Construction of solutions to the subcritical gKdV equations with a given asymptotical behavior.

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Abstract

We consider the sub-critical generalized Korteweg-de Vries equation :

$$u_t + (u_{xx} + u^4)_x = 0, \quad t, x \in \mathbb{R}.$$

Let $R_j(t,x) = Q_{c_j}(x - x_j - c_j t)$ $(j = 1, \ldots, N)$ be N soliton solutions to this equation. Denote $U(t)$ the KdV linear group, and let V be in an adequate weighted Sobolev space.

We construct a solution $u(t)$ to the generalized Korteweg-de Vries equation such that :

$$\lim_{t\to\infty} \left\| u(t) - U(t)V - \sum_{j=1}^{N} R_j(t) \right\|_{H^1} = 0.$$

1 Introduction

1.1 General setting

We consider the following sub-critical generalized Korteweg-de Vries equation :

$$u_t + (u_{xx} + u^4)_x = 0, \quad t, x \in \mathbb{R}. \tag{1}$$

It is a special case of the generalized Korteweg-de Vries equation :

$$u_t + (u_{xx} + u^p)_x = 0, \quad t, x \in \mathbb{R}, \tag{2}$$

where $p \geq 2$. The case $p = 2$ corresponds to the original equation introduced by Korteweg and de Vries [9] in the context of shallow water waves. For both $p = 2$ and $p = 3$, this equation has many applications to Physics : see for example Miura [21], Lamb [11].

There are two formally conserved quantities for solutions to (2) :

$$\int u^2(t) = \int u^2(0) \qquad (L^2 \text{ mass}), \tag{3}$$

$$E(u(t)) = \frac{1}{2}\int u_x^2(t) - \frac{1}{p+1}\int u^{p+1}(t) = E(u(0)) \qquad (\text{energy}). \tag{4}$$

2000 Mathematical Subject Classification : Primary 35Q53, Secondary 35B40, 35Q51.
Keywords : Generalized Korteweg-de Vries equations, wave operator, large data, solitons, linear-non linear interaction.

The local Cauchy problem for (2) has been intensively studied by many authors. Kenig, Ponce and Vega [7] proved the following existence and uniqueness result in $H^1(\mathbb{R})$: for $u_0 \in H^1(\mathbb{R})$, there exist $T = T(\|u_0\|_{H^1}) > 0$ and a solution $u \in C([0,T], H^1(\mathbb{R}))$ to (1) satisfying $u(0) = u_0$, which is unique in some class $Y_T \subset C([0,T], H^1(\mathbb{R}))$. For such a solution, one has conservation of mass and energy. Moreover, if T_1 denotes the maximal time of existence for u, then either $T_1 = +\infty$ (global solution) or $T_1 < \infty$ and $\|u(t)\|_{H^1} \to \infty$ as $t \uparrow T_1$ (blow-up solution).

In the case $2 \le p < 5$, all solutions to (2) in H^1 are global and uniformly bounded thanks to the conservation laws and the Gagliardo-Nirenberg inequality :

$$\forall v \in H^1(\mathbb{R}), \quad \int |v|^{p+1} \le C(p) \left(\int v^2 \right)^{\frac{p+3}{4}} \left(\int v_x^2 \right)^{\frac{p-1}{4}}. \tag{5}$$

The case $p = 5$ is L^2-critical, in the sense that the mass remains unaffected by scaling. If

$$u_t + (u_{xx} + u^5)_x = 0, \quad t, x, \in \mathbb{R}, \tag{6}$$

then $u_\lambda(t,x) = \lambda^{1/6} u(\lambda t, \lambda^{1/3} x)$ is also a solution to (6), and $\|u_\lambda\|_{L^2} = \|u\|_{L^2}$. In this case, the local existence result of [7] is improved to initial data in L^2 (instead of H^1). However, existence of finite time blow-up solutions was proved by Merle [20] and Martel and Merle [17]. Therefore $p = 5$ also appears as a critical exponent for the long time behavior of solution to (2).

A fundamental property of (2) is the existence of a family of explicit traveling wave solutions. If Q denotes the only solution (up to translation) of :

$$Q > 0, \quad Q \in H^1(\mathbb{R}), \quad Q_{xx} + Q^p = Q, \quad \text{i.e.} \quad Q(x) = \left(\frac{p+1}{2\cosh^2(\frac{p-1}{2}x)} \right)^{\frac{1}{p-1}},$$

then for $c > 0$ the soliton

$$R_{c,x_0} = c^{\frac{1}{p-1}} Q(\sqrt{c}(x - x_0 - ct)) \text{ is a solution to (2).}$$

For $p = 2$ and $p = 3$, equation (2) is completely integrable, and thus has very special features. The inverse scattering transform method allows to solve the Cauchy problem in an appropriate space (for example if $u_0 \in H^4$ and $xu_0 \in L^1$) and the qualitative behaviour of solutions is well understood. For example, given u_0 smooth and with rapid decay, there exist N solitons R_{c_j,x_j} such that

$$\left\| u(t) - \sum_{j=1}^N R_{c_j,x_j}(t) \right\|_{L^\infty(x \ge -t^{1/3})} \le \frac{C}{t^{1/3}} \quad (\text{as } t \to \infty).$$

See for example Schuur [23], Eckhaus and Schuur [5], Miura [21].

However, if $p \neq 2$ or 3, the inverse scattering transform method does not longer apply, and the description of solutions in the general, non-integrable case is an open problem. It can be decomposed in two types of problems.

Problem 1 : Asymptotic behaviour. Given an initial data u_0, does the out coming solution $u(t)$ to (2) exists for all time ? If it does (for example in the subcritical case), can its behavior be described, as $t \to \infty$? If blow up happens, can the blow up rate and profile be determined ?

Problem 2 : Non-linear wave operator. Given some reasonable behaviour as $t \to \infty$, can we find a solution $u(t)$ to (2) defined for large enough t, with this behaviour ? Is there uniqueness for $u(t)$?

1.2 Recent results on Problems 1 and 2

Let us now develop some recent results which will be the base to our result. We denote $U(t)$ the linear operator for KdV equation, i.e. $v(t) = U(t)V$ satisfies $v_t + v_{xxx} = 0$, $v(0) = V$.

The first result deals with scattering for small initial data, a problem studied by many authors (see for example [24], [22], [2], [6]). Let us remind the result of Hayashi and Naumkin [6]. Introduce the following weighted Sobolev spaces :

$$H^{s,m} = \{\phi \in \mathcal{S}' | \ \|\phi\|_{H^{s,m}} = \|(1+|x|^2)^{m/2}(1-\partial_x^2)^{s/2}\phi\|_{L^2} < \infty\}. \qquad (7)$$

Scattering operator. Let $p > 3$. Given u_0 small enough in $H^{1,1}$, the out-coming solution $u(t)$ to (2) is global in time, and there is scattering, in the sense that there exists a function $V \in L^2$ so that :

$$\|u(t) - U(t)V\|_{L^2} \to 0 \quad as \quad t \to \infty.$$

Furthermore, $\|u(t)\|_{L^\infty} \leq Ct^{-1/3}$ (linear decay rate).

This is the description of solutions with initial data around 0 (in $H^{1,1}$), a result which can be understood as stability around 0.

The second type of results we want to focus on is that which describes the solutions around solitons or a sum of solitons. The following result of Martel, Merle, Tsai [18] solves the problem of stability in H^1 of a sum of N decoupled solitons (see also Martel and Merle [14]).

Stability of the sum of N solitons. Suppose $p = 2, 3$ or 4. Let $N \in \mathbb{N}$, and $0 < c_1 < \ldots < c_N$. There exist γ_0 and α_0 (small) and A, L_0 (large), so that the following is true. Assume that there exist $L \geq L_0$, $\alpha < \alpha_0$ and $x_1^0 < \ldots < x_N^0$ such that :

$$\left\| u(0) - \sum_{j=1}^{N} Q_{c_j}(\cdot - x_j^0) \right\|_{H^1} \leq \alpha, \quad with \ \ x_j^0 \geq x_{j-1}^0 + L, \quad for \ j = 2, \ldots, N.$$

Then there exist $x_1(t), \ldots, x_N(t) \in \mathbb{R}$ such that :

$$\forall t \geq 0, \qquad \left\| u(t) - \sum_{j=1}^{N} Q_{c_j}(\cdot - x_j(t) - c_j t) \right\|_{H^1} \leq A(\alpha + e^{-\gamma_0 L}).$$

These results are related to Problem 1. Let us now turn to results concerning Problem 2. First, Martel [12] proved the existence and uniqueness of N-solitons in the cases $p = 2, 3, 4$ or 5 :

Existence and uniqueness of the N-soliton. Let $p \in [2, 5]$. Let $N \in \mathbb{N}$, $0 < c_1 < \ldots < c_N$, and $x_1, \ldots, x_N \in \mathbb{R}$. There exist $T_0 \in \mathbb{R}$ and a unique function

$u \in C([T_0, +\infty), \mathbb{R})$, which is a H^1 solution to (1), and such that :

$$\left\| u(t) - \sum_{j=1}^{N} Q_{c_j}(\cdot - x_j - c_j t) \right\|_{H^1} = 0 \quad as \quad t \to \infty.$$

Furthermore, $u \in C^\infty([T_0, \infty) \times \mathbb{R})$ and convergence takes place in H^s for all $s \geq 0$, with an exponential decay :

$$\exists \gamma > 0,\ \forall s \geq 0, \exists A_s \quad / \quad \left\| u(t) - \sum_{j=1}^{N} Q_{c_j}(\cdot - x_j - c_j t) \right\|_{H^1} \leq A_s e^{-\gamma t}.$$

This result appears as a development of monotonicity properties and a dynamical argument, ideas which where used by Martel and Merle [14] and Martel, Merle and Tsai [18].

However, it is a surprise that the method could be adapted even to the critical case $p = 5$, although it is well known that solitons are unstable in $H^1(\mathbb{R})$: there is in fact blow-up for a large class of initial data and the blow-up profile is stable, see [15], [17], [20], [16]. Another surprise is uniqueness of the N-soliton.

Notice that in view of this result, the stability of a sum of N solitons can be interpreted as stability of the N-soliton (solution to (2)).

The last result solves the case of a linear behavior, that is the existence of a wave operator :

Large data wave operator. Let $p > 3$, and $V \in H^{2,2}$. There exist $T_0 \in \mathbb{R}$ and $u \in C([T_0, \infty), H^1)$ solution to (2) such that :

$$\|u(t) - U(t)V\|_{H^1} \to 0 \quad as \quad t \to \infty.$$

Furthermore u is unique in an adapted class.

In the same way that the result of Martel [12] was based on considerations of Martel, Merle and Tsai [18], this result strongly relies on the analysis of Hayashi and Naumkin [6].

1.3 Statement of the main result

Our goal is to construct solutions which behave like a sum of a linear term $U(t)V$, and of N solitons, in the subcritical $p < 5$ case. Notice that in [3] such solutions are constructed in the critical case $p = 5$. More precisely, given $0 < c_1 < \ldots < c_n$ and $x_1, \ldots x_N \in \mathbb{R}$, we would like to construct solutions $u(t)$ to (2), defined for large enough times and such that

$$\left\| u(t) - U(t)V - \sum_{j=1}^{N} R_{c_j, x_j}(t) \right\|_{H^1} \to 0 \quad \text{as} \quad t \to \infty.$$

In this article, we construct such solutions in the case $p = 4$ (that is, for equation (1)), provided that V is smooth enough, with sufficient decay on the right. From now on and throughout the rest of the article,

$$\text{we focus on the sub-critical case } p = 4. \tag{8}$$

Let us first remind the functional setting which will be used throughout the proofs. Fix once for all the three constants :

$$\gamma \in (0, 1/3), \quad \alpha = \frac{1}{2} - \gamma \in (0, 1/2) \text{ and } \delta = \frac{1 - 2\gamma}{3} > 0. \tag{9}$$

(γ is arbitrary). These constants are those of [6] in the case $p = 4$.

Again as in [6], we will use the notation $D = \partial_x = \frac{\partial}{\partial x}$ for the partial differentiation with respect to the space variable x, and

$$D^\alpha f = \mathcal{F}^{-1} \xi^\alpha e^{-(i\pi/2)(1+\alpha)} \hat{f},$$

along with the two following operators

$$J^t f = U(t) x U(-t) f = (x - 3t\partial_x^2) f, \text{ and } I^t \phi = x\phi + 3t \int_{-\infty}^x \partial_t f(t, y) dy.$$

We write J^t and I^t so as to emphasize that we will always consider norms at a fixed time t although J^t and I^t are space-time operators.

Our working spaces will be defined through the time dependent M_0^t norm :

$$\mathcal{H}_t = \{f \in L^2(\mathbb{R}) |\, M_0^t(f) = \|f\|_{H^1} + \|DJ^t f\|_{L^2} + \|D^\alpha J^t f\|_{L^2} < \infty\}.$$

J^t only appears in the norm, as it is convenient to do linear estimates (see [6], Lemma 2.3). But we introduced I^t because it is easier to handle when doing energy methods estimates. Notice that M_0^0 is very similar to $\| \cdot \|_{H^{1,1}}$.

We will finally use the following notation for weighted spaces : for a positive function h,

$$\|f\|_{H^s(h)}^2 = \int |(Id - \Delta)^{s/2} f|^2(x) h(x) dx.$$

Following a usual convention, different positive constants might be denoted by the same letter C.

Our main result is the following.

Theorem 1 (Nonlinear wave operator). *Let $V \in H^{5,1} \cap H^{2,2}$ be such that :*

$$x_+^{4/3} D^5 V \in L^2, \qquad x_+^8 V \in H^1,$$

(where $x_+ = \max\{0, x\}$). Let $N \in \mathbb{N}$, $0 < c_1 < \ldots < c_N$ and $x_1, \ldots, x_N \in \mathbb{R}$. Denote $R_j(t, x) = Q_{c_j}(x - x_j - c_j t)$ N solitons.

Then there exists $u^* \in C([T_0, +\infty), H^4 \cap \mathcal{H}_0^t)$, for some $T_0 \in \mathbb{R}$, solution to (1), such that if we introduce :

$$w^*(t) = u^*(t) - U(t)V - \sum_{j=1}^N R_j(t),$$

we have

$$\|w^*(t)\|_{H^4} + M_0^t(w^*(t)) \to 0 \quad \text{as} \quad t \to \infty.$$

Furthermore, we have the following decay rate :

$$\|w^*(t)\|_{H^4} \leq C t^{-1/3}, \qquad M_0^t(w^*(t)) \leq C t^{-\delta}.$$

***Remark* 1.** This result allows to work with large data (V large in L^2), which is both surprising and satisfactory. However, it deals with smooth and decaying data. A natural setting would be a result with $V \in H^1$, and some decay on the right to ensure low interaction with the solitons. Theorem 1 should be viewed as a step in the solving process of Problem 2.

An important change in the method of proof when considering [12] is the following. Solitons have an exponential decay, and so integrability (in time) is always automatic. Here the linear term $U(t)V$ will interfere with the solitons to produce a polynomial decay in time, and this will require taking care of.

Similarly, when handling the linear term $U(t)V$ (following the framework of [4]), we will have to take care of the interference of the solitons.

***Remark* 2.** This result is similar to [3], where a non-linear wave operator is constructed in the L^2 critical case $p = 5$.

In both cases, the scheme of proof first dwells on the interaction with the solitons, and in a second step uses arguments from the linear scattering theory to control the interaction with the linear term (along with the results obtained in the first step). The argument for the soliton interaction is very similar in the case $p = 4$ and in the case $p = 5$. However, the second step is very different.

For $p = 5$, the linear scattering theory of Kenig, Ponce and Vega [8] is available : it is done in L^2, and so requires much less smoothness and decay on V. The main difficulty is to mix both approaches, as the soliton theory relies on an analysis in $C^0_t H^1_x$, and the natural space in the theory of [8] is $L^5_x L^{10}_t$: in particular, solitons do not belong to this space (nor to $L^5_x L^{10}_{t \geq T}$ for any T). The problem is then to separate the linear analysis from the non-linear one, and when considering the interference of one over the other, to be able to interchange integrals in time and in space in an adequate way. This can be done with a small loss in the decay, with respect to the optimal result one can expect using this method.

In the non-critical case, the scattering analysis of [8] is no longer available, and we have to relie on the theory of Naumkin and Hayashi [6]. Their method break down at some point, when taking care of the interference between the solitons and the linear term. However, we manage to recover the leap by energy method arguments, and this is why we have to reinforce the assumptions on V, and obtain a stronger convergence (H^4). Our method could be adapted also to the critical case, but would give a much less sharp result than what is obtained in [3].

The problem of the uniqueness of solutions behaving as the sum of a linear term and N soliton is an open question, in both the critical and sub-critical case. Remind that if $V = 0$, one has uniqueness in H^1 (see [12]) : this result is linked with very fast convergence of the constructed solution to its profile not only in H^1 but in H^4. However, it seems that one can not derive easily from this work a proof for $V \neq 0$.

***Remark* 3.** Theorem 1 is valid only for $p = 4$ for two main reasons. First, it contains the existence of a scattering operator, so that $p > 3$. Second, it also contains the existence of a N-soliton, which is only true for $p \leq 5$. The fact that our setting only deals with integer p comes from our crucial use of the regularity of the non-linearity function $x \mapsto x^p$ and also from better integrability properties (if $p \geq 4$ instead of $p > 3$).

However, one can prove an analoguous result for $p = 5$, but that one would be much less precise than we is stated in [3].

