \nu)$ denotes the collection of all measures on $X \times X$ with first and second marginal respectively equal to μ and ν.

Proposition 9.1 (W$_1$ metrizes weak-$\star$ convergence)**.** *Fix $m > 0$. Given $(\mu_n)_n \in \mathcal{M}_{1,m}(X)^{\mathbb{N}}$ and $\mu \in \mathcal{M}_{1,m}(X)$, the two following facts are equivalent*

(i) For all $f \in \mathscr{C}_b^0(X)$,

$$\int_X (f(z) + \mathrm{d}_X(z, 0))\, \mathrm{d}\mu_n(z) \underset{n\to+\infty}{\longrightarrow} \int_X (f(z) + \mathrm{d}_X(z, 0))\, \mathrm{d}\mu(z).$$

(ii) $(\mathrm{W}_1(\mu_n, \mu))_n \to_n 0$.

Proposition 9.2 (Monge-Kantorovitch duality)**.** *Fix $m > 0$ and consider μ and ν in $\mathcal{M}_{1,m}(X)$. Then*

$$\mathrm{W}_1(\mu, \nu) = \sup_{\|\nabla_z \phi\|_\infty \leq 1} \left\{ \int_X \phi(z)\mathrm{d}\mu(z) - \int_X \phi(z)\mathrm{d}\nu(z) \right\},$$

9.2 Hadamard's global inversion Theorem

Theorem 9.3 (Hadamard)**.** *Let $n \geq 1$. Let Z be $\mathscr{C}^1$ map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Then Z is a $\mathscr{C}^1$ diffeomorphism if and only if* $\det \nabla Z$ *never vanishes and f is proper, i.e.*

$$|Z(y)| \to +\infty, \quad \text{as } |y| \to +\infty.$$

9.3 Interpolation

The following classical interpolation estimate can be for instance found in [8, Thm 1.5.2].

Theorem 9.4 (Gagliardo-Nirenberg-Sobolev)**.** *Consider $1 \leq p, q, r \leq \infty$ and $m \in \mathbb{N}$. Assume that $j \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ satisfy*

$$\frac{1}{p} = \frac{j}{d} + \left(\frac{1}{r} - \frac{m}{d}\right)\alpha + \frac{1-\alpha}{q},$$
$$\frac{j}{m} \leq \alpha \leq 1,$$

with the exception $\alpha < 1$ if $m - j - d/r \in \mathbb{N}$. Then, the following holds. For any $g \in \mathrm{L}^q(\mathbb{T}^d)$, if $\mathrm{D}^m g \in \mathrm{L}^r(\mathbb{T}^d)$, then $\mathrm{D}^j g \in \mathrm{L}^p(\mathbb{T}^d)$ and we have the following estimate for g

$$\|\mathrm{D}^j g\|_{\mathrm{L}^p(\mathbb{T}^d)} \lesssim \|\mathrm{D}^m g\|^{\alpha}_{\mathrm{L}^r(\mathbb{T}^d)} \|g\|^{1-\alpha}_{\mathrm{L}^q(\mathbb{T}^d)} + \|g\|_{\mathrm{L}^q(\mathbb{T}^d)},$$

where the constant behind $\lesssim$ does not depend on g. If $\langle \mathrm{D}^j g\rangle = 0$, then the term $\|g\|_{\mathrm{L}^q(\mathbb{T}^d)}$ in the rhs can be dispensed with.

9.4 Parabolic regularity estimates for the Navier-Stokes equations with a source term: proof of Propositions 4.2 and 4.3

We provide in this section a complete proof of Propositions 4.2 and 4.3. To this aim, we rely on the following standard approximation procedure: we consider, for $\chi \in \mathscr{C}^\infty(\mathbb{T}^d)$, the regularized system:

$$\partial_t u + (\widetilde{u}_\chi \cdot \nabla) u - \nu \Delta u + \nabla p = F, \tag{9.1}$$

$$\operatorname{div} u = 0, \tag{9.2}$$

$$u(0,\cdot) = u_0, \tag{9.3}$$

where $\widetilde{u}_\chi := u \star \chi$. When u_0 and F are smooth, the existence of a unique smooth solution to system (9.1) – (9.3) is standard.