***Remark* 4.** There are some analogous results for the (critical) non-linear Schrödinger equation. See Bourgain and Wang [1], Krieger and Schlag [10], Merle [19]. In [1], a solution to the critical NLS equation with a given blow-up behaviour is constructed : due to the conformal transform, this is in fact equivalent to construct a solution to the critical NLS equation which behaves like the sum of a soliton and a linear term. High smoothness and low interaction with the soliton are required on the linear term.

In Section 2, we give a detailed outline of the proof of Theorem 1, decomposing it into steps : each of these step is summarized in a proposition. In Section 3, we give some preliminary results and each of the following sections is devoted to the proof of one of the propositions stated in Section 2.

Acknowledgment

I am grateful to my advisor Frank Merle for numerous discussions and his constant encouragements.

2 Outline of the proof

Let $V \in H^{5,1} \cap H^{2,2}$ such that $x_+^8 D^5 V \in L^2$ and $x_+^{4/3} V \in H^1$. Let $0 < c_1 < \ldots < c_N$ and $x_1, \ldots, x_N \in \mathbb{R}$. Denote the soliton with speed c_j and shift x_j :

$$R_j(t,x) = Q_{c_j}(x - x_j - c_j t).$$

Define also $R(t) = \sum_{j=1}^N R_j(t)$.

Let S_n be an increasing sequence of time, so that $S_n \to \infty$ as $n \to \infty$. For $n > 0$, we define $u_n(t)$, the solution to

$$\left\{ \begin{array}{l} u_{n\,t} + (u_{n\,xx} + u_n^4)_x = 0, \\ u_n(S_n) = U(S_n) + R(S_n). \end{array} \right. \tag{10}$$

Equivalently, we introduce the error term

$$w_n(t) = u_n(t) - U(t)V - R(t),$$

so that $w_n(t)$ satisfies the equation

$$\left\{ \begin{array}{l} w_{n\,t} + w_{n\,xxx} + \left(u_n^4 - \sum_{j=1}^N R_j^4 \right)_x = 0, \\ w_n(S_n) = 0. \end{array} \right. \tag{11}$$

As $u_n(S_n) \in H^1$, $u_n n \in C_b(\mathbb{R}, \mathcal{H}^1)$; the same thing is true for $w_n(t)$.

The heart of the proof of Theorem 1 is the following result.

Proposition 1 (Uniform estimates). *There exists T_0 such that for all n such that $S_n \geq T_0$, the solution $u_n(t)$ to (10) and the solution $w_n(t)$ to (11) belong to $C([T_0, S_n], \mathcal{H}_0^t \cap H^4)$. Furthermore, we have*

$$\forall t \in [T_0, S_n], \quad \|w_n(t)\|_{H^4} \leq C_0 t^{-1/3}, \quad M_0^t(w_n(t)) \leq C_0 t^{-\delta}, \tag{12}$$

for some constant C_0 not depending on n (recall $\delta > 0$ is introduced in (9)).

The proof of this proposition requires several steps.

The first remark allows us to further assume smallness on $w_n(t)$, in order to get the decay (12).

Proposition 1' (Reduction of proof). *There exist $\varepsilon_0 > 0$, C_0, and $T_0 \geq 1$ with $2C_0T_0^{-\delta} \leq \varepsilon_0$ such that the following is true, for all $n \in \mathbb{N}$. Suppose that there exists $I_n \in [T_0, S_n]$ such that*

$$\forall t \in [I_n, S_n], \quad \|w_n(t)\|_{H^4} + M_0^t(w_n(t)) \leq \varepsilon_0.$$

Then in fact

$$\forall t \in [I_n, S_n], \quad \|w_n(t)\|_{H^4} \leq C_0 t^{-1/3}, \quad M_0^t(w_n(t)) \leq C_0 t^{-\delta}.$$

Proof of Proposition 1 assuming Proposition 1'. Let $T_0 = \max\{1, C_0^{1/\delta}\varepsilon_0\}$, and define

$$I_n^* = \inf_{t^* \in [1,S_n]} \{t^* | \ \forall t \in [t^*, S_n], \quad \|w_n(t)\|_{H^4} + M_0^t(w_n(t)) \leq \varepsilon_0\}.$$

As $w_n(S_n = 0)$, by upper semi-continuity of the norm of the flow (see [4, Appendix B]), we obtain that the set on which we do the infimum is non-empty, so that $I_n^* < S_n$.

Then of course, for all $t \in (I_n^*, S_n]$, $\|w_n(t)\|_{H^4} + M_0^t(w_n(t)) \leq \varepsilon_0$. This allows us to apply Proposition 1' so that

$$\forall t \in (I_n^*, S_n], \qquad \|w_n(t)\|_{H^4} \leq C_0 t^{-1/3}, \quad M_0^t(w_n(t)) \leq C_0 t^{-\delta}. \tag{13}$$

If $I_n^* > 1$, we also get that $\limsup_{t \downarrow I_n^*} \|w_n(t)\|_{H^4} + M_0^t(w_n(t)) \geq \varepsilon_0$ (from the minimality of I_n^*). In particular, this gives

$$\varepsilon_0 \leq \limsup_{t \downarrow I_n^*} \|w_n(t)\|_{H^4} + M_0^t(w_n(t)) \leq 2C_0 {I_n^*}^{-\delta}.$$

So that $I_n^* \leq \varepsilon_0/(2C_0))^{-1/\delta}$. In any case, we get that $I_n^* \leq T_0$: (13) allows us to conclude. □

Thus, our goal is now to prove Proposition 1'.

Proof of Proposition 1'.
Step 1 : Monotonicity and non-linear tools. We obtain H^1 estimates on the right. Let us intoduce the cut-off speed

$$\sigma_0 \in (0, \min\{c_1, c_2 - c_1, \dots, c_N - c_{N-1}\}), \tag{14}$$

to be determined in the proof of the following Proposition 2, and the cut-off function

$$\psi(x) = \frac{2}{\pi} \arctan\left(\exp\left(-\frac{\sqrt{\sigma_0}}{2}x\right)\right), \qquad \psi_0(t,x) = \psi(x - \sigma_0 t - 2|x_1|). \tag{15}$$

$\psi_0(t)$ allows us to separate the solitons interaction from the $U(t)V$ interaction.

Proposition 2 (Interaction with the solitons). *There exist $\sigma_1 > 0$, $\varepsilon_1 > 0$, C_1 and T_1 such that the following is true. If $\sigma_0 \leq \sigma_1$, $\varepsilon_0 \leq \varepsilon_1$ and $T_0 \geq T_1$, then, for all $n \in \mathbb{N}$ and all $t \in [I_n, S_n]$,*

$$\begin{aligned}\|w\|_{H^1(1-\psi_0(t))} &\leq C_1 e^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t} + C_1\|U(t)V\|_{H^1(1-\psi_0(t))}\\ &+ C_1(S_n - t + 1)\|U(t)V\|_{L^2(1-\psi_0(S_n))} + C_1\int_t^{S_n} \|U(t)V\|_{H^1(1-\psi_0(t))}dt.\end{aligned}$$

Observe that this proposition in fact holds for all $p \in [2,5]$; however, we will only do it for $p = 4$.

Essentially we obtain a polynomial decay on $\|w_n(t)\|_{H^1(1-\psi_0(t))}$ (instead of an exponential decay in the case of solely soliton). However the good point is that we can choose this polynomial decay to be as fast as we want by lowering the interaction of $U(t)V$ with the solitons, that is, by requiring sufficicient decay on the right for V.

Now we would like to complete the M_0^t estimate. But it happens that the construction of [6] relies on a very nice cancelation involving the operators J^t and I^t, which allows a bootstrap in $\mathcal{H}_0^t$. Here, this nice clockwork breaks down because of the interaction with the solitons R_j (the precise term that arise will be treated in full detail in the proof of the final step 4). We therefore are forced to work in H^3 which is the more natural space where all the computations of [6] are done (of course in H^3, the bootstrap of [6] doesn't work anymore because of a lack of information).

We need a good control on the interaction with the soliton at the H^3 level : more precisely (this will be done in full detail in subsection, we need $t\|w_n\|_{H^3(1-\psi_0(t))}$ be integrable in time. This can not be achieved by improving Proposition 2 to H^3, as its proof is done through considerations at H^1 level. This is why we go up to H^4 : with a weak control on $\|w_n\|_{H^4}$, and a strong control on $\|w_n\|_{H^1(1-\psi_0(t))}$, we obtain by interpolation the desired control on $\|w_n\|_{H^3(1-\psi_0(t))}$. Indeed, we have the following corollary to Proposition 2, in which we estimate some quantities which we will need later on.

Corollary 1. *Suppose $V \in H^{5,1} \cap H^{2,2}$ is such that*

$$x_+^{4/3}D^5V \in L^2, \quad \text{and} \quad x_+^8 V \in H^1.$$

Then for some $C_1' > 0$, we have, for all $n \in \mathbb{N}$ and for all $t \in [I_n, S_n]$,

$$\begin{aligned}t\|w_n(t)\|_{H^3(1-\psi_0(t))} &+ t\|U(t)V\|_{H^2(1-\psi_0(t))}\\ &+ \|U(t)V\|_{H^5(1-\psi_0(t))} + \|U(t)(xV_x)\|_{H^1(1-\psi_0(t))} \leq \frac{C_1'}{t^{4/3}}.\end{aligned}$$

Proof. We combine the result of Proposition 2 and Lemma 3. First observe that from Lemma 3, our assumptions translate to

$$\|D^5U(t)V\|_{L^2(1-\psi_0(t))} \leq Ct^{-4/3}, \tag{16}$$

$$\|U(t)V\|_{L^2(1-\psi_0(t))} + \|U(t)V_x\|_{L^2(1-\psi_0(t))} \leq Ct^{-8}. \tag{17}$$

So that by interpolation of (16) and (17),

$$\|U(t)V\|_{H^5(1-\psi_0(t))} \leq Ct^{-4/3}.$$

Again by interpolation, we get

$$\begin{aligned}\|U(t)V\|_{H^2(1-\psi_0(t))} &\leq \|U(t)V\|_{H^1(1-\psi_0(t))}^{3/4}\|U(t)V\|_{H^5(1-\psi_0(t))}^{1/4}\\ &\leq \frac{C}{t^{\frac{3}{4}\cdot 8}}\cdot\frac{C}{t^{\frac{1}{4}\cdot\frac{4}{3}}} \leq \frac{C}{t^{19/3}} \leq \frac{C}{t^{7/3}}.\end{aligned}$$

Now, by Proposition 2 and (17), we get

$$\|w_n(t)\|_{H^1(1-\psi_0(t))} \leq \frac{C}{t^7}.$$

Now recall that $\|w_n(t)\|_{H^4} \leq \varepsilon_0$, so that by interpolation

$$\begin{aligned}\|w_n(t)\|_{H^3(1-\psi_0(t))} &\leq C\|w_n(t)\|_{H^1(1-\psi_0(t))}^{1/3}\|w_n(t)\|_{H^4(1-\psi_0(t))}^{2/3}\\ &\leq \frac{C}{t^{7/3}}\|w_n(t)\|_{H^4}^{2/3} \leq \frac{C}{t^{7/3}}.\end{aligned}$$

For the xV_x estimate : first notice that

$$\begin{aligned}\int V_{xx}^2(x)x_+^{14/3}dx &= \int_0^\infty V_{xx}^2 x^{14/3}dx\\ &= -\int_0^\infty V_{xxx}V_x x^{14/3}dx - \int_0^\infty V_{xx}V_x x^{11/3}dx\\ &\leq \left(\int_0^\infty V_{xxx}^2 x^{8/3}dx\int_0^\infty V_x^2 x^{20/3}dx\right)^{1/2}\\ &\qquad + \left(\int_0^\infty V_{xx}^2 x^{8/3}dx\int_0^\infty V_x^2 x^{14/3}dx\right)^{1/2}\\ &\leq \|x_+^{4/3}V_{xxx}\|_{L^2}\|x_+^{10/3}V_x\|_{L^2} + \|x_+^{4/3}V_{xx}\|_{L^2}\|x_+^{7/3}V_x\|_{L^2}.\end{aligned}$$

As $V \in H^{2,2}$, $xV_x \in H^1$, and moreover,

$$\int \left((xV_x)^2 + |D(xV_x)|^2\right)x_+^{8/3}dx \leq \int \left(V_x^2 + V_{xx}^2\right)(1+x_+^{14/3})dx,$$

so that

$$\|(1+x_+^{7/3})(xV_x)\|_{H^1}^2 \leq \|(1+x_+^{10/3})V\|_{H^1}\|(1+x_+^{4/3})V\|_{H^3}.$$

From our H^5 estimate and $(1+x_+^8)V \in H^1$, we get

$$\|U(t)(xV_x)\|_{H^1(1-\psi_0(t))} \leq Ct^{-7/3}\|(1+x_+^{7/3})(xV_x)\|_{H^1} \leq Ct^{-7/3}. \qquad \square$$

Step 2 : Energy method estimates. Now that we have assumed H^4 control, we have to obtain H^4 uniform decay.

Proposition 3 (Interaction with the linear term, H^4 bounds). *There exists C_2 such that $\forall n \in \mathbb{N}$, $\forall t \in [I_n, S_n]$,*

$$\|w_n(t)\|_{H^4} \leq \frac{C_2}{t^{1/3}}.$$

First consider L^2 and H^1 estimates. We want to control what happens in the zone $x < \sigma_0 t$, that is the interaction with the linear term $U(t)V$: we follow the framework of [4]. The crucial point is to use our a priori control on $M_0^t(w(t))$. We have

$$|w_n(t,x)| \le \frac{C}{t^{1/3}} \left(1 + \frac{|x|}{\sqrt[3]{t}}\right)^{-1/4} M_0^t(w_n(t)),$$
$$|w_{nx}(t,x)| \le \frac{C}{t^{2/3}} \left(1 + \frac{|x|}{\sqrt[3]{t}}\right)^{1/4} M_0^t(w_n(t)).$$

These, along with Proposition 2 allow to obtain the H^1 decay estimate, in a very similar way to [4].

For the higher order estimates, i.e. H^2, H^3 and H^4, the pointwise control that we have on w_n and w_{nx} is not enough. If we wanted to improve our control to $M_0^t(w_{nx})$, we would always face the same problem for the higher order derivatives. The path that we will follow to avoid this is to use almost conservation quantities at level H^2 etc. For example, let u be a solution to (1), then

$$\frac{d}{dt}\left(\int u_{xx}^2 - \frac{20}{3}\int u_x^2 u^3\right) = 2\int u_x^5 + 80\int u_x^3 u^5.$$

Three elements are to be noticed. First, there is a corrective term $\int u_x^2 u^3$ to prevent the apparition of $\int u_{xx}^2 u_x u^2$, which we could not control, as noted in [12]. Second, $\int u_x^3 u^5$ comes from the corrective term, and will never be harmful, as it has a better integrability than the others (power 8 instead of 5). Third, $\int u_x^5$ has a more than quadratic term in u_x (when u_x appear less than twice, we can use directly our control on $\|u_x\|_{L^2}$ already obtained). To control this kind of terms, we use the Gagliardo-Nirenberg inequality :

$$\forall q \ge 2,\ \forall v \in H^1, \quad \|v\|_{L^q}^q \le C(q)\|v\|_{L^2}^{\frac{q+2}{2}} \|v_x\|_{L^2}^{\frac{q-2}{2}}. \tag{5}$$

As the maximal exponent on the term with highest derivatives is 5 or less, exponent on $\|v_x\|_{L^2}$ will always be less than 2, which means that we will always be in the position to apply Lemma 4. Assume for now that, when estimating the derivative in time of the H^{s+1} norm (squared) of $w_n(t)$, all terms have appropriate control except for ($\beta \in [0,3]$)

$$\int |D^s w_n|^{2+\beta} |D^{s-1} w_n|^{3-\beta}.$$

Further assume that all previous estimates gave a decay $\|w_n\|_{H^s} \le Ct^{-1/3}$. Thus, as our term has power 5, from (5) we would get a control :

$$\left|\frac{d}{dt}\|w_n\|_{H^{s+1}}^2\right| \le \|w_n\|_{H^s}^{5-\beta} \|w_n\|_{H^{s+1}}^{\beta} \le \frac{\|w_n\|_{H^{s+1}}^{\beta}}{t^{(5-\beta)/3}}.$$

With $\mu = \beta/2$, $\lambda = (5-\beta)/3$, Lemma 4 gives the decay $\|w\|_{H^{s+1}} \le Ct^{-\nu}$, with

$$\nu = \frac{1}{2}\frac{(5-\beta)/3 - 1}{1 - \beta/2} = \frac{1}{3}.$$

This means that the rate of decay $t^{-1/3}$ is likely to propagate as the regularity index s increases (in fact, for $p \geq 4$, similar computations show that the rate of decay $t^{-(p-3)/3}$ propagates). p integer is interesting regarding the regularity of the non-linearity function : to obtain the H^2 formula quoted, we already need a C^4 regularity, which translates to $p \geq 4$. In any case, our assumption $p = 4$ is now crucial. Of course we will need the estimate of Corollary 1 to handle some interaction terms.

Observe finally that this decay rate of $t^{-1/3}$ is the best one can expect, due to the slow decay of the linear term $U(t)V$.

Step 3 : Linear tools from scattering theory. We can now complete the decay estimate, by controling the remaining of the M_0^t norm.