This section is organized as follows: first we perform the energy estimates for the regularized system in Proposition 9.5. We deduce in a second time the corresponding results for the true Navier-Stokes system in Proposition 4.2 (for $d = 2$) and Proposition 4.3 (for $d = 3$).

Proposition 9.5. *Consider a nondecreasing function $\gamma \in \mathscr{C}^1_b(\mathbb{R})$ vanishing at 0 and such that $\|\gamma\|_{\mathrm{W}^{1,\infty}(\mathbb{R})} \leq 1$. There exists $\mathrm{C} > 0$ and an onto nondecreasing continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, such that the following holds for $d \in \{2,3\}$, any $u_0 \in \mathscr{C}^\infty_{\mathrm{div}}(\mathbb{T}^d)$, $F \in \mathscr{C}^\infty(\mathbb{R}_+ \times \mathbb{T}^d)$ and any $\chi \in \mathscr{C}^\infty(\mathbb{T}^d)$ such that $\|\chi\|_1 = 1$, the unique solution u of* (9.1) – (9.3) *satisfies for $t \geq 0$,*

$$\gamma(t)\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^d)} + \nu \int_0^t \gamma(s)\|\Delta u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^d)}\,\mathrm{d}s \\ \leq \varphi\left(\mathrm{A}(t) + \frac{1}{\nu}\right)\left(1 + \int_0^t \gamma(s)\|F(s)\|^2_{\mathrm{L}^2(\mathbb{T}^d)}\,\mathrm{d}s\right)\Phi(h_d(t)), \tag{9.4}$$

where φ is a continuous nonnegative increasing function cancelling at 0, $\Phi(z) := (1+z)e^z$, $h_2 = 0$ and

$$h_3(t) := \frac{\mathrm{C}}{\nu}\int_0^t \|\nabla u(s)\|^2_{\mathrm{L}^3(\mathbb{T}^d)}\,\mathrm{d}s, \tag{9.5}$$

$$\mathrm{A}(t) := \frac{1}{2}\sup_{[0,t]}\|u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^d)} + \nu\int_0^t \|\nabla u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^d)}\,\mathrm{d}s. \tag{9.6}$$

Proof. We multiply the equation by $-\gamma(t)\Delta u$, and use adequate integrations by parts on together with Young's and Hölder's inequality, to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\gamma(t)\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^d)}\Big\} + \frac{\nu\gamma(t)}{2}\|\Delta u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^d)} \\ \leq \frac{1}{2}\gamma'(t)\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^d)} \\ + \frac{\gamma(t)}{2\nu}\|F(t)\|^2_{\mathrm{L}^2(\mathbb{T}^d)} + \gamma(t)\|\Delta u(t)\|_{\mathrm{L}^2(\mathbb{T}^d)}\|u(t)\|_{\mathrm{L}^6(\mathbb{T}^d)}\|\nabla u(t)\|_{\mathrm{L}^3(\mathbb{T}^d)}. \tag{9.7}$$

We start with the case $d = 3$ which is more straightforward as (9.4) involves in the right-hand side an extra term compared to $d = 2$.. We use another time Young's inequality and the Sobolev embedding $\mathrm{H}^1(\mathbb{T}^3) \hookrightarrow \mathrm{L}^6(\mathbb{T}^3)$ to write

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\gamma(t)\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\Big\} + \nu\gamma(t)\|\Delta u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^3)} \\ \lesssim \gamma'(t)\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^3)} + \frac{\gamma(t)}{\nu}\|F(t)\|^2_{\mathrm{L}^2(\mathbb{T}^3)} + \frac{\gamma(t)}{\nu}\|u(t)\|^2_{\mathrm{H}^1(\mathbb{T}^3)}\|\nabla u(t)\|^2_{\mathrm{L}^3(\mathbb{T}^3)}.$$