Proposition 4 (Interaction with the linear term, M_0^t bound). *There exists C_3 such that $\forall n \in \mathbb{N}$, $t \in [I_n, S_n]$*

$$M_0^t(w_n(t)) \leq \frac{C_3}{t^\delta}.$$

Remind that $M_0^t(w_n(t)) = \|w_n(t)\|_{H^1} + \|D^\alpha J^t w_n(t)\|_{L^2} + \|DJ^t w_n(t)\|_{L^2}$. $\|w_n(t)\|_{H^1}$ has already been estimated, so we only need to focus on the last two terms. We follow the framework of [6] and [4]. First, we estimate $\|D^\alpha I^t w_n(t)\|_{L^2}$ and $\|I^t w_{nx}(t)\|_{L^2}$. For this, we use the usual $\frac{1}{2}\frac{d}{dt}\|f\|_{L^2}^2 = (Lf, f)$, and plug in Lf the equation satisfied by f : here $f = D^\alpha I^t w_n(t)$ or $DI^t w_n(t)$.

When doing the computations on $(LI^t w_{nx}(t), I^t w_{nx}(t))$, we encounter a term of the type

$$\int (I^t w_{nx}(t))^2 R^2. \tag{18}$$

This is localized term in space, but in H^3 regularity instead of H^1 regularity. This fact explains that we needed to get decay for higher regularity norms than just H^1. Ideally, we would try to obtain directly H^3 on the right decay. However, this seems not to be possible. One easy way is to obtain low decay rate for the global space norms H^s, which we did up to H^4. Corollary 1 allows us to bound this troublesome term (18).

This explains how to obtain

$$\|D^\alpha I^t w_n(t)\|_{L^2} + \|I^t w_{nx}(t)\|_{L^2} \leq Ct^{-\delta}.$$

It remains to go back to J^t, which is done in a similar way as in [6] and [4], and does not raise more difficulties than those treated earlier.

This concludes the proof of Proposition 1', and thus of Proposition 1. $\square$

We can now conclude :

Proof of Theorem 1.
Step 1 : A compactness result. From Proposition 1, we dispose of a sequence $u_n(t)$ defined on $[T_0, S_n]$, solution to (1), such that

$$u(S_n) = U(S_n)V + \sum_{j=1}^{N} R_j(S_n) = U(S_n) + R(S_n),$$

and that the uniform estimates hold $(w_n(t) = u_n(t) - U(t)V - R(t))$:

$$\exists T_0 \geq 1, \exists C_0 > 0, \quad \forall n \in \mathbb{N},\ \forall t \in [T_0, S_n], \qquad \|w_n(t)\|_{H^4} + M_0^t(w_n(t)) \leq \frac{C_0}{t^\delta}.$$

Let us prove the following compactness result on the sequence $u_n(T_0)$.

Claim. We have

$$\lim_{A\to\infty} \sup_{n\in\mathbb{N}} \int_{|x|\geq A} u_n^2(T_0, x)dx = 0.$$

Proof. Indeed, let $\varepsilon > 0$, and $T(\varepsilon)$ such that $C_0 T(\varepsilon)^{-\delta} \leq \sqrt{\varepsilon}$. Then

$$\int (u_n(T(\varepsilon)) - U(T(\varepsilon))V - R(T(\varepsilon))^2 \leq \varepsilon.$$

Let $A(\varepsilon)$ be such that $\int_{|x|\geq A(\varepsilon)} (U(T(\varepsilon))V + R(T(\varepsilon)))^2(x)dx \leq \varepsilon$; we get

$$\int_{|x|\geq A(\varepsilon)} u_n^2(T(\varepsilon), x)dx \leq 2\varepsilon.$$

Let $g \in C^3$ a function such that $g(x) = 0$ if $x \leq 0$, $g(x) = 1$ if $x \geq 2$, and furthermore $0 \leq g'(x) \leq 1$, $0 \leq g'''(x) \leq 1$.

Remind that if $f \in C^3$ does only depend on x, we have

$$\frac{d}{dt}\int u_n^2 f = -3\int {u_n}_x^2 f_x + \int u_n^2 f_{xxx} + \frac{8}{5}\int u_n^5 f_x.$$

(See Lemma 7 and its proof). For $C(\varepsilon)$ to be determined later, we then have :

$$\begin{aligned} \frac{d}{dt}\int u_n^2(t,x) g\left(\frac{x - A(\varepsilon)}{C(\varepsilon)}\right) &= -\frac{3}{C(\varepsilon)}\int {u_n}_x^2 g'\left(\frac{x - A(\varepsilon)}{C(\varepsilon)}\right) \\ &+ \frac{1}{C(\varepsilon)^3}\int u_n^2 g'''\left(\frac{x - A(\varepsilon)}{C(\varepsilon)}\right) + \frac{8}{5C(\varepsilon)}\int u_n^5 g'\left(\frac{x - A(\varepsilon)}{C(\varepsilon)}\right). \end{aligned}$$

As $t \geq T_0 \geq 1$, u_n satisfies $\|u_n(t)\|_{H^1} \leq C_0 + \|V\|_{H^1} + \sum_{j=1}^N \|Q_{c_j}\|_{H^1} \leq C^0$. So that :

$$\begin{aligned} &\left|\frac{d}{dt}\int u_n^2(t,x) g\left(\frac{x - A(\varepsilon)}{C(\varepsilon)}\right)\right| \\ &\qquad \leq \frac{1}{C(\varepsilon)}\left(3\int {u_n}_x^2(t) + \int u_n^2(t) + \frac{8}{5}\|u_n\|_{L^\infty}^3 \int u_n^2(t)\right) \\ &\qquad \leq \frac{1}{C(\varepsilon)}\left(3{C^0}^2 + \frac{8}{5}2^{3/2}{C^0}^5\right). \end{aligned}$$

Now choose $C(\varepsilon) = \max\left\{1, \frac{T(\varepsilon)-T_0}{\varepsilon}\left(3{C^0}^2 + \frac{8}{5}2^{3/2}{C^0}^5\right)\right\}$, from which we derive

$$\left|\frac{d}{dt}\int u_n^2(t,x) g\left(\frac{x - A(\varepsilon)}{C(\varepsilon)}\right)\right| \leq \frac{\varepsilon}{T(\varepsilon) - T_0}.$$

And after integration in time between T_0 and $T(\varepsilon)$:

$$\int_{x\geq 2C(\varepsilon)+A(\varepsilon)} u_n^2(T_0, x) \leq \int u_n^2(T_0, x) g\left(\frac{x - A(\varepsilon)}{C(\varepsilon)}\right) \leq 3\varepsilon.$$

Now considering $\frac{d}{dt}\int u_n^2(t,x)g\left(\frac{-A(\varepsilon)-x}{C(\varepsilon)}\right)$, we get in a similar way

$$\int_{x\leq -2C(\varepsilon)-A(\varepsilon)} u_n^2(T_0,x)\leq 3\varepsilon.$$

So that if we denote $A_\varepsilon = 2C(\varepsilon/6)+A(\varepsilon/6)$, we obtain :

$$\forall n\in\mathbb{N},\qquad \int_{|x|\geq A_\varepsilon} u_n^2(T_0,x)\leq\varepsilon,$$

as claimed. □

Step 2 : Construction of u^*. $u_n(T_0)$ is a bounded sequence in $H^4\cap\mathcal{H}_0^{T_0}$, so that it converges weakly to $\varphi_0\in H^4(\mathbb{R})\cap\mathcal{H}_0^{T_0}(\mathbb{R})$ (up to a subsequence). The previous compactness result ensures that the convergence is strong in $L^2(\mathbb{R})$. Indeed, let $\varepsilon>0$, and A such that $\int_{|x|\geq A}\varphi_0^2(x)dx\leq\varepsilon$ and

$$\forall n\in\mathbb{N},\qquad \int_{|x|\geq A} u_n^2(T_0,x)\leq\varepsilon.$$

The injection $H^1([-A,A])\hookrightarrow L^2([-A,A])$ is compact, so that $\int_{|x|\leq A}|u_n(T_0,x)-\varphi_0(x)|^2dx\to 0$. We thus derive that

$$\limsup_{n\in\mathbb{N}}\|u_n(T_0)-\varphi_0\|^2_{L^2(\mathbb{R})}\leq 4\varepsilon.$$

As this is true for all $\varepsilon>0$, $u_n(T_0)\to\varphi_0$ in $L^2(\mathbb{R})$. By interpolation, $u_n(T_0)$ converges strongly to φ_0 in H^3. Denote $u^*(t)$ the solution to

$$\begin{cases} u_t^*+(u_{xx}^*+u^{*4})_x=0,\\ u^*(T_0)=\varphi_0.\end{cases}$$

The Cauchy problem being globally well-posed in H^1, u^* is well defined. Now the flow is continuous in H^3, so that for all $t\in\mathbb{R}$, $u_n(t)\to u^*(t)$ in H^3, and we can pass to the limit in the H^3 estimates, to get

$$\forall t\in\mathbb{R},\qquad \|u^*(t)-U(t)V-R(t)\|_{H^3}\leq C_0t^{-1/3}.$$

Denote $w^*(t)=u^*(t)-U(t)V-R(t)$. $w_n(t)\to w^*(t)$ in H^1 so that $w^*(t)$ is the only possible weak limit of $w_n(t)$ in $H^4\cap\mathcal{H}_0^t$. In particular, the convergence is strong in H^3 and

$$\|w^*(t)\|_{H^4}\leq\liminf_{n\to\infty}\|w_n(t)\|_{H^4}\leq\frac{C_0}{t^{1/3}},\quad M_0^t(w^*(t))\leq\liminf_{n\to\infty}M_0^t(w_n(t))\leq\frac{C_0}{t^\delta}.$$

This completes the proof of Theorem 1. □

This scheme of proof is similar to that of [12], [4]. Steps 2, 3 and 4 of the proof of Proposition 1' remain to be completed.

In Section 3, we present some preliminary results. In Section 4, we prove Proposition 2. In Section 5, we prove Proposition 3. Finally, in Section 6, we prove Proposition 4. This completes the proof of Proposition 1', and thus, the proof of Theorem 1.

3 Preliminaries

3.1 Cut-off functions and notation for localized quantities

We already introduced $\sigma_0 \in (0, 1/2 \min\{c_1, c_2 - c_1, \dots, c_N - c_{N-1}\})$, and the cut off function

$$\psi(x) = \frac{2}{\pi} \arctan\left(\exp\left(-\frac{\sqrt{\sigma_0}}{2} x\right)\right). \tag{15}$$

We can check that $\lim_{+\infty} \psi = 0$, $\lim_{-\infty} \psi = 1$, ψ is decreasing. Furthermore, by direct computations :

$$\psi'(x) = -\frac{\sqrt{\sigma_0}}{2\pi \cosh\left(\frac{\sqrt{\sigma_0}}{2} x\right)}, \quad \psi''' = \frac{\sigma_0}{4} \psi'(x) \left(1 - \frac{2}{\cosh\left(\frac{\sqrt{\sigma_0}}{2} x\right)}\right),$$

so that

$$|\psi'''(x)| \leq -\frac{\sigma_0}{4} \psi'(x). \tag{19}$$

We introduce, for $j = 1, \dots, N-1$,

$$m_j(t) = \frac{c_j + c_{j+1}}{2} t + \frac{x_j + x_{j+1}}{2}, \quad m_0(t) = \sigma_0 t - 2|x_1|.$$

So that we can define, for $j = 0, \dots, N-1$,

$$\psi_j(t, x) = \psi(x - m_j(t)), \quad \psi_N(t, x) = 1.$$

Then we set, for $j = 1, \dots, N-1$,

$$\phi_0(t) = \psi_0(t), \quad \phi_j(t) = \psi_j(t) - \psi_{j-1}(t), \quad \phi_N(t) = 1 - \psi_{N-1}(t).$$

By construction, $\sum_{k=1}^{j} \phi_k = \psi_j$. Finally, we define some local quantities related to mass and energy :

$$M_j(t) = \int u_t^2(t) \phi_j(t), \quad E_j(t) = \int \left(\frac{1}{2} u_x^2(t) - \frac{1}{5} u^5(t)\right) \phi_j(t),$$
$$F_j(t) = E_j(t) + \frac{1}{100} M_j(t).$$

3.2 $\mathcal{H}_0^t$ estimates

Remind our notations

$$\gamma \in \left(0, \frac{1}{3}\right), \quad \alpha = \frac{1}{2} - \gamma, \quad \delta = \frac{1 - 2\gamma}{3} > 0, \tag{9}$$

the operator $J^t f = xf - 3t\partial_x^2 f = U(t) x U(-t) f$, and our working norm

$$M_0^t(f) = \|f\|_{H^1} + \|D^\alpha J^t f\|_{L^2} + \|\partial J^t f\|_{L^2}.$$

First a few remarks on M_0^t. Of course $M_0^0(f) \leq C\|f\|_{H^{1,1}}$. Second, note that $J^t U(t) V = U(t) x V$ (and $U(t)$ is a H^s isometry), so that if $V \in H^{1,1}$, we have the uniform control in t :

$$M_0^t(U(t)V) \leq C\|V\|_{H^{1,1}}. \tag{20}$$

We now remind the linear results obtained in [6] (Lemma 2.2), in a slightly improved form.

Lemma 1 ([6]). *Let $t > 0$ and f be a function so that $M_0^t(f)$ is bounded. Then for $r > 4$,*

$$\|f\|_{L^r} \le \frac{C}{(1+t)^{1/3-1/(3r)}} M_0^t(f).$$

And one also has the point wise inequalities

$$|f(x)| \le \frac{CM_0^t(f)}{(1+t)^{1/3}} \left(1 + \left|\frac{x}{t^{1/3}}\right|\right)^{-\frac{1}{4}}, \quad |f_x(x)| \le \frac{CM_0^t(f)}{t^{2/3}} \left(1 + \left|\frac{x}{t^{1/3}}\right|\right)^{\frac{1}{4}}.$$

As a simple consequence, for $V \in H^{1,1}$, we have similar decay estimates on $U(t)V$.

Proof. See [6], Lemma 2.2 and its proof (especially inequalities (2.16), (2.17) and (2.18)). The proof of refinement can be found in [4], Appendix A. □

We will also need the polarized version of Lemma 2.3 of [6] (in the case $p = 4$) :

Lemma 2. *Let $p \ge 3$ and $g, h : \mathbb{R} \to \mathbb{R}$. Then the following inequalities are hold if their right-hand side is bounded :*

$$\|D^\alpha g^p\|_{L^2} \le C\|g^{p-1}\|_{L^2}(\|gg_x\|_{L^\infty}^{1/2} + \|g\|_{L^\infty}^{3\gamma}\|gg_x\|_{L^\infty}^{(1-3\gamma)/2}),$$
$$\|D^\alpha |g|^{p-1}h_x\|_{L^2} \le C(\|D^\alpha h\|_{L^2} + \|h_x\|_{L^2})(\|g\|_{L^\infty}^{p-3}\|gg_x\|_{L^\infty}$$
$$+ \|g\|_{L^\infty}^{p-3-2\gamma}\|g\|_{L^2}^{2\gamma}\|gg_x\|_{L^\infty} + \|g\|_{L^\infty}^{p-3+2\gamma}\|gg_x\|_{L^\infty}^{1-\gamma}).$$

Proof. See [6], Lemma 2.3 and its proof (case $\sigma = 0$). □

3.3 Estimates of $U(t)V$ on the right

Recall our definition of $\psi_0(t)$ (15), given $\sigma_0 > 0$. We will often need estimates of the type $\|U(t)V\|_{H^1(1-\psi_0(t))}$, as it is a measure of the interaction between the linear term $U(t)V$ and the solitons.

Let us denote $x_+ = \max\{x, 0\}$.

Lemma 3 ($U(t)V$ estimates on the right). *Let $f \in L^2$, then*

$$\|U(t)f\|_{L^2(1-\psi_0(t))} \le \|f\|_{L^2(1-\psi_0(t/2))} \to 0 \quad \text{as} \quad t \to \infty. \tag{21}$$

Assume in addition that $(1 + x_+^q)f(x) \in L^2$, for some $q > 0$. Then there exists a constant $C = C(\sigma_0, x_1)$ independent of f such that

$$\forall t \ge 1, \quad \|U(t)f\|_{L^2(1-\psi_0(t))} \le \frac{C}{t^q}\|(1 + x_+^q)f(x)\|_{L^2}. \tag{22}$$

We will apply this result to V and its derivatives (see Corollary 1).

Proof. The key remark is that $U(t)$ "pushes" the L^2-mass on the left. We compute :

$$\frac{d}{d\tau}\int |U(2\tau - t)f|^2\psi_0(\tau)$$
$$= 2\int (U(2\tau - t)f)_\tau U(2\tau - t)f\psi_0(\tau) + \int |U(2\tau - t)f|^2\psi_{0\tau}(\tau)$$

$$
\begin{aligned}
&= -4\int U(2\tau-t)f_{xxx}U(2\tau-t)f\psi_0(\tau) + \int |U(2\tau-t)f|^2 \psi_{0\tau}(\tau) \\
&= 4\int U(2\tau-t)f_{xx}U(2\tau-t)f_x\psi_0(\tau) + 4\int U(2\tau-t)f_{xx}U(2\tau-t)f\psi_{0x}(\tau) \\
&\qquad + \int |U(2\tau-t)f|^2\psi_{0\tau}(\tau) \\
&= -6\int |U(2\tau-t)f_x|^2\psi_{0x}(\tau) - 4\int U(2\tau-t)f_xU(2\tau-t)f\psi_{0xx}(\tau) \\
&\qquad\quad + \int |U(2\tau-t)f|^2\psi_{0\tau}(\tau) \\
&= -6\int |U(2\tau-t)f_x|^2\psi_{0x}(\tau) + \int |U(2\tau-t)f|^2(2\psi_{0xxx}(\tau)+\psi_{0\tau}(\tau)).
\end{aligned}
$$

As $\psi_{xxx} \le \frac{\sigma_0}{4}|\psi_x|$, $\psi_{0\tau} = -\sigma_0\psi_{0x}$, and $\psi_x < 0$, we have that,

$$
\psi_{0x}(\tau) < 0 \quad \text{and} \quad 2\psi_{0xxx}(\tau) + \psi_{0\tau}(\tau) \ge 0.
$$

So that $\tau \mapsto \int U(2\tau-t)f(x)^2\psi_0(\tau,x)dx$ is an increasing function of τ. In particular, when comparing for $\tau = t$ and $\tau = t/2$ ($t \ge 0$), we have :

$$
\forall t \ge 0, \qquad \int |U(t)f|^2\psi_0(t) \ge \int f^2\psi_0(t/2).
$$

As the flow $U(t)$ preserves the L^2-mass, we get

$$
\int |U(t)f|^2(x)(1-\psi_0(t,x))dx \le \int f^2(x)(1-\psi_0(t/2,x))dx. \tag{23}
$$

Suppose that for some $q > 0$, $(1+x_+^q)f(x) \in L^2$. Then for $t \ge 1$,

$$
\begin{aligned}
\int f^2(1-\psi_0(t/2)) &= \int_{x\le\sigma_0 t/4} f^2(1-\psi_0(t/2)) + \int_{x\ge\sigma_0 t/4} f^2(1-\psi_0(t/2)) \\
&\le \sup_{x\le\sigma_0 t/4}(1-\psi_0(t/2,x))\int f^2 + \left(\frac{\sigma_0 t}{4}\right)^{-2q}\int_{x\ge\sigma_0 t/4} x^{2q}f^2 \\
&\le C(x_0)e^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|f\|_{L^2}^2 + C(\sigma_0)t^{-2q}\|x_+^q f\|_{L^2}^2.
\end{aligned}
$$

And we get

$$
\forall t \ge 1, \quad \|U(t)f\|_{L^2(1-\psi_0(t))} \le \frac{C}{t^q}\|(1+x_+)^q f\|_{L^2},
$$

which is (22). □

3.4 An ODE lemma

Lemma 4 (Booster). *Let $\kappa > 0$, $\lambda > 1$, $\mu \in (0,1)$, and $f \in L^\mu([a,b])$ $(0 < a < b < +\infty)$ be a non-negative upper semi-continuous function satisfying*

$$
\forall t \in [a,b], \quad f(t) \le \frac{C}{t^\kappa} + C\int_t^b \frac{f^\mu(\tau)}{\tau^\lambda}d\tau,
$$

Define $\nu = \min\{\kappa, \frac{\lambda-1}{1-\mu}\}$. Then there exists $k = k(C,\kappa,\lambda,\mu)$ not depending on b such that

$$
\forall t \in [a,b], \quad f(t) \le \frac{kC}{t^\nu}.
$$

***Remark* 5.** Of course, if instead we have

$$f(t) \leq \frac{C}{t^\kappa} + \sum_{i=1}^{I} C_i \int_t^b \frac{f^{\mu_i}(\tau)}{\tau^{\lambda_i}} d\tau,$$

the final decay estimate is still valid, with $\nu = \min\{\kappa, (\frac{\lambda_i - 1}{1-\mu_i})_i\}$ being the least favorable exponent.

Proof. Let $k > 1$ to be determined later. Let us consider

$$T = \inf \left\{ \tau \geq a \middle| \forall t \in [\tau, b], \quad f(t) \leq \frac{kC}{t^\nu} \right\}.$$

Observe that T is in fact minimal for the property. As $b > 0$, $f(b) \leq \frac{C}{t^\nu} < \frac{kC}{t^\nu}$, so that by upper semi continuity, $T < b$. Then, if $t \in [T, b]$, we have $(t \geq a > 0)$

$$f(t) \leq \frac{C}{t^\nu} + \frac{C(kC)^\mu}{(\lambda - 1 + c\nu)t^{\lambda - 1 + \mu\nu}}.$$

If $\nu = \frac{\lambda-1}{1-\mu}$, $\lambda - 1 + \mu\nu = (\lambda - 1)\left(1 + \frac{\mu}{1-\mu}\right) = \frac{\lambda-1}{1-\mu} = \nu$. Else $\nu = \kappa$, $\frac{\lambda-1}{1-\mu} \geq \kappa = \nu$ so that $\lambda - 1 \geq (1-\mu)\nu$ and $\lambda - 1 + \mu\nu \geq \nu$. In any case, we get

$$f(t) \leq C \frac{1 + \frac{(kC)^\mu}{\lambda - 1 + \mu\nu}}{t^\nu}.$$

Let us now choose k such that $2\left(1 + \frac{(kC)^\mu}{\lambda-1+\mu\nu}\right) \leq k$, which is possible as $\mu < 1$ (notice that $k > 2$). We get finally $f(t) \leq \frac{kC}{2t^\nu}$. By a standard continuity argument, we deduce that $T = a$. $\square$

4 Estimates on the right : proof of Proposition 2

We follow the framework of [12]. The hypothesis we will use in this section is :

$$\forall t \in [I_n, S_n], \quad \|w_n(t)\|_{H^1} \leq \varepsilon_0.$$

4.1 Modulation close to asymptotic profile

Let us remind that $Q_c(x) = c^{\frac{1}{p-1}} Q(\sqrt{c}x)$.

Lemma 5 (Modulation of $w_n(t)$). *There exist T_2 and ε_2 such that if $I_n \geq T_2$ and $\varepsilon_0 \leq \varepsilon_2$, the following is true. For all $t \in [I_n, S_n]$, there exist $y_j(t)$ and $\gamma_j(t)$ such that if we denote*

$$\tilde{R}_j(t,x) = Q_{\gamma_j(t)}(x - y_j(t)), \qquad \tilde{R}(t,x) = \sum_{j=1}^{N} \tilde{R}_j(t,x),$$

$$\tilde{w_n}(t) = u_n(t,x) - U(t)V - \tilde{R}(t,x),$$

we have for all $j = 1, \dots, N$,

$$\int \tilde{w}_n(t,x) \tilde{R}_{j_x}(t,x) dx = 0 \quad \textit{and} \quad \int \tilde{w}_n(t,x)\tilde{R}_j(t,x) dx = 0.$$

Moreover, there exists C_1^2 such that :

$$\|\tilde{w}_n(t)\|_{H^1} + \sum_{j=1}^N |\gamma_j(t) - c_j| + \sum_{j=1}^N |y_j(t) - x_j - c_j t| \leq C_1^2 \varepsilon_0,$$

$$|y_j'(t) - c_j| + |\gamma_j'(t)| \leq C_1^2 e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C_1^2 \|U(t)V\|_{L^2(1-\psi_0(t))} + C_1^2 \left(\int \tilde{w}_n^2(t) e^{-\sqrt{\sigma_0}|x-c_j t|}\right)^{1/2}.$$

Proof. The construction of the modulated parameters (and the first estimate) essentially relies on the implicit function theorem by a standard arguement : we refer to [25] and [26].

Let us focus on the second estimate (local estimate). We begin by computing the equation satisfied by $\tilde{w}_n$. The equation satisfied by $\tilde{R}_k$ (using $-c_k R_{kx} + R_{kxxx} + R_k^4)_x = 0$) is

$$\begin{aligned}\tilde{R}_{kt} + \tilde{R}_{kxxx} &= (-y_k'(t) + c_k)\tilde{R}_{kx} + \frac{\gamma_k'(t)}{\gamma_k(t)}\left(\frac{\tilde{R}_k(t)}{3} + (x - y_k(t))\frac{\tilde{R}_{kx}(t)}{2}\right) \\ &\qquad - c_k \tilde{R}_{kx} + \tilde{R}_{kxxx} \\ &= (-y_k'(t) + c_k)\tilde{R}_{kx} + \frac{\gamma_k'(t)}{\gamma_k(t)}\left(\frac{\tilde{R}_k(t)}{3} + (x - y_k(t))\frac{\tilde{R}_{kx}(t)}{2}\right) - (\tilde{R}_k^4)_x.\end{aligned}$$

So that $\tilde{w}_n = u_n(t) - U(t)V - \tilde{R}(t)$ satisfies

$$\begin{aligned}\tilde{w}_{nt} + \tilde{w}_{nxxx} &= \sum_{k=1}^N (y_k'(t) - c_k)\tilde{R}_{kx} - \sum_{k=1}^N \frac{\gamma_k'}{\gamma_k}\left(\frac{\tilde{R}_k}{3} + (x - y_k(t))\frac{\tilde{R}_{kx}}{2}\right) \\ &\qquad - \left((\tilde{w}_n + U(t)V + \tilde{R})^4 - \sum_{k=1}^N \tilde{R}_k^4\right)_x. \qquad (24)\end{aligned}$$

Now, if we express $\tilde{R}_j$ in terms of R_j :

$$\tilde{R}_{j_{xt}} = -y_j'(t)\tilde{R}_{j_{xx}} + \frac{\gamma_j'(t)}{\gamma_j(t)}\left(\frac{\tilde{R}_{j_x}(t)}{3} + (x - y_j(t))\frac{\tilde{R}_{j_{xx}}(t)}{2} + \frac{\tilde{R}_{j_x}(t)}{2}\right).$$

And keeping in mind that $\frac{d}{dt}\int \tilde{w}_n \tilde{R}_{j_x} = \int \tilde{w}_n \tilde{R}_{j_x} = 0$, we get

$$\int \tilde{w}_{nt}\tilde{R}_{j_x} = -\int \tilde{w}_n \tilde{R}_{j_{xt}} = \int \tilde{w}_n \left(y_j'(t) - \frac{\gamma_j'(t)}{\gamma_j(t)}\frac{x - y_j(t)}{2}\right)\tilde{R}_{j_{xx}}.$$

We multiply (24) by $\tilde{R}_{j_x}$ and integrate in x, and do integration by parts :

$$\begin{aligned}(y_j'(t) - c_j)\int \tilde{R}_{j_x}^{\,2} &= -y_j'(t)\int \tilde{w}(t)\tilde{R}_{j_{xx}} + \frac{\gamma_j'(t)}{2\gamma_j(t)}\int \tilde{w}_n(t)(x - y_k(t))\tilde{R}_{j_{xx}} \\ &\qquad - \int \tilde{w}_n(t)\tilde{R}_{j_{xxxx}} - \sum_{k,k\neq j}(c_k - y_k'(t))\int \tilde{R}_{j_x}\tilde{R}_{kx}\end{aligned}$$

$$+\sum_{k=1}^{N}\frac{\gamma_k'}{\gamma_k}\int \tilde{R}_{j\,x}\left(\frac{\tilde{R}_k}{3}+(x-y_k(t))\frac{\tilde{R}_{k\,x}}{2}\right)$$
$$-\int\left((\tilde{w}_n+U(t)V+\tilde{R})^4-\sum_{k=1}^{N}\tilde{R}_k^4\right)\tilde{R}_{j\,xx}.$$

First consider the 3 first terms. Remind that for all $j=1,\dots,N$:

$$|\tilde{R}_j(t,x)|+|\tilde{R}_{j\,x}(t,x)|\le Ce^{-\sqrt{\sigma_0}|x-c_jt|}.$$

Furthermore, as $Q_{xx}=Q-Q^4$, we can express $\tilde{R}_{j\,xx}$ and $\tilde{R}_{j\,xxxx}$ in terms of powers of $\tilde{R}_j$. Hence, the integral part of these term is bounded by

$$\int|\tilde{w}_n(t)|(1+|x-c_jt|)e^{-\sqrt{\sigma_0}|x-c_jt|}\le C\left(\int|\tilde{w}_n(t)|^2e^{-\sqrt{\sigma_0}|x-c_jt|}\right)^{1/2}.$$

For the fourth term, $\int|\tilde{R}_{j\,x}R_{kx}|\le e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}$. This also apply to the fifth term for $k\neq j$, and for $k=j$:

$$\int\tilde{R}_{j\,x}\left(\frac{\tilde{R}_j}{3}+(x-y_j(t))\frac{\tilde{R}_{j\,x}}{2}\right)=0.$$

And for the non-linear last term, when developing, the large terms cancel one another, so that we can control the rest by

$$C\int(|\tilde{w}_n(t)|+|U(t)V|)e^{-\sqrt{\sigma_0}|x-c_jt|}.$$

Finally, we have altogether

$$\begin{aligned}|y_j'(t)-c_j|\le{}& C\left(1+\left|\frac{\gamma_j'(t)}{\gamma_j(t)}\right|\right)\left(\int|\tilde{w}_n(t)|^2e^{-\sqrt{\sigma_0}|x-c_jt|}\right)^{1/2}\\&+Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\sum_{k,k\neq j}|y_k'(t)-c_k|+Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\sum_{k,k\neq j}\left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right|\\&+Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}+C\|U(t)V\|_{L^2(1-\psi_0(t))}.\end{aligned}\tag{25}$$

Now, we have to do the same kind of argument on γ_j. Let us multiply (24) by $\tilde{R}_j$, using

$$\int\tilde{w}_{n\,t}(t)\tilde{R}_j=-\int\tilde{w}_n(t)\tilde{R}_{j\,t}(t)=-\frac{\gamma_j'(t)}{2\gamma_j(t)}\int(x-y_j(t))\tilde{w}_n\tilde{R}_{j\,x}.$$

We obtain (after an integration by parts $\int(x-y_j(t))\tilde{R}_j\tilde{R}_{j\,x}=-\frac{1}{2}\int\tilde{R}_j^2$) :

$$\begin{aligned}\frac{1}{12}\frac{\gamma_j'(t)}{\gamma_j(t)}\int\tilde{R}_j^2={}&\frac{\gamma_j'(t)}{2\gamma_j(t)}\int\tilde{w}_n(t)(x-y_k(t))\tilde{R}_{j\,x}-\int\tilde{w}_n(t)\tilde{R}_{j\,xxx}\\&-\sum_{k=1}^{N}(c_k-y_k'(t))\int\tilde{R}_j\tilde{R}_{kx}+\sum_{k\neq j}\frac{\gamma_k'}{\gamma_k}\int\tilde{R}_j\left(\frac{\tilde{R}_k}{3}+(x-y_k(t))\frac{\tilde{R}_{kx}}{2}\right)\end{aligned}$$

$$-\int\left((\tilde{w}_n+U(t)V+\tilde{R})^4-\sum_{k,k\neq j}\tilde{R}_k^4\right)\tilde{R}_{j_x}.$$

Let us notice again that the only possibly large term (in the first sum) is in fact $\int \tilde{R}_j\tilde{R}_{j_x}=0$. If we argue like before, we get

$$\begin{aligned}\left|\frac{\gamma_j'(t)}{\gamma_j(t)}\right| &\le C\left(1+\frac{\gamma_j'(t)|}{\gamma_j(t)}\right)\left(\int \tilde{w}_n^2(t)e^{-\sqrt{\sigma_0}|x-c_jt|}\right)^{1/2}\\ &+Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\sum_{k,k\neq j}|y_k'(t)-c_k|+e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\sum_{k,k\neq j}\left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right|\\ &+Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}+C\|U(t)V\|_{L^2(1-\psi_0(t))}. \end{aligned} \tag{26}$$

We can now do some computations. Let us sum our $2N$ estimates (25) and (26) together :

$$\begin{aligned}\sum_{k=1}^N\left(|y_k'(t)-c_k|+\left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right|\right)\le C\left(1+\sum_{k=1}^N|y_k'(t)|+\sum_{k=1}^N\left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right|\right)\|\tilde{w}_n\|_{L^2}\\ +Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\left(\sum_{k=1}^N|y_k'(t)|+\left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right|\right)+Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}+C\|U(t)V\|_{L^2(1-\psi_0(t))}.\end{aligned}$$

So that for ε_0 small enough, as $\|\tilde{w}_n\|_{L^2}\le\varepsilon_0$, and t large enough, we get

$$\sum_{k=1}^N|y_k'(t)-c_k|+\left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right|\le C.$$

Let us now go back to (25) : we get exactly what we want on $|y_j'(t)-c_j|$. In the same way, as $\gamma_k>\sigma_0$ for ε_0 small enough (first estimate), we get the result for $|\gamma_j'(t)|$ (plugging in (26)). $\square$

Let us remind that by construction

$$\tilde{w}(S_n)=w(S_n)=0,\quad y_j(S_n)=x_j+c_jS_n,\quad \gamma_j(S_n)=c_j,\quad \tilde{R}_j(S_n)=R_j(S_n). \tag{27}$$

Naturally, the geometric parameters $y_j(t)$ and $\gamma_j(t)$ control the distance between $R_j(t)$ and $\tilde{R}_j(t)$:

$$\|\tilde{R}_j(t)-R_j(t)\|_{H^s}^2\le C(s)(|y_j(t)-x_j-c_jt|^2+|\gamma_j(t)-c_j(t)|^2).$$

For simplicity of notation, let us denote

$$\tilde{v}_n(t)=\tilde{w}_n(t)+U(t)V=u_n(t)-\tilde{R}(t).$$

Lemma 6 (Main terms in M_j and E_j, $j\ge 1$). *We have, for all $t\in[I_n,S_n]$,*

(1) $$\left|M_j(t)-\left(\int Q_{\gamma_j(t)}^2+2\int\tilde{v}_n(t)\tilde{R}_j(t)+\int\tilde{v}_n^2(t)\phi_j(t)\right)\right|\le C_1^3e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t},$$

(2) $$\left|E_j(t)-\left[\frac{1}{2}\int(\tilde{v}_{n_x}^{\,2}(t)-4\tilde{R}_j^3(t)\tilde{v}_n^2(t))\phi_j(t)-\gamma_j(t)\int\tilde{v}_n(t)\tilde{R}_j(t)\right.\right.$$

$$+E(Q_{\gamma_j(t)})\Big]\Big| \le C_1^3 e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C_1^3\varepsilon_0 \int \tilde{v}_n^2(t)\phi_j(t),$$

$$(3)\quad \left|\left(E_j(t)+\frac{\gamma_j(t)}{2}M_j(t)\right)-\left(E(Q_{\gamma_j(t)})+\frac{\gamma_j(t)}{2}\int Q^2_{\gamma_j(t)}\right)-\frac{1}{2}H_j(t)\right| \le C_1^3 e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C_1^3\varepsilon_0 \int \tilde{v}_n^2(t)\phi_j(t),$$

where $H_j(t) = \int (\tilde{v}_{nx}^{\ 2}(t) - 4\tilde{R}_j^3(t)\tilde{v}_n^2(t) + \gamma_j(t)\tilde{v}_n^2(t))\phi_j(t)$.