Using the definition (9.6) of $\mathrm{A}(t)$ and the fact that $\|\gamma\|_{\mathrm{W}^{1,\infty}(\mathbb{R})} \le 1$, we infer, introducing $\ell(t) := \gamma(t)\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}$

$$\begin{aligned}
&\ell'(t) + \nu\gamma(t)\|\Delta u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^3)} \\
&\qquad \lesssim \|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^3)} + \frac{\mathrm{A}(t)}{\nu}\|\nabla u(t)\|^2_{\mathrm{L}^3(\mathbb{T}^3)} + \frac{\gamma(t)}{\nu}\|F(t)\|^2_{\mathrm{L}^2(\mathbb{T}^3)} + \frac{1}{\nu}\ell(t)\|\nabla u(t)\|^2_{\mathrm{L}^3(\mathbb{T}^3)},
\end{aligned}$$

which implies by Gronwall's inequality (since $\ell(0) = 0$), using once again the definition of $\mathrm{A}(t)$,

$$\begin{aligned}
&\ell(t) + \int_0^t \nu\gamma(s)\|\Delta u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s \\
&\qquad\qquad \lesssim \left(\frac{\mathrm{A}(t)}{\nu} + h_3(t) + \frac{1}{\nu}\int_0^t \gamma(s)\|F(s)\|^2_{\mathrm{L}^2(\mathbb{T}^3)}\,\mathrm{d}s\right)\exp(h_3(t)),
\end{aligned}$$

where h_3 is given by (9.5), for some univeral constant $\mathrm{C} > 0$. Using Young's inequality, this last estimate can be recasted into (9.4) for some function φ.

For the case $d = 2$ we go back to estimate (9.7) and invoke the Gagliardo-Nirenberg-Sobolev estimate of Theorem 9.4, applied for $(j, p, q, r, m) = (0, 3, 2, 2, 1)$, which writes

$$\|\nabla g\|_{\mathrm{L}^3(\mathbb{T}^2)} \lesssim \|\mathrm{D}^2 g\|^{1/3}_{\mathrm{L}^2(\mathbb{T}^2)}\|\nabla g\|^{2/3}_{\mathrm{L}^2(\mathbb{T}^2)}.$$

Since on the torus we have $\|\mathrm{D}^2\, g\|_{\mathrm{L}^2(\mathbb{T}^2)} \lesssim \|\Delta g\|_{\mathrm{L}^2(\mathbb{T}^2)}$, we therefore get, using Young's inequality with the pair of conjugate exponents $(3/2, 3)$,

$$\|\Delta u(t)\|_{\mathrm{L}^2(\mathbb{T}^2)}\|u(t)\|_{\mathrm{L}^6(\mathbb{T}^2)}\|\nabla u(t)\|_3 - \frac{\nu}{4}\|\Delta u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^2)} \lesssim \frac{1}{\nu^2}\|u(t)\|^3_{\mathrm{L}^6(\mathbb{T}^2)}\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^2)}.$$

Now, the Gagliardo-Nirenberg-Sobolev estimate for $(j, p, q, r, m) = (0, 6, 2, 2, 1)$ gives

$$\|g\|_{\mathrm{L}^6(\mathbb{T}^2)} \lesssim \|\nabla g\|^{2/3}_{\mathrm{L}^2(\mathbb{T}^2)}\|g\|^{1/3}_{\mathrm{L}^2(\mathbb{T}^2)} + \|g\|_{\mathrm{L}^2(\mathbb{T}^2)},$$

so that we have

$$\begin{aligned}
&\|\Delta u(t)\|_{\mathrm{L}^2(\mathbb{T}^2)}\|u(t)\|_{\mathrm{L}^6(\mathbb{T}^2)}\|\nabla u(t)\|_{\mathrm{L}^3(\mathbb{T}^2)} - \frac{\nu}{4}\|\Delta u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^2)} \\
&\quad \lesssim \frac{1}{\nu^2}\|u(t)\|_{\mathrm{L}^2(\mathbb{T}^2)}\|\nabla u(t)\|^4_{\mathrm{L}^2(\mathbb{T}^2)} + \frac{1}{\nu^2}\|u(t)\|^3_{\mathrm{L}^2(\mathbb{T}^2)}\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^2)} \\
&\quad \le \frac{\mathrm{A}(t)}{\nu^2}\|\nabla u(t)\|^4_{\mathrm{L}^2(\mathbb{T}^2)} + \frac{\mathrm{A}(t)^3}{\nu^2}\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^2)}.
\end{aligned}$$