Proof. (1) We compute $(u_n = \tilde{v}_n + \tilde{R})$:

$$M_j(t) = \int u_n^2\phi_j(t) = \int \left(\tilde{v}_n^2 + 2\tilde{v}_n\tilde{R}(t) + \sum_{k=1}^N \tilde{R}_k^2(t)\right)\phi_j(t).$$

As $\phi_j(t)$ is localized in the interval $[m_{j-1}(t), m_j(t)]$ like $\tilde{R}_j(t)$, we get for $k \neq j$

$$\int \tilde{R}_k^2(t)\phi_j(t) \le Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}, \quad \text{and} \quad \left|\int \tilde{R}_j^2(t)\phi_j(t) - \int Q^2_{\gamma_j(t)}\right| \le Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}.$$

(2) In the same way,

$$\begin{aligned}
E_j(t) &= \int \left(\frac{1}{2}(\tilde{v}_{nx}^{\ 2}(t) + 2\tilde{v}_{nx}(t)\tilde{R}_x + \tilde{R}_x^2) - \frac{1}{5}(\tilde{v}_n(t)+\tilde{R}(t))^5\right)\phi_j(t)\\
&= \int (\frac{1}{2}\tilde{v}_{nx}^{\ 2}(t) - 2\tilde{R}^3\tilde{v}_n^2(t))\phi_j + \int (\frac{1}{2}\tilde{R}_x^2 - \frac{1}{5}\tilde{R}^5)\phi_j(t)\\
&\quad - \int \tilde{v}_n(t)(\tilde{R}_{xx} + \tilde{R}^4)\phi_j - \int \tilde{R}_x\tilde{v}_n(t){\phi_j}_x\\
&\quad + \int \left[\frac{(-(\tilde{v}_n(t)+\tilde{R})^5 + \tilde{R}^5)}{5} + \tilde{v}_n(t)\tilde{R}^4 + 2\tilde{R}^3\tilde{v}_n^2(t)\right]\phi_j.
\end{aligned}$$

We keep the first integral untouched. The second one is $E(Q_{\gamma_j(t)})$ up to an exponential correction. For the third one, recall that $Q_{xx} + Q^4 = Q$, so that again

$$\int \tilde{v}_n(t)(\tilde{R}_{xx} + \tilde{R}^4)\phi_j = \gamma_j(t)\int \tilde{v}_n(t)\tilde{R}_j(t) + O(e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}).$$

The fourth one is exponentially small (with $\tilde{R}$ and ${\phi_j}_x$). Finally the fifth is of order at least 3 in v_n, so that we control it by

$$\int \tilde{v}_n(t)^k\phi_j(t) \le \|\tilde{v}_n(t)\|_{L^\infty}\int \tilde{v}_n(t)^2\phi_j(t).$$

This gives the desired result.

(3) is the sum of (1) and (2). Notice that the scalar product $\int \tilde{v}_n(t)\tilde{R}_j(t)$ cancels in H_j : the linear combination has been constructed for this. □

As usual, we now need definite positiveness on the quadratic form linked to the linearized operator of (1) around the soliton R_j.

Proposition 5 (Positivity of a quadratic form, sub-critical case). *There exists $\sigma_1 > 0$ small enough so that the following is true. For $\sigma_0 \le \sigma_1$, there exist $T_3 = T_3(\sigma_0)$ and $\lambda_1 > 0$ (not depending on σ_0), so that for all $t \ge T_3$, for all $j = 1, \dots, N$, and for all $v \in H^1$,*

$$\int (v_x^2 - 4\tilde{R}_j(t)^3 v^2 + \gamma_j(t) v^2)\phi_j(t)$$
$$\ge \lambda_1 \int (v_x^2 + v^2)\phi_j(t) - \frac{1}{\lambda_1}\left(\left(\int v\tilde{R}_j(t)\right)^2 + \left(\int v\tilde{R}_{j_x}(t)\right)^2\right).$$

Proof. A similar result can be found in [18, Lemma 4], [17, Appendix A] and [3, Appendix], to which we refer for the proof. □

From now on and throughout the rest of the proof, $\sigma_0 < \sigma_1$ is fixed.

4.2 Monotonicity properties

The next step is a surprising and crucial almost-monotonicity lemma.

Lemma 7 (Monotonicity property [13]). *There exists $C_1^1 > 0$ such that for all $j = 0, \dots, N$ and $t \in [I_n, S_n]$,*

$$\sum_{k=0}^{j} (M_k(S_n) - M_k(t)) \ge -C_1^1 e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t},$$
$$\sum_{k=0}^{j} (F_k(S_n) - F_k(t)) \ge -C_1^1 e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}.$$

Proof. This lemma is very similar to the monotonicity Lemma of [18] and [12]. The only difference is the presence of the term $U(t)V$: this will be taken care of essentially due to the smallness of $\|U(t)V\|_{L^\infty}$. Let us now do the computations. First notice that

$$\sum_{k=0}^{j} M_k(t) = \int {u_n}_t^2(t)\psi_j(t), \quad \sum_{k=0}^{j} E_k(t) = \int \left(\frac{1}{2}{u_n}_x^2(t) - \frac{1}{5}u_n^5(t)\right)\psi_j(t).$$

For $j = N$, the result is the conservation of mass and energy. Otherwise we compute for $f(t,x) \in C^3$:

$$\begin{aligned}
\frac{d}{dt}\int u_n^2 f - \int u_n^2 f_t &= 2\int u_{n\,t} u_n f = -2\int (u_{n\,xx} + u_n^4)_x u_n f \\
&= 2\int (u_{n\,xx} + u_n^4)(u_{n\,x} f + u_n f_x) \\
&= \int \left(-3{u_n}_x^2 + \frac{8}{5}u_n^5\right) f_x - 2\int u_{n\,x} u_n f_{xx} \\
&= \int \left(-3{u_n}_x^2 + \frac{8}{5}u_n^5\right) f_x + \int u_n^2 f_{xxx}.
\end{aligned}$$

So that we get

$$\frac{d}{dt}\int u_n^2 \psi_j(t) = -\int \left(3{u_n}_x^2 + m_j'(t)u_n^2 - \frac{8}{5}u_n^5\right)\psi_{j\,x} + \int u_n^2 \psi_{j\,xxx}.$$

But $m_j'(t) \geq \sigma_0$ so that by (19), and $\psi_{j_x} \leq 0$:

$$\frac{d}{dt}\int u_n^2\psi_j(t) \geq \int \left(3{u_n}_x^2 + \frac{3\sigma_0}{4} - \frac{8}{5}u_n^5\right)|\psi_{j_x}(t)|$$

It remains to bound the third term. We consider two cases : let $R_0 > 0$ be chosen later. When $x \in [c_jt + x_j + R_0, c_{j+1}t + x_{j+1} - R_0]$, ψ_{j_x} is big but $R(t)$ is small so that u_n too. More precisely,

$$\begin{aligned}\left|\frac{8}{5}u_n^3(t,x)\right| &\leq C(\|w_n(t)\|_{L^\infty}^3 + \|U(t)V\|_{L^\infty}^3 + |R(t,x)|^3)\\ &\leq C(\varepsilon_0^3 + t^{-1} + e^{-\sqrt{\sigma_0}R_0}) \leq \frac{\sigma_0}{4},\end{aligned} \tag{28}$$

if R_0 and T_0 are large enough, and ε_0 is small enough. On this interval, the second term is larger than the third.

When x is not on the previously considered interval, then $x \notin [m_j(t) - \sigma_0 t, m_j(t) + \sigma_0 t]$ (for T_0 large enough), so that

$$|\psi_{j_x}(t,x)| \leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}.$$

Now by interpolation between L^2 and H^1, we have a uniform control $\int |u_n|^5 \leq C$. So that finally

$$\frac{d}{dt}\int u_n^2\psi_j(t) \geq \int \left(3{u_n}_x^2 + \frac{\sigma_0}{2}u_n^2\right)|\psi_{j_x}(t)| - Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} \geq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}. \tag{29}$$

We integrate this last estimate between t and S_n, and this gives the estimates on M_j.

For the estimates on F_j, we compute in a similar way

$$\begin{aligned}&\frac{d}{dt}\int \left({u_n}_x^2 - \frac{2}{5}u_n^5\right)f - \int \left({u_n}_x^2 - \frac{2}{5}u_n^5\right)f_t\\ &= 2\int (u_{nxt}u_{nx} - u_n^4 u_{nt})f = -2\int u_{nt}(u_{nxx} + u_n^4)f - 2\int u_{nt}u_{nx}f_x\\ &= -\int (u_{nxx} + u_n^4)^2 f_x + 2\int (u_{nxx} + u_n^4)_x u_{nx}f_x\\ &= -\int \left((u_{nxx} + u_n^4)^2 + 2{u_n}_{xx}^2 - 8{u_n}_x^2u_n^3\right)f_x - 2\int u_{nxx}u_{nx}f_{xx}\\ &= -\int \left((u_{nxx} + u_n^4)^2 + 2{u_n}_{xx}^2 - 8{u_n}_x^2u_n^3\right)f_x + \int {u_n}_x^2 f_{xxx}.\end{aligned}$$

So that

$$\begin{aligned}&\frac{d}{dt}\int \left({u_n}_x^2 - \frac{2}{5}u_n^5\right)\psi_j(t)\\ &= -\int \left((u_{nxx} + u_n^4)^2 + 2{u_n}_{xx}^2 - 8{u_n}_x^2u_n^3\right)\psi_{j_x}(t)\\ &\quad - m_j'(t)\int \left({u_n}_x^2 - \frac{2}{5}u_n^5\right)\psi_{j_x}(t) + \int {u_n}_x^2\psi_{j_{xxx}}(t).\end{aligned}$$

Again $m_j'(t) \geq \sigma_0$ and $|m_j'(t)| \leq c_N$, so that $\int {u_n}_x^2 {\psi_j}_{xxx}(t) - \frac{\sigma_0}{4} \int {u_n}_x^2 {\psi_j}_x(t) \geq 0$ and

$$\frac{d}{dt} \int \left({u_n}_x^2 - \frac{2u_n^5}{5} \right) \psi_j(t) \geq \frac{3\sigma_0}{4} \int {u_n}_x^2 |{\psi_j}_x(t)| \\ - \int \left(8{u_n}_x^2 |u_n|^3 - \frac{2c_N}{5} |u_n|^5 \right) |{\psi_j}_x(t)|. \quad (30)$$

To bound $\int {u_n}_x^2 |u_n|^3 |{\psi_j}_x(t)|$, we proceed like before and get

$$8 \int {u_n}_x^2 |u_n|^3 |{\psi_j}_x(t)| \geq -\frac{\sigma_0}{2} \int |{u_n}_x^2 |{\psi_j}_x(t) - Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}. \quad (31)$$

However for $\frac{2c_N}{5} \int u_n^5 |{\psi_j}_x(t)|$, some L^2 norm is needed (which is why we introduced F_j, as in [12]). Choosing ε_1 small enough and R_0 large enough, we can improve (28) to $\sigma_0/400$, and so obtain :

$$\frac{2c_N}{5} \int u_n^5 \geq -\frac{\sigma_0}{100} \int u_n^2 |{\psi_j}_x(t)| - Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}. \quad (32)$$

Now adding up (30) and $1/50 \cdot (29)$, and using (31) and (32), we get

$$\frac{d}{dt} \int \left({u_n}_x^2 - \frac{2}{5}u_n^5 + \frac{1}{50}u_n^2 \right) \psi_j(t) \geq \frac{\sigma_0}{2} \int {u_n}_x^2 |\psi_x(t)| - Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}.$$

And the estimate on F_j comes by integration between t and S_n. $\square$

4.3 Abel transform and conclusion of the proof of Proposition 2

Proof of Proposition 2. We can now conclude the H^1 estimates on the right for w_n. First let us obtain some estimates for $\tilde{w}_n(t)$. We compute

$$\sum_{j=1}^{N} \frac{1}{\gamma_j^2(t)} \left(E_j + \frac{\gamma_j(t)}{2} M_j \right) = \sum_{j=1}^{N-1} \left(\left(\frac{1}{\gamma_j^2(t)} - \frac{1}{\gamma_{j+1}^2(t)} \right) \sum_{k=1}^{j} F_k \right) \\ + \sum_{j=1}^{N-1} \left(\frac{1}{2} \left(\frac{1}{\gamma_j(t)} - \frac{1}{\gamma_{j+1}(t)} \right) \left(1 - \frac{\sigma_0}{50} \left(\frac{1}{\gamma_j(t)} + \frac{1}{\gamma_{j+1}(t)} \right) \right) \sum_{k=1}^{j} M_k \right) \\ + \frac{1}{\gamma_N^2(t)} \sum_{k=1}^{N} F_k + \frac{1}{2\gamma_N(t)} \left(1 - \frac{\sigma_0}{50c_N} \right) \sum_{j=1}^{N} M_k.$$

All the terms in the right hand side are positive, so that we can apply Lemma 7 :

$$\sum_{j=1}^{N} \frac{1}{\gamma_j^2(t)} \left(E_j(t) + \frac{\gamma_j(t)}{2} M_j(t) \right) - \sum_{j=1}^{N} \frac{1}{\gamma_j^2(t)} \left(E_j(S_n) + \frac{\gamma_j(t)}{2} M_j(S_n) \right) \\ \leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}.$$

Now we use fact 3. of Lemma 6 at time t and at time S_n (remind that $|\gamma_j(t) - c_j| \le C\varepsilon_0$, so that $c_N + \varepsilon_0 \ge \gamma_j(t) \ge \sigma_0$)

$$\begin{aligned}\sum_{j=1}^{N} \frac{1}{\gamma_j^2(t)} H_j(t) &\le Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C_3\varepsilon_0 \int \left(\tilde{v}_n^2(t) + \tilde{v}_n^2(S_n)\right) \sum_{j=1}^{N} \phi_j(t) \\ &\le Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C\varepsilon_0 \|\tilde{v}_n(t)\|^2_{L^2(1-\psi_0(t))} \\ &\quad + C\varepsilon_0 \|U(S_n)V\|^2_{L^2(1-\psi_0(S_n))}. \end{aligned} \tag{33}$$

By Proposition 5, we have for $j = 1, \dots, N$,

$$H_j(t) \ge \lambda_1 \int (\tilde{v}_n^2(t) + \tilde{v}_{n\,x}^{\,2}(t))\phi_j(t) - \frac{1}{\lambda_1}\left(\left(\int \tilde{v}_n(t)Q\right)^2 + \left(\int \tilde{v}_n(t)Q_x\right)^2\right).$$

So that if we sum up those N inequalities, there exists $\lambda_0 > 0$, neither depending on σ_0 nor ε_0, such that

$$\begin{aligned}&\sum_{j=1}^{N} \frac{1}{\gamma_j^2(t)} H_j(t) \\ &\ge \lambda_0 \|\tilde{v}_n(t))\|^2_{H^1(1-\psi_0(t))} - \frac{1}{\lambda_0} \sum_{j=1}^{N} \left(\left(\int \tilde{v}_n(t)\tilde{R}_j(t)\right)^2 + \left(\int \tilde{v}_n(t)\tilde{R}_{j\,x}(t)\right)^2\right) \\ &\ge \lambda_0 \|\tilde{v}_n(t))\|^2_{H^1(1-\psi_0(t))} - \frac{1}{\lambda_0} \sum_{j=1}^{N} \left(\left(\int U(t)V\tilde{R}_j\right)^2 + \left(\int \tilde{U}(t)VQ_x\right)^2\right) \\ &\ge \lambda_0 \|\tilde{v}_n(t))\|^2_{H^1(1-\psi_0(t))} - \frac{C}{\lambda_0} \|U(t)V\|^2_{L^2(1-\psi_0(t))}. \end{aligned} \tag{34}$$

Note that our control is only on the right because we summed up for $j \ge 1$, which is coherent : we do not expect to obtain somme control in the domain $x < \sigma_0 t$, where $U(t)V$ has its L^2-mass.