All in all, keeping the notation $\ell(t) = \gamma(t)\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^2)}$, we infer from (9.7)

$$\begin{aligned}
&\ell'(t) + \nu\gamma(t)\|\Delta u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^2)} \\
&\lesssim \|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \frac{\gamma(t)}{\nu}\|F(t)\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \frac{\mathrm{A}(t)}{\nu^2}\gamma(t)\|\nabla u(t)\|^4_{\mathrm{L}^2(\mathbb{T}^2)} + \frac{\mathrm{A}(t)^3}{\nu^2}\gamma(t)\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^2)} \\
&\lesssim \left(1 + \frac{\mathrm{A}(t)^3}{\nu^2}\right)\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \frac{\gamma(t)}{\nu}\|F(t)\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \frac{\mathrm{A}(t)}{\nu^2}\ell(t)\|\nabla u(t)\|^2_{\mathrm{L}^2(\mathbb{T}^2)}.
\end{aligned}$$

which leads, by Gronwall's inequality, to the following estimate (changing the definition of the universal constant C)

$$\ell(t) + \nu\int_0^t \|\Delta u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^2)}\,\mathrm{d}s \lesssim \left(1 + \frac{1}{\nu}\int_0^t \gamma(s)\|F(s)\|^2_{\mathrm{L}^2(\mathbb{T}^2)}\,\mathrm{d}s\right)\psi\left(\mathrm{A}(t) + \frac{1}{\nu}\right)$$

for some nonnegative increasing function ψ, where we have repeatedly used

$$\int_0^t \|\nabla u(s)\|^2_{\mathrm{L}^2(\mathbb{T}^2)}\, \mathrm{d}s \leq \frac{\mathrm{A}(t)}{\nu}.$$

As in the three dimensional case, the last estimate can be written in the form (9.4) for some other function φ using Young's inequality. ☐

Proof of Proposition 4.2. In 2D, choosing $\gamma \in \mathscr{C}^1(\mathbb{R})$ that coincides with the identity on $[0,1]$ and that is constant equal to 1 on $[2,+\infty)$, Proposition 9.5 allows, up to a standard compactness argument, to prove Proposition 4.2. ☐

Proof of Proposition 4.3. In 3D, appropriate modifications have to be done, resting either on the smallness of the initial data and the forcing term or the largeness of the viscosity. Recall the notation $\|\cdot\|_{\dot{\mathrm{H}}^s(\mathbb{T}^3)}$ for the L^2 norm associated to the multiplier $|\xi|^s$.

Proposition 9.6. *There exists a universal constant* $\mathrm{C}_\star$ *such that the following holds. For any* $\nu > 0$, $u_0 \in \mathscr{C}^\infty_{\mathrm{div}}(\mathbb{T}^3)$, $F \in \mathscr{C}^\infty(\mathbb{R}_+ \times \mathbb{T}^3)$ *and any* $\chi \in \mathscr{C}^\infty(\mathbb{T}^3)$ *such that* $\|\chi\|_1 = 1$, *if for some* $T > 0$ *one has*

$$\|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \frac{\mathrm{C}_\star}{\nu}\int_0^T \|F(s)\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)}\, \mathrm{d}s \leq \frac{\nu^2}{\mathrm{C}_\star^2}, \tag{9.8}$$

then the unique solution u *of* (9.1) – (9.3) *satisfies for* $t \in [0,T]$,

$$\|u(t)\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \nu\int_0^t \|\nabla u(s)\|^2_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)}\, \mathrm{d}s \leq \|u_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \frac{\mathrm{C}_\star}{\nu}\int_0^t \|F(s)\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)}\, \mathrm{d}s.$$

In particular, recalling the definition (9.5), *on* $[0,T]$ *we have* $h_3 \leq \mathrm{C}/\mathrm{C}_\star$ *where* C *is the universal constant given in Proposition 9.5.*

Proof. Let us first recall the fundamental energy estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\mathrm{L}^2(\mathbb{T}^3)} + \nu\|u\|^2_{\dot{\mathrm{H}}^1(\mathbb{T}^3)} \lesssim \frac{1}{\nu}\|F\|^2_{\mathrm{H}^{-1}(\mathbb{T}^3)}. \tag{9.9}$$