Combining (34) and (33), provided that ε_0 is small enough so that $C_3\varepsilon_0 < \lambda_0/2$, we deduce :

$$\frac{1}{C}\|\tilde{v}_n(t))\|^2_{H^1(1-\psi_0(t))} \le e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + \|U(t)V\|^2_{L^2(1-\psi_0(t))} + \|U(S_n)V\|^2_{L^2(1-\psi_0(S_n))}.$$

Finally, recall $\tilde{v}_n(t) = \tilde{w}_n(t) + U(t)V$, thus

$$\begin{aligned}\|\tilde{w}_n(t))\|^2_{H^1(1-\psi_0(t))} &\le 2\|\tilde{v}_n(t))\|^2_{H^1(1-\psi_0(t))} + 2\|U(t)V\|^2_{H^1(1-\psi_0(t))} \\ &\le Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C\|U(t)V\|^2_{L^2(1-\psi_0(t))} + C\|U(S_n)V\|^2_{H^1(1-\psi_0(S_n))}. \end{aligned} \tag{35}$$

Now that we have an appropriate estimate on $\|\tilde{w}_n(t)\|_{H^1(1-\psi_0(t)}$, we have only to go back to $w_n(t) = \tilde{w}_n(t) + R(t) - \tilde{R}(t)$. As we noticed after the proof of Lemma 5 :

$$\begin{aligned}\|w_n(t)\|_{H^1(1-\psi_0(t))} &\le \|R(t) - \tilde{R}(t)\|_{H^1} + \|\tilde{w}_n(t))\|_{H^1(1-\psi_0(t))} \\ &\le C\sum_{k=1}^{N} |y_j(t) - x_j - c_j t| + |\gamma_j(t) - c_j| + Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t} \end{aligned}$$

$$+ C\|U(t)V\|_{H^1(1-\psi_0(t))} + C\|U(t)V\|_{L^2(1-\psi_0(S_n))}. \quad (36)$$

Now, using the L^2_{loc} estimate of Lemma 5, and then the estimate 35 :

$$\begin{aligned}
|y_j'(t) - c_j| + |\gamma_j'(t)| &\le C_2 e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C_2\|U(t)V\|_{L^2(1-\psi_0(t))} \\
&\quad + C_2\left(\int \tilde{w}_n^2(t)e^{-\sqrt{\sigma_0}|x-c_jt|}\right)^{1/2} \\
&\le Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t} + C\|U(t)V\|_{H^1(1-\psi_0(t))} \\
&\quad + C\|U(S_n)V\|_{L^2(1-\psi_0(S_n))}.
\end{aligned}$$

Let us integrate this between t and S_n. Remind the initial conditions $y_j(S_n) = x_j + c_jS_n$, $\gamma_j(S_n) = c_j$, we obtain

$$\begin{aligned}
|y_j(t) - x_j - c_jt| + |\gamma_j(t) - c_j| &\le Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t} + C\int_t^{S_n} \|U(t)V\|_{H^1(1-\psi_0(t))}dt \\
&\quad + C(S_n - t)\|U(S_n)V\|_{L^2(1-\psi_0(S_n))}.
\end{aligned}$$

This, together with (36), concludes the proof of Proposition 2. □

5 Global estimates : proof of Proposition 3

We now want to control what happens in the zone $x < \sigma_0 t$, that is the interaction with the linear term $U(t)V$. We follow the path of [4]. As our a priori estimates only concern w_n, we cannot use $\tilde{w}_n$, which has a better H^1 decay on the right : we don't have any available control on $M_0^t(\tilde{w}_n)$. The second point is that it appears to be difficult to control only $\|w_n\|_{H^s(\psi_0(t))}$, and this is why we do computation on the whole space, to obtain the decay estimate :

$$\|w_n(t)\|_{H^4} \le \frac{C}{t^{1/3}}.$$

(Some terms that appear in the integration by part behave badly, but vanish when integrating on the whole space).

Recall our pointwise estimates on $w_n(t)$ ($M_0^t(w_n(t) \le \varepsilon_0)$: we have

$$\begin{aligned}
|w_n(t,x)| &\le \frac{C}{t^{1/3}}\left(1 + \frac{|x|}{\sqrt[3]{t}}\right)^{-1/4} M_0^t(w_n(t)), \\
|w_{nx}(t,x)| &\le \frac{C}{t^{2/3}}\left(1 + \frac{|x|}{\sqrt[3]{t}}\right)^{1/4} M_0^t(w_n(t)).
\end{aligned}$$

We proceed in two subsections : one for the H^1 estimate, which is very similar to that of [6] or [4], and one for H^s, $s > 1$, which requires high integrability and smoothness of the non-linearity ($p \ge 4$).

5.1 H^1 estimate

Proof of Proposition 3, H^1 estimate. L^2 estimate.

Here, no monotonicity is involved (it is essentially a linear theory). We bound the absolute value of the derivative in time of the L^2 norm of $w_n(t)$, and then integrate our estimate backward in time, with $w_n(S_n) = 0$.

We use the equation for w_n

$$w_{nt} + w_{nxxx} + \left(u_n^4 - \sum_{j=1}^{N} R_j^4\right)_x = 0. \tag{11}$$

We multiply by w_n, and integrate in x. After an integration by part, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int w_n^2 &= \int \left(u_n^4 - \sum_{j=1}^{N} R_j^4\right) w_{nx} \\
&= \int (w_n + U(t)V + R)^4 - R(t)^4 - (w_n + U(t)V)^4) w_{nx} \\
&\quad - \int \left(R^4 - \sum_{j=1}^{N} R_j^4\right)_x w_n - \int ((w_n + U(t)V)^4)_x w_n.
\end{aligned}$$

Let us develop

$$(w_n + U(t)V + R)^4 - R^4 - (w_n + U(t)V)^4 = \sum_{k=1}^{3} C_4^k (w_n + U(t)V)^k R^{4-k}.$$

So that

$$\begin{aligned}
&\left| \int ((w_n + U(t)V + R)^4 - R^4 - (w_n + U(t)V))^4 w_{nx} \right| \\
&\leq \sum_{k=1}^{3} C_4^k \int |w_n + U(t)V|^k R^{4-k} |w_{nx}| \\
&\leq C \|w_{nx}\|_{L^2} \|w_n + U(t)V\|_{L^2(1-\psi_0(t))} \sum_{k=1}^{3} \|(w_n + U(t)V)^{k-1} R^{3-k}\|_{L^\infty} \\
&\leq C\varepsilon_0 \|w_n + U(t)V\|_{L^2(1-\psi_0(t))}.
\end{aligned}$$

Note that our control is essentially $\|w_{nx}\|_{L^2}\|w_n + U(t)V\|_{L^2(1-\psi_0(t))}$, and so relies on a priori estimate on $\|w_n\|_{H^1}$ to control the L^2 level. In fact this problem will only be acute for H^4, but let us explain now how to avoid it. We need to fully develop the term $(w_n + U(t)V + R)^4$. We do integration by part in this way :

$$\int w_n^i U(t) V^j R^{4-i-j} w_{nx} = -\frac{1}{i+1} \int w_n^{i+1} (U(t) V^j R^{4-i-j})_x,$$

so that all derivatives go on R or on $U(t)V$. It is then clear that in the L^2 case, our control improves to :

$$C\|w\|_{L^2(1-\psi_0(t)}^2 + C\|w\|_{L^2} \|U(t)V\|_{H^1(1-\psi_0)}.$$

The point being that the estimate only involves $\|w_n\|_{L^2}$, and we are safe if we assume enough regularity and decay on V. For now, the direct method is simpler, so we will use it up to the H^3 estimate. Let us now go back to rest of the terms.

Of course, the purely solitons-interaction is exponentially small :

$$\left|\int \left(R^4 - \sum_{j=1}^{N} R_j^4\right)_x w_n\right| \le \left\|R^4 - \sum_{j=1}^{N} R_j^4\right\|_{H^1} \|w_n\|_{L^2} \le Ce^{-\frac{-\sigma_0\sqrt{\sigma_0}}{4}t}\|w_n(t)\|_{L^2}.$$

And to complete, we have to treat the purely linear interaction, which we control as in [4] :

$$\begin{aligned}
&\left|4\int (w_n + U(t)V)^3(w_n + U(t)V)_x w_n\right| \\
&\qquad\qquad \le \|(w_n + U(t)V)_x(w_n + U(t)V)\|_{L^\infty}\|w_n + U(t)V\|_{L^\infty} \\
&\qquad\qquad\quad \times \|w_n + U(t)V\|_{L^2}\|w_n\|_{L^2} \\
&\qquad\qquad \le \frac{C}{t^{4/3}}\|w_n\|_{L^2}.
\end{aligned}$$

So that we get

$$\frac{d}{dt}\|w_n(t)\|_{L^2}^2 \le \left(\frac{C}{t^{4/3}} + e^{-\frac{-\sigma_0\sqrt{\sigma_0}}{4}t}\right)\|w_n(t)\|_{L^2} + C\|w_n(t) + U(t)V\|_{L^2(1-\psi_0(t))}.$$

We integrate between t and S_n, and obtain ($w_n(S_n) = 0$)

$$\|w(t)\|_{L^2} \le \frac{C}{t^{1/3}}. \tag{37}$$

as soon as $\|w_n(t) + U(t)V\|_{L^2(1-\psi_0(t))} \le Ct^{-5/3}$.

$\dot{H}^1$ estimate.

We differentiate (11) with respect to x :

$$w_{nxt} + w_{nxxxx} + \left(u_n^4 - \sum_{j=1}^{N} R_j^4\right)_{xx} = 0.$$

Now we multiply by w_{nx} and integrate in x. After an integration by parts, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int {w_n}_x^2 &= \int \left(u_n^4 - \sum_{j=1}^{N} R_j^4\right)_x w_{nxx} \\
&= \int (w_n + U(t)V + R)^4 - R^4 - (w_n + U(t)V)^4))_x w_{nxx} \\
&\quad - \int \left(R^4 - \sum_{j=1}^{N} R_j^4\right)_{xx} w_{nx} - \int ((w_n + U(t)V)^4)_x w_{nxx}.
\end{aligned}$$

Let us first treat the second line. As in the L^2 case,

$$\left|\int \left(R^4 - \sum_{j=1}^{N} R_j^4\right)_{xx} w_{nx}\right| \le C\varepsilon^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|w_{nx}\|_{L^2}.$$

And for the purely linear interaction term,

$$\begin{aligned}\int ((w_n+U(t)V)^4)_x w_{nxx} &= 4\int w_{nxx}(w_{nx}+U(t)V_x)(w_n+U(t)V)^3\\ &= -6\int w_{nx}^2(w_n+U(t)V)_x(w_n+U(t)V)^2\\ &\quad -4\int w_{nx}U(t)V_{xx}(w_n+U(t)V)^3\\ &\quad -12\int w_{nx}(w_n+U(t)V)_x^2(w_n+U(t)V)^2.\end{aligned}$$

Now, we control each of the terms with $\|w_{nx}\|_{L^2}$, $\|(w_n+U(t)V)_x\|_{L^2}$ (first and third term), $\|w_n+U(t)V\|_{L^2}$ (second term) and the rest in L^∞, noticing for the second term that :

$$|U(t)V_{xx}(x)| \leq \frac{C}{t^{2/3}}\left(1+\frac{|x|}{\sqrt[3]{t}}\right)^{1/4}\|V_x\|_{H^{1,1}}.$$

So that as previously

$$\left|\int ((w_n+U(t)V)^4)_x w_{nxx}\right| \leq \frac{C}{t^{4/3}}\|w_{nx}\|_{L^2}.$$

We now turn to

$$\begin{aligned}&\int (w_n+U(t)V+R)^4-R^4-(w_n+U(t)V)^4))_x w_{nxx}\\ &\qquad = \sum_{k=1}^{3} kC_4^k\int (w_{nx}+U(t)V_x)(w_n+U(t)V)^{k-1}R^{4-k}w_{nxx}\\ &\qquad\quad + \sum_{k=1}^{3}(4-k)\int (w_n+U(t)V)^k R_x R^{3-k})w_{nxx}.\end{aligned}$$

Hence this interaction term is controlled by

$$C\|w_n+U(t)V\|_{H^1(1-\psi_0(t))}\|w_{xx}\|_{L^2} \leq C\|w_n\|_{H^1(1-\psi_0(t))}.$$

(remind that $w, U(t)V, R, R_x \in L^\infty$). Again, we obtain

$$\frac{d}{dt}\|w_{nx}\|_{L^2}^2 \leq C\left(\frac{1}{t^{4/3}}+e^{-\frac{-\sigma_0\sqrt{\sigma_0}}{4}t}\right)\|w_{nx}\|_{L^2} + C\|w_n\|_{H^1(1-\psi_0(t))}.$$

We integrate between t and S_n, and derive ($w_n(S_n)=0$)

$$\|w_{nx}(t)\|_{L^2} \leq \frac{C}{t^{1/3}}, \tag{38}$$

as soon as $\|w_n(t)\|_{H^1(1-\psi_0(t))} \leq Ct^{-5/3}$. □

Notice that this proof extends to the case $p>3$.

5.2 H^4 estimate

Proof of Proposition 3. We only present here the proof for the H^2 estimate, as the higher estimate will be treated in the same way, and will raise in fact less difficulties. The H^3 and H^4 estimates are done in full detail in the Appendix.

The proof goes in two steps : the first step is to derive a satisfactory H^2 type relation, the second step is to do the appropriate estimates on this relation

Step 1. Obtaining the relation (40). We now derive a satisfactory relation on $\frac{d}{dt}\int {w_n}_{xx}^2$. As before, we use (11), twice differentiated :

$$w_{nxxt} + w_{nxxxxx} + \left(u_n^4 - \sum_{j=1}^N R_j^4\right)_{xxx} = 0.$$

We multiply it by w_{nxx}, and do an integration by parts, to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int {w_n}_{xx}^2 &= \int \left(u_n^4 - \sum_{j=1}^N R_j^4\right)_{xx} w_{nxxx}\\ &= \int \left(u_n^4 - R^4\right)_{xx} w_{nxxx} + \int \left(R^4 - \sum_{j=1}^N R_j^4\right)_{xxx} w_{nxx}.\end{aligned}$$

The second integral is harmless. Let us develop the first term :

$$\begin{aligned}\left(u_n^4 - R^4\right)_{xx} &= 4(u_{nxx}u_n^3 - R_{xx}R^3) + 12({u_n}_x^2 u_n^2 - R_x^2 R^2)\\ &= 4w_{nxx}u_n^3 + 4((U(t)V+R)_{xx}u_n^3 - R_{xx}R^3)\\ &\quad + 12({u_n}_x^2 u_n^2 - R_x^2R^2).\end{aligned}$$

We put in front the factor w_{nxx}, in view of an integration by parts. Indeed, we want to get rid of the 3 derivative term w_{nxxx}. We compute :

$$\begin{aligned}\int \left(u_n^4 - R^4\right)_{xx} w_{nxxx} &= -6\int {w_n}_{xx}^2 u_{nx}u_n^2\\ &\quad - 4\int((U(t)V+R)_{xx}u_n^3 - R_{xx}R^3)_x w_{nxx} - 12\int({u_n}_x^2u_n^2 - R_x^2R^2)_x w_{nxx}\\ &= -6\int {w_n}_{xx}^2 u_{nx}u_n^2 - 4\int((U(t)V+R)_{xxx}u_n^3 - R_{xxx}R^3)w_{nxx}\\ &\quad - 12\int((U(t)V+R)_{xx}u_{nx}u_n^2 - R_{xx}R_xR^2)w_{nxx}\\ &\quad - 24\int(u_{nxx}u_{nx}u_n^2 - R_{xx}R_xR^2)w_{nxx} - 24\int({u_n}_x^3u_n - R_x^3R)w_{nxx}.\end{aligned}$$

Let us focus on the first term on the last line, to get :

$$\begin{aligned}&= -30\int {w_n}_{xx}^2 u_{nx}u_n^2 - 4\int((U(t)V+R)_{xxx}u_n^3 - R_{xxx}R^3)w_{nxx}\\ &\quad - 36\int((U(t)V+R)_{xx}u_{nx}u_n^2 - R_{xx}R_xR^2)w_{nxx}\\ &\quad - 24\int({u_n}_x^3u_n - R_x^3R)w_{nxx}.\end{aligned}$$

The first term $\int w_{n_{xx}}^2 u_{nx} u_n^2$ is troublesome, as when developing it contains $\int w_{n_{xx}}^2 R_x R^2$, which we do not control yet. This is why we will correct this by considering :

$$\frac{d}{dt}\int w_{n_x}^2 u_n^3 = 2\int w_{nxt} w_{nx} u_n^3 + 3\int w_{n_x}^2 u_{nt} u_n^2$$
$$= -2\int w_{nxxxx} w_{nx} u_n^3 - \int \left(u_n^4 - \sum_{j=1}^{N} R_j^4\right)_{xx} w_{nx} u_n^3$$
$$- 3\int w_{n_x}^2 u_{nxxx} u_n^2 - 12\int w_{n_x}^2 u_{nx} u_n^5.$$

Remark that :

$$-\int \left(u_n^4 - \sum_{j=1}^{n} R_j^4\right)_{xx} w_{nx} u_n^3$$
$$= -\int (u_n^4 - R^4)_{xx} w_{nx} u_n^3 - \int \left(R^4 - \sum_{j=1}^{n} R_j^4\right)_{xx} w_{nx} u_n^3,$$

where the second integral will be treated as usual. Two terms are to be rearranged in the previous expression : those with high derivative. The first one is

$$-2\int w_{nxxxx} w_{nx} u_n^3$$
$$= 2\int w_{nxxx} w_{nxx} u_n^3 + 6\int w_{nxxx} w_{nx} u_{nx} u_n^2$$
$$= -9\int w_{n_{xx}}^2 u_{nx} u_n^4 - 6\int w_{nxx} w_{nx} u_{nxx} u_n^2 - 12\int w_{nxx} w_{nx} u_{n_x}^2 u_n$$
$$= -15\int w_{n_{xx}}^2 u_{nx} u_n^2 + 6\int w_{n_{xx}}^2 (U(t)V + R(t))_x u_n^2$$
$$- 6\int w_{nxx} w_{nx} (U(t)V + R)_{xx} u_n^2 - 6\int w_{nxx} w_{nx} u_{n_x}^2 u_n,$$

and the second one

$$-3\int w_{n_x}^2 u_{nxxx} u_n^2$$
$$= 6\int w_{nxx} w_{nx} u_{nxx} u_n^4 + 6\int w_{n_x}^2 u_{nxx} u_n^2$$
$$= 6\int w_{n_{xx}}^2 u_{nx} u_n^2 - 6\int w_{n_{xx}}^2 (U(t)V + R)_x u_n^2$$
$$+ 6\int w_{nxx} w_{nx} (U(t)V + R)_{xx} u_n^2 - 12\int w_{nxx} w_{nx} u_{n_x}^2 u_n.$$

So that we get

$$\frac{d}{dt}\int w_{n_x}^2 u_n^3 = -9\int w_{n_{xx}}^2 u_{nx} u_n^2 - 24\int w_{nxx} w_{nx} u_{n_x}^2 u_n$$
$$- \int (u_n^4 - R^4)_{xx} w_{nx} u_n^3 - \int \left(R^4 - \sum_{j=1}^{n} R_j^4\right)_{xx} w_{nx} u_n^3$$

$$- 12 \int {w_n}_x^2 u_{nx} u_n^5. \tag{39}$$

If we put everything together, we obtain the desired equality, on which we will do all our estimates :

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\int {w_n}_{xx}^2 - \frac{20}{3}\int {w_n}_x^2 u_n^3\right) \\
&= -4\int ((U(t)V+R)_{xxx}u_n^3 - R_{xxx}R^3)w_{nxx} \\
&\quad - 36\int ((U(t)V+R)_{xx}u_{nx}u_n^2 - R_{xx}R_xR^2)w_{nxx} \\
&\quad - 24\int ({u_n}_x^3 u_n - R_x^3R)w_{nxx} + 40\int w_{nxx}w_{nx}{u_n}_x^2 u_n \\
&\quad + \frac{20}{3}\int (u_n^4 - R^4)_{xx}w_{nx}u_n^3 + 80\int {w_n}_x^2 u_{nx}u_n^5 \\
&\quad + \int \left(R^4 - \sum_{j=1}^N R_j^4\right)_{xxx} w_{nxx} + \frac{20}{3}\int \left(R^4 - \sum_{j=1}^n R_j^4\right)_{xx} w_{nx}u_n^3.
\end{aligned} \tag{40}$$

Step 2. Estimating terms in (40). We now estimate separately every term appearing in the right hand side of (40).

• First let us bound the 2 terms of (40) with $R^4 - \sum_j R_j^4$.

$$\left|\int \left(R^4 - \sum_{j=1}^N R_j^4\right)_{xxx} w_{nxx}\right| \leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|w_{nxx}\|_{L^2}. \tag{41}$$

• And (remind $\|u_n\|_{L^\infty} \leq C$) :

$$\left|\int (R^4 - \sum_{j=1}^n R_j^4)_{xx} w_{nx}u_n^3\right| \leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|w_{nx}\|_{L^2}. \tag{42}$$

• Let us now consider the terms with exponent 8 in (40).

$$\int (u_n^4 - R^4)_{xx}w_{nx}u_n^3 = \int w_{nx}\left(\sum_{k=1}^4 C_4^k (w_n + U(t)V)^k R^{4-k}\right)_{xx} u_n^3.$$

So that all terms are at least quadratic in w or $(w+U(t)V)$. We do an integration by parts on the (unique) term with $w_{nxx}w_{nx}$. Thus, all the terms with at least one R are controlled by

$$C\|w\|_{H^1(1-\psi_0(t))}^2 + C\|U(t)V\|_{H^2(1-\psi_0(t))}^2.$$

It remains to treat

$$\int ((w_n + U(t)V)^4)_{xx} w_{nx}(w_n + U(t)V)^3.$$

Again, the term containing w_{nxx} is treated with an integration by parts, to have 3 terms with 1 derivative. The term with $U(t)V_{xx}$ is in some sense the worst, although the fact $V \in H^{2,2}$ allows to bound it (this is similar to what happens in the purely linear case [4]) :

$$\|U(t)V_{xx}(w_n + U(t)V)\|_{L^\infty} \|w_{nx}(w_n + U(t)V)\|_{L^\infty} \|w_n + U(t)V\|_{L^4}^4 \leq \frac{C}{t^{8/3}}.$$

For the terms with three terms with one derivative, one of these is controlled in L^2, which gives the same decay rate $Ct^{-8/3}$. Finally, we get

$$\left| \int (u_n^4 - R^4)_{xx} w_{nx} u_n^3 \right| \leq C\|w\|_{H^1(1-\psi_0(t))}^2 + C\|U(t)V\|_{H^2(1-\psi_0(t))}^2 + \frac{C}{t^{8/3}}. \quad (43)$$

Arguing similarly allows us to bound the second term

$$\left| \int {w_n}_x^2 u_{nx} u_n^5 \right| \leq \|w_n\|_{H^1(1-\psi_0(t))}^2 + \frac{C}{t^{8/3}}. \quad (44)$$

We will now consider each of the 4 remaining terms of (40) separately. However, one constant in the treatment will be that the term w_{nxx} always appear exactly once, and will be controlled in L^2. The second point will be that all terms where only w_n and $U(t)V$ appear (not R) will be controlled by

$$\frac{C}{t^{4/3}} \|w_{nxx}\|_{L^2}.$$

All the terms that do not fall in this category will be bounded by a control of the type "estimates on the right", as they contain both R and $w_n + U(t)V$ (there is no terms with only R).

To do this, we develop each term in a "purely linear" part and a "linear-non linear" interaction part.

- $\int ((U(t)V + R)_{xxx} u_n^3 - R_{xxx} R^3) w_{nxx}$. We develop our main term :

$$\begin{aligned}
&(U(t)V + R)_{xxx} u_n^3 - R_{xxx} R^3 \\
&\quad = U(t)V_{xxx}(w_n + U(t)V)^3 + U(t)V_{xxx} R \cdot \sum_{k=0}^{2} C_3^k (w_n + U(t)V)^k R^{2-k} \\
&\qquad + R_{xxx}(w + U(t)V) \cdot \sum_{k=1}^{3} C_3^k (w_n + U(t)V)^{k-1} R^{3-k}.
\end{aligned}$$

Remember $V \in H^{3,1}$ so that $V_{xx} \in H^{1,1}$, and we get

$$\begin{aligned}
&\left| \int ((U(t)V + R)_{xxx} u_n^3 - R_{xxx} R^3) w_{nxx} \right| \\
&\leq C\Big(\|U(t)V_{xxx}(w + U(t)V)\|_{L^\infty} \|w_n + U(t)V\|_{L^\infty} \|w_n + U(t)V\|_{L^2} \\
&\quad + \|U(t)V\|_{H^3(1-\psi_0(t))} + \|w\|_{H^1(1-\psi_0(t))} \Big) \|w_{nxx}\|_{L^2} \\
&\leq C\left(\frac{1}{t^{4/3}} + \|U(t)V\|_{H^3(1-\psi_0(t))} + \|w\|_{H^1(1-\psi_0(t))} \right) \|w_{nxx}\|_{L^2}. \quad (45)
\end{aligned}$$

• $\displaystyle\int ((U(t)V+R)_{xx}u_{nx}u_n^2 - R_{xx}R_xR^2)w_{nxx}$. We develop as before

$$
\begin{aligned}
&(U(t)V+R)_{xx}u_{nx}u_n^2 - R_{xx}R_xR^2\\
&= U(t)V_{xx}(w_n+U(t)V)_x(w_n+U(t)V)^2 + U(t)V_{xx}R_xu_n^2\\
&\quad + R_{xx}(w+U(t)V)_xu_n^2 + U(t)V_{xx}(w_n+U(t)V)_x(2(w_n+U(t)V)+R)R\\
&\quad + R_{xx}R_x(w_n+U(t)V)(w_n+U(t)V+2R).
\end{aligned}
$$

So that :

$$
\begin{aligned}
&\left|\int (U(t)V+R)_{xx}u_{nx}u_n^2 - R_{xx}R_xR^2 w_{nxx}\right|\\
&\qquad \le \Big(\|U(t)V_{xx}\|_{L^2}\|(w_n+U(t)V)_x(w_n+U(t)V)\|_{L^\infty}\|w_n+U(t)V\|_{L^\infty}\\
&\qquad\quad + C\|U(t)V\|_{H^2(1-\psi_0(t))} + \|w_n\|_{H^1(1-\psi_0(t))}\Big)\|w_{nxx}\|_{L^2}. \qquad (46)
\end{aligned}
$$

The last two terms are the hardest : the assumption of high integrability ($p \ge 4$) is crucially used. Indeed, these terms contain the information on $\int u_x^5u^{p-4} = -4\int u_{xx}u_x^3u^{p-3}$.

• $\displaystyle\int ({u_n}_x^3 u_n - R_x^3R)w_{nxx}$. We develop as usual

$$
\begin{aligned}
&{u_n}_x^3u_n - R_x^3R\\
&\quad = (w_n+U(t)V)_x^3(w_n+U(t)V) + (w_n+U(t)V)_x^3R + R_x^3(w_n+U(t)V)\\
&\qquad + (w_n+U(t)V)_xR_x\cdot\left(\sum_{k=1}^{2} C_3^k(w_n+U(t)V)_x^{k-1}R^{2-k}\right)\cdot u_n.
\end{aligned}
$$

First let us forget the first term with no soliton term, and focus on the last three. Remind that $w_{nx}, U(t)V_x \in L^\infty$. All these term have R and $w_n+U(t)V$ (with at most 1 derivative) in factor, so that they are bounded by

$$
C\|w_n\|_{H^1(1-\psi_0(t))}\|w_{nxx}\|_{L^2}.
$$

Let us now turn to the remaining term

$$
\begin{aligned}
&\int (w_n+U(t)V)_x^3(w_n+U(t)V)w_{nxx}\\
&\qquad = \int (w_n+U(t)V)_x^2(w_n+U(t)V)w_{nx}w_{nxx}\\
&\qquad\quad + \int U(t)V_x(w_n+U(t)V)_x(w_n+U(t)V)(w_n+U(t)V)_xw_{nxx}.
\end{aligned}
$$

We use our previously obtained decay $\|w_{nx}\|_{L^2} \le Ct^{-1/3}$, and the a priori estimate $\|w_{nx}\|_{L^\infty} \le \varepsilon_0$ in the first integral, and $\|U(t)V_x\|_{L^\infty} \le Ct^{-1/3}$ (as $V_x \in H^{1,1}$) for the second integral, to get the bound

$$
\left|\int (w_n+U(t)V)_x^3(w_n+U(t)V)w_{nxx}\right|
$$

$$
\begin{aligned}
&\le \|w_{xx}\|_{L^2}\|w_x\|_{L^2}\|(w_n+U(t)V)_x\|_{L^\infty}\|(w_n+U(t)V)_x(w_n+U(t)V)\|_{L^\infty}\\
&\quad+\|w_{xx}\|_{L^2}\|w_{nx}+U(t)V_x\|_{L^2}\|(w_n+U(t)V)_x(w_n+U(t)V)\|_{L^\infty}\|U(t)V_x\|_{L^\infty}\\
&\le \frac{C}{t^{4/3}}\|w_{nxx}\|_{L^2}.
\end{aligned}
$$

So that we obtain in the end

$$
\left|\int ({u_n}_x^3 u_n - R_x^3 R) w_{nxx}\right| \le C\left(\frac{1}{t^{4/3}} + \|w_n\|_{H^1(1-\psi_0(t))}\right)\|w_{nxx}\|_{L^2}. \tag{47}
$$

- $\int w_{nxx} w_{nx} {u_n}_x^2 u_n$. We develop as usual

$$
\begin{aligned}
\int w_{nxx} w_{nx} {u_n}_x^2 u_n &= \int w_{nxx} w_{nx} (w_n+U(t)V)_x^2 (w_n+U(t)V)\\
&\quad+\int w_{nxx} w_{nx} (w_n+U(t)V)_x^2 R + \int w_{nxx} w_{nx} R_x u_{nx} u_n.
\end{aligned}
$$

The last two terms are clearly controlled as in the previous case by

$$
\|w_n\|_{H^1(1-\psi_0(t))}\|w_{nxx}\|_{L^2}.
$$

And for the term on the first line :

$$
\begin{aligned}
&\left|\int w_{nxx} w_{nx} (w_n+U(t)V)_x^2 (w_n+U(t)V)\right|\\
&\le \|w_{nxx}\|_{L^2}\|w_{nx}\|_{L^2}\|(w_n+U(t)V)_x\|_{L^\infty}\|(w_n+U(t)V)_x(w_n+U(t)V)\|_{L^\infty}\\
&\le \frac{C}{t^{4/3}}\|w_{nxx}\|_{L^2}.
\end{aligned}
$$

And we get for this last term :

$$
\left|\int w_{nxx} w_{nx} {u_n}_x^2 u_n\right| \le C\left(\frac{1}{t^{4/3}} + \|w_n\|_{H^1(1-\psi_0(t))}\right)\|w_{nxx}\|_{L^2}. \tag{48}
$$

Step 3. Conclusion of the H^2 bound. All the terms on the right hand side in (40) were estimated. As we would like to have a bound on $\|w_{nxx}\|_{L^2}$ (without the corrective term), we have to use an integral form for these bounds, and we have to estimate the corrective term $\int {w_n}_x^2 u_n^3$. When developing u_n^3, treating the term with R on one side and the purely “linear” term on the other side, we get

$$
\begin{aligned}
\left|\int {w_n}_x^2 u_n^3\right| &\le \|w_n\|_{H^1(1-\psi_0(t)}^2 + \int {w_n}_x^2 |w_n+U(t)V|^3\\
&\le \|w_n\|_{H^1(1-\psi_0(t)}^2 + \frac{C}{t^{5/3}}.
\end{aligned}
$$

If we put everything together, for this H^2 estimate, starting from the equation (40), and the bounds for each term (41), (42), (43), (44), (45), (46), (47), and (48), we get

$$
\left|\frac{d}{dt}\left(\frac{1}{2}\int {w_n}_{xx}^2 - \int {w_n}_x^2 u_n^3\right)\right|
$$

$$\leq C\left(\frac{\|w_n\|_{H^2}}{e^{\frac{\sigma_0\sqrt{\sigma_0}}{4}t}}+\|w_n\|^2_{H^1(1-\psi_0(t))}+\|U(t)V\|^2_{H^2(1-\psi_0(t))}+\frac{(1+\|V\|_{H^{2,2}})}{t^{8/3}}\right.$$
$$\left.+\left(\frac{1+\|V\|_{H^{3,1}}}{t^{4/3}}+\|w_n\|_{H^1(1-\psi_0(t))}+\|U(t)V\|_{H^3(1-\psi_0(t))}\right)\|w_{nxx}\|_{L^2}\right).$$

Let us integrate in time between t and S_n, so that as soon as

$$\|w_n\|_{H^1(1-\psi_0(t))}+\|U(t)V\|_{H^3(1-\psi_0(t))}\leq\frac{C}{t^{4/3}},$$

we get, for all $t\in[I_n,S_n]$,

$$\|w_{nxx}(t)\|^2_{L^2}\leq\frac{C}{t^{5/3}}+\int_t^{S_n}\frac{\|w_{nxx}(\tau)\|_{L^2}}{\tau^{4/3}}d\tau.$$

With Lemma 4, we derive :

$$\forall t\in[I_n,S_n],\qquad \|w_{nxx}(t)\|_{L^2}\leq\frac{C}{t^{1/3}}.$$

□

6 M_0^t estimate : proof of Proposition 4

We now want to conclude the proof of Proposition 1', that is to prove that for $t\in[I_n,S_n]$,

$$M_0^t(w_n(t))=\|w_n(t)\|_{H^1}+\|D^\alpha Jw_n(t)\|_{L^2}+\|DJw_n(t)\|_{L^2}\leq\frac{C}{t^\delta}.$$

$\delta<\frac{1}{3}$: it remains to estimate $\|D^\alpha J^t w_n\|_{L^2}$ and $\|DJ^t w_n\|_{L^2}$. As in [6] and [4], we do the computation on the dilation operator

$$I^t f=xf+3t\int_{-\infty}^x f_t dx,$$

as it is easier to compute with. So we will proceed in two lemmas, one concerning $I^t w_n$, and then coming back from $I^t w_n$ to $J^t w_n$. Let us first do a short reminder of commutation properties of these operators. Let us note $L=\partial_t+\partial_{xxx}$ the linear KdV operator. Then

$$I^t f-J^t f=3t\int_{-\infty}^x Lf dx.$$

We have the following commutation relations :

$$[L,J^t]=0,\quad [L,I^t]f=3\int_{-\infty}^x Lf dx,\quad [J^t,\partial_x]=[I^t,\partial_x]=-Id.$$

Notice that $I^tU(t)V-J^tU(t)V=3t\int_{-\infty}^x LU(t)V dx=0$, hence

$$\|D^\alpha I^tU(t)V\|_{L^2}+\|DI^tU(t)V\|_{L^2}\leq C\|V\|_{H^{1,1}}.$$

6.1 $I^t w_n$ estimates

Let f so that the following has a sense and $\Phi : \mathbb{R} \to \mathbb{R}$ a C^1 function. Then we have the chain rule relation :

$$I^t(\Phi(f)_x) = x\Phi(f)_x + 3t\Phi(f)_t = x\Phi'(f)f_x + 3t\Phi'(f)f_t = \Phi'(f)I^t f_x. \tag{49}$$

We will use this formula for $\Phi(x) = x^4$ and $f = u_n$ or $f = R$.