Consider Λ the Fourier multiplier associated to $|\xi|$. After taking the scalar product with Λu, thanks to Plancherel's formula, Hölder's inequality, to the continuity of the Leray projector $\mathbb{P}$ on $\mathrm{L}^{3/2}(\mathbb{T}^3)$, we can obtain as well

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)} &+ \nu\|\nabla u\|^2_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)} \\ &\lesssim \|\Lambda u\|_{\mathrm{L}^3(\mathbb{T}^3)}\|\nabla u\|_{\mathrm{L}^3(\mathbb{T}^3)}\|u\|_{\mathrm{L}^3(\mathbb{T}^3)} + \|\Lambda u\|_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)}\|F\|_{\dot{\mathrm{H}}^{-1/2}(\mathbb{T}^3)}.\end{aligned}$$

Using Young's inequality and combining with (9.9) we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \nu\|\nabla u\|^2_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)} \lesssim \|\Lambda u\|_{\mathrm{L}^3(\mathbb{T}^3)}\|\nabla u\|_{\mathrm{L}^3(\mathbb{T}^3)}\|u\|_{\mathrm{L}^3(\mathbb{T}^3)} + \frac{1}{\nu}\|F\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)}.$$

we therefore have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \nu\|\nabla u\|^2_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)} \lesssim \|\Lambda u\|_{\mathrm{L}^3(\mathbb{T}^3)}\|\nabla u\|_{\mathrm{L}^3(\mathbb{T}^3)}\|u\|_{\mathrm{L}^3(\mathbb{T}^3)} + \frac{1}{\nu}\|F\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)}.$$

We have by Sobolev embedding

$$\begin{aligned}\|g - \langle g\rangle\|_{\mathrm{L}^3(\mathbb{T}^3)} &\lesssim \|g\|_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)},\\ \|g\|_{\mathrm{L}^3(\mathbb{T}^3)} &\lesssim \|g\|_{\mathrm{H}^{1/2}(\mathbb{T}^3)}.\end{aligned}$$

Since Λu and ∇u have a vanishing mean, we therefore have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \nu\|\nabla u\|^2_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)} \lesssim \|\nabla u\|^2_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)}\|u\|_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \frac{1}{\nu}\|F\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)}.$$

This is a differential inequality of the form

$$x'(t) + 2\nu y(t) \leq \mathrm{C}\Big(x(t)^{1/2}y(t) + \frac{1}{\nu}z(t)\Big),$$

where C is some universal constant and

$$x(t) = \|u(t)\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)}, \quad y(t) = \|\nabla u(t)\|^2_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)}, \quad z(t) = \|F(t)\|^2_{\dot{\mathrm{H}}^{-1/2}(\mathbb{T}^3)}. \tag{9.10}$$

After integration, we hence have

$$x(t) + \nu\int_0^t y(s)\,\mathrm{d}s \leq x(0) + \int_0^t y(s)\left(\mathrm{C}x(s)^{1/2} - \nu\right)\,\mathrm{d}s + \frac{\mathrm{C}}{\nu}\int_0^t z(s)\,\mathrm{d}s.$$

In particular, if for some $T > 0$ one has (this precisely corresponds to the assumption (4.1))

$$x(0) + \frac{\mathrm{C}}{\nu}\int_0^T z(s)\,\mathrm{d}s \leq \frac{\nu^2}{\mathrm{C}^2}, \tag{9.11}$$

then by a standard continuity argument we chan show that

$$t^\star := \sup\left\{t \leq T,\, x(t)^{1/2} \leq \frac{\nu}{\mathrm{C}}\right\},$$

must satisfy $t^\star = T$, and finally on $[0,T]$,

$$x(t) + \nu\int_0^t y(s)\,\mathrm{d}s \leq x(0) + \frac{\mathrm{C}}{\nu}\int_0^t z(s)\,\mathrm{d}s, \tag{9.12}$$

which corresponds to the desired inequality, recalling (9.10). □

Combining the estimates of Proposition 9.6 and Proposition 9.5 and choosing γ as we did for Proposition 4.2, we finally prove Proposition 4.3 by a compactness argument. □

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