Let us start with $\|I^t w_{nx}\|_{L^2}$ as the result obtained will then be used for $\|D^\alpha I w_n\|_{L^2}$. We proceed in a very analoguous way as for the H^2 estimate, in 3 similar steps.

$\|I^t w_{nx}\|_{L^2}$ *estimate.*

Step 1. Notice that $(LI^t f, f) = \frac{1}{2}\frac{d}{dt}\|f(t)\|_{L^2}$, which is why we compute :

$$\begin{aligned}
LI^t w_{nx} &= I^t L w_{nx} + L w_n = -I^t\left(\left(u_n^4 - \sum_{j=1}^N R_j^4\right)_{xx}\right) - \left(u_n^4 - \sum_{j=1}^N R_j^4\right)_x \\
&= -I^t(u_n^4 - R^4)_{xx}) - (u_n^4 - R^4)_x \\
&\quad + I^t\left(R^4 - \sum_{j=1}^N R_j^4\right)_{xx} - \left(R^4 - \sum_{j=1}^N R_j^4\right)_x.
\end{aligned} \tag{50}$$

Let us can focus on

$$\begin{aligned}
-I^t(u_n^4 - R^4)_{xx}) - (u_n^4 - R^4)_x &= (I^t(u_n^4 - R^4)_x)_x - 2(u_n^4 - R^4)_x \\
&= -4(u_n^3 I^t u_{nx} - R^3 I^t R_x)_x - 2(u_n^4 - R^4)_x \\
&= -12(u_{nx}u_n^2 I^t u_{nx} - R_x R^2 I^t R_x) \\
&\quad - 4(u_n^3(I^t u_{nx})_x - R^3(I^t R_x)_x) - 8(u_{nx}u_n^3 - R_x R^3).
\end{aligned}$$

So that

$$\begin{aligned}
LI^t w_{nx} =& - 12(u_{nx}u_n^2 I^t u_{nx} - R_x R^2 I^t R_x) \\
& - 4(u_n^3(I^t u_{nx})_x - R^3(I^t R_x)_x) - 8(u_{nx}u_n^3 - R_x R^3) \\
& + I^t\left(R^4 - \sum_{j=1}^N R_j^4\right)_{xx} - \left(R^4 - \sum_{j=1}^N R_j^4\right)_x.
\end{aligned} \tag{51}$$

This expression of $LI^t w_{nx}$ is the one will develop.

Step 2. As previously, for every term in (51), we take the "purely linear" term apart, and all the remaining terms contain both $w_n + U(t)V$ and R, and so will be bounded using estimates "on the right" obtain in Section 4.

- Of course the terms on the last line will be negligible :

$$\left|\int \left(I^t\left(R^4 - \sum_{j=1}^N R_j^4\right)_{xx} - \left(R^4 - \sum_{j=1}^N R_j^4\right)_x\right) I^t w_{nx}\right| \leq Cte^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|I^t w_{nx}\|_{L^2}.$$

- Then consider

$$u_{nx}u_n^3 - R_x R^3 = (w_n + U(t)V)_x (w_n + U(t)V)^3$$

$$
\begin{aligned}
&+ (w_n + U(t)V)_x R \cdot \left(\sum_{k=0}^{2} C_3^k (w_n + U(t)V)^k R^{2-k} \right) \\
&+ R_x (w_n + U(t)V) \cdot \left(\sum_{k=1}^{3} C_3^k (w_n + U(t)V)^{k-1} R^{3-k} \right).
\end{aligned}
$$

The last two lines have both a localizing term R or R_x, and $w_n + U(t)V$ with at most 1 derivative ; for the first term we use the argument of the linear case, the L^2 norm going on one $(w_n + U(t)V)$, so that

$$
\begin{aligned}
&\left| \int (u_{nx} u_n^3 - R_x R^3) I^t w_{nx} \right| \\
&\leq \Big(\|(w_n + U(t)V)_x (w_n + U(t)V)\|_{L^\infty} \|w_n + U(t)V\|_{L^\infty} \|w_n + U(t)V\|_{L^2} \\
&\quad + C \|w_n + U(t)V\|_{H^1(1-\psi_0(t))} \Big) \|I^t w_{nx}\|_{L^2} \\
&\leq C \left(\frac{1}{t^{4/3}} + \|w_n\|_{H^1(1-\psi_0(t))} \right) \|I^t w_{nx}\|_{L^2}.
\end{aligned} \tag{52}
$$

For the two other terms, we have to be a little more careful.

- We develop

$$
\begin{aligned}
u_n^3 (I^t u_{nx})_x - R^3 (I^t R_x)_x &= (w_n + U(t)V)^3 ((I^t w_{nx})_x + (I^t U(t) V_x)_x) \\
&\quad + (w_n + U(t)V)^3 (I^t R_x)_x + R^3 ((I^t w_{nx})_x + (I^t U(t) V_x)_x) \\
&\quad + 3(w_n + U(t)V) R (w + U(t)V + R)((I^t w_{nx})_x + (I^t (U(t)V + R)_x)_x).
\end{aligned}
$$

First, split all the terms between those containing $(I w_{nx})_x$ and those with $(IU(t)V_x)_x$ or $(IU(t)V_x)_x$. Now multiply all by $I w_{nx}$, and integrate in x. For the terms containing $(I w_{nx})_x$, further integrate by parts. We get

$$
\begin{aligned}
&\int (u_n^3 (I^t u_{nx})_x - R^3 (I^t R_x)_x) I^t w_{nx} \\
&= -\frac{3}{2} \int (w_n + U(t)V)_x (w_n + U(t)V)^2 (I^t w_{nx})^2 \\
&\quad + \int (w_n + U(t)V)^3 (I^t U(t) V_x)_x I^t w_{nx} + \int (w_n + U(t)V)^3 (I^t R_x)_x I^t w_{nx} \\
&\quad - \frac{1}{2} \int (R^3)_x (I^t w_{nx})^2 + \int R^3 (I^t U(t) V_x)_x I^t w_{nx} \\
&\quad - \frac{1}{2} \int A_x (I^t w_{nx})^2 + \int A (I^t (U(t)V + R)_x)_x) I^t w_{nx},
\end{aligned}
$$

where $A = 3(w_n + U(t)V) R (w_n + U(t)V + R)$. Then the first line is bounded as a regular "linear" term by

$$
\frac{C}{t^{4/3}} \|I^t w_{nx}\|_{L^2}^2 \leq \frac{C}{t^{4/3}} \|I^t w_{nx}\|_{L^2}.
$$

Observe that $(I^t U(t) V_x)_x = (J^t U(t) V_x)_x = (U(t) x V_x)_x$. As $V \in H^{2,2}$, $x V_x \in H^{1,1}$ and $(U(t) x V_x)_x$ has the "almost $t^{-2/3}$" decay of Lemma 1. So that the first term of the second line is bounded by

$$
\frac{C}{t^{4/3}} \|I^t w_{nx}\|_{L^2}.
$$

Notice that uniformly for $t \geq 1$,

$$|I^t R|(x) \leq Ct(1 - \psi_0(t,x)). \tag{53}$$

And the same is true with derivatives on R etc. So that the second term of the second line is bounded by

$$Ct\|w + U(t)V\|_{L^2(1-\psi_0(t))}\|I^t w_{nx}\|_{L^2}.$$

We now have to bound $\int R^3(I^t w_{nx})^2$. This is the key point where we need some result on a H^3 decay on the right for w_n. Indeed, remind that by definition

$$I^t w_{nx} = xw_{nx} + 3tw_{nt} = xw_{nx} - 3tw_{nxxx} - 3t\left(u_n^4 - \sum_{j=1}^{N} R_j^4\right)_x.$$

Proceeding as previously, we naturally obtain ($t \geq I_n \geq 1$)

$$\|RI^t w_{nx}\|_{L^2} \leq Ct\|w_n\|_{H^3(1-\psi_0(t))}. \tag{54}$$

So that :

$$\left|\int R^3(I^t w_{nx})^2\right| \leq Ct\|w_n\|_{H^3(1-\psi_0(t))}\|I^t w_{nx}\|_{L^2}.$$

We go on treating our terms :

$$\left|\int R^3(I^t U(t)V_x)_x I^t w_{nx}\right| \leq \|U(t)(xV_x)_x\|_{L^2(1-\psi_0(t))}\|I^t w_{nx}\|_{L^2}.$$

And for the last line, we have the bound

$$\begin{aligned}&C\|(w_n + U(t)V)R\|_{W^{1,\infty}}\|I^t w_{nx}\|_{L^2}^2\\ &\quad + C\|(w_n + U(t)V)R\|_{L^\infty}(\|(I^t U(t)V_x)_x\|_{L^2} + \|(I^t R_x)_x\|_{L^2})\|I^t w_{nx}\|_{L^2}.\end{aligned}$$

But $\|(I^t U(t)V_x)_x\|_{L^2} = \|U(t)(xV_x)_x\|_{L^2} = \|(xV_x)_x\|_{L^2}$, and $\|(I^t R_x)_x\|_{L^2} \leq Ct$. And of course

$$\|(w + U(t)V)R\|_{W^{1,\infty}} \leq C\|w_n\|_{H^2(1-\psi_0(t))} + \|U(t)V\|_{H^2(1-\psi_0(t))},$$

so that our bound for this last line rewrites

$$C(\|w_n\|_{H^2(1-\psi_0(t))} + \|U(t)V\|_{H^2(1-\psi_0(t))})(\|V\|_{H^{2,2}} + t + 1)\|I^t w_{nx}\|_{L^2}.$$

And for the second term of our main expression, we get ($t \geq 1$)

$$\begin{aligned}&\left|\int (u_n^3(I^t u_{nx})_x - R^3(I^t R_x)_x)I^t w_{nx}\right|\\ &\leq C\left(\frac{1}{t^{4/3}} + t\|w_n\|_{H^3(1-\psi_0(t))} + t\|U(t)V\|_{H^2(1-\psi_0(t))}\right)\|I^t w_{nx}\|_{L^2}.\end{aligned} \tag{55}$$

- We can now turn to the last term :

$$\begin{aligned}&u_{nx}u_n^2 I^t u_{nx} - R_x R^2 I^t R_x\\ &= (w_n + U(t)V)_x(w_n + U(t)V)^2 I^t(w_n + U(t)V)_x\end{aligned}$$

$$
\begin{aligned}
&+ (w_n + U(t)V)_x (w_n + U(t)V)^2 I^t R_x \\
&+ (w_n + U(t)V)_x R (2(w_n + U(t)V)R) \cdot I^t u_{nx} \\
&+ R_x R^2 I^t (w_n + U(t)V)_x + R_x (w_n + U(t)V)(2(w_n + U(t)V) + R) \cdot I^t u_{nx}.
\end{aligned}
$$

Multiply by $I^t w_{nx}$, and integrate in x. Remember that $\|I^t w_{nx}\|_{L^2} \leq \varepsilon$ by assumption on $[I_n, S_n]$, $\|I^t U(t) V_x\|_{L^2} \leq C\|V\|_{H^{1,1}}$ and $I^t R_x\|_{L^2} \leq Ct$, so that $\|I^t u_{nx}\|_{L^2} \leq Ct$. We obtain

$$
\begin{aligned}
\int & (u_{nx} u_n^2 I^t u_{nx} - R_x R^2 I^t R_x) I^t w_{nx} \\
&\leq \|(w_n + U(t)V)_x (w_n + U(t)V)\|_{L^\infty} \|w_n + U(t)V\|_{L^\infty} \\
&\qquad\qquad \times \|I^t (w_n + U(t)V)_x\|_{L^2} \|I^t w_{nx}\|_{L^2} \\
&\quad + \|w_n + U(t)V\|_{L^\infty}^2 \|(w_n + U(t)V)_x I^t R_x\|_{L^2} \|I^t w_{nx}\|_{L^2} \\
&\quad + C\|(w_n + U(t)V)_x R\|_{L^\infty} \|I^t u_{nx}\|_{L^2} \|I^t w_{nx}\|_{L^2} \\
&\quad + \|R_x R^2 I^t (w_n + U(t)V)_x\|_{L^2} \|I^t w_{nx}\|_{L^2} \\
&\quad + C\|(w_n + U(t)V) R_x\|_{L^2} \|I^t u_{nx}\|_{L^2} \|I^t w_{nx}\|_{L^2}.
\end{aligned}
$$

The only non straightforward term is $R_x R^2 I^t (w_n + U(t)V)_x$. Now, analogously to (54), we have

$$
\|R_x R^2 I^t w_{nx}\|_{L^2} \leq Ct \|w_n\|_{H^3(1-\psi_0(t))}.
$$

And we directly get

$$
\|R_x R^2 I^t U(t) V_x\|_{L^2} \leq C\|U(t)(xV_x)_x\|_{L^2(1-\psi_0(t))}.
$$

So that when rewriting the previous estimate, we obtain

$$
\begin{aligned}
\int & (u_{nx} u_n^2 I^t u_{nx} - R_x R^2 I^t R_x) I^t w_{nx} \\
&\leq \frac{C}{t^{p/3}} \|I^t w_{nx}\|_{L^2} + Ct^{1/3} \|w_n\|_{H^1(1-\psi_0(t))} \|I^t w_{nx}\|_{L^2} \\
&\quad + Ct(\|w_n\|_{H^2(1-\psi_0(t))} + \|U(t)V\|_{H^2(1-\psi_0(t))}) \|I^t w_{nx}\|_{L^2} \\
&\quad + C(t\|w_n\|_{H^3(1-\psi_0(t))} + \|U(t)(xV_x)_x\|_{L^2(1-\psi_0(t))}) \|I^t w_{nx}\|_{L^2} \\
&\quad + Ct\|w_n + U(t)V\|_{L^2(1-\psi_0(t))} \|I^t w_{nx}\|_{L^2}.
\end{aligned}
\tag{56}
$$

Step 3. Let us now conclude the $I^t w_{nx}$ estimate : we add up the results of (52), (55), and (56), plug them in (51), and get

$$
\begin{aligned}
\left| \frac{1}{2} \frac{d}{dt} \|I^t w_{nx}\|_{L^2}^2 \right| \leq C \Bigg(& \frac{1}{t^{4/3}} + t\|w_n\|_{H^3(1-\psi_0(t))} + t\|U(t)V\|_{H^2(1-\psi_0(t))} \\
& + \|U(t)(xV_x)_x\|_{L^2(1-\psi_0(t))} \Bigg) \|I^t w_{nx}\|_{L^2}.
\end{aligned}
$$

So that after integration in time between t and S_n, we have

$$
\|I^t w_{nx}\|_{L^2} \leq \frac{C}{t^{4/3}}, \tag{57}
$$

as soon as

$$t\|w_n\|_{H^3(1-\psi_0(t))} + t\|U(t)V\|_{H^2(1-\psi_0(t))} + \|U(t)(xV_x)_x\|_{L^2(1-\psi_0(t))} \le \frac{C}{t^{4/3}}.$$

Notice that thanks to $DI^t w_n = I^t w_{nx} + w_{nx}$, we also have

$$\|DI^t w_n\|_{L^2} \le \frac{C}{t^{1/3}}.$$

This will be useful for the following of the proof.

$\|D^\alpha I^t w_n\|_{L^2}$ *estimate.*
Step 1 and 2. Let us compute

$$\begin{aligned} LI^t w_n = I^t L w_n + 3\int L w_n &= -I^t(u_n^4 - \sum_{j=1}^N R_j^4)_x - 3(u_n^4 - \sum_{j=1}^N R_j^4) \\ &= -4(u_n^3 I^t u_{nx} - R^3 I^t R_x) - 3(u_n^4 - R^4) \\ &\quad - 4(R^3 I^t R_x - \sum_{j=1}^N R_j^3 I^t R_{j\,x}) - 3(R^4 - \sum_{j=1}^N R_j^4). \end{aligned} \tag{58}$$

What we want is then to apply operator D^α to our equation, multiply both sides by $D^\alpha I^t w_n$ and integrate in x : we get $\frac{1}{2}\|D^\alpha I^t w_n\|_{L^2}^2$ on the left hand side, and we are to do some estimations on the right hand side. As we already have an estimate on $DI^t w_n$ we can avoid a discussion on the behavior of D^α with respect to a product of functions. Indeed, apart from the purely "linear term" which is treated as in [4], we will use

$$|(D^\alpha h, D^\alpha I^t w_n)| = |(h, D^{2\alpha} I^t w_n)| \le \|h\|_{L^2}(\|D^\alpha I^t w_n\| + \|DI^t w_n\|_{L^2}). \tag{59}$$

(as $\alpha < 1/2$). Now, let us bound the terms in (58).
• First :

$$\left\| -4(R^3 I^t R_x - \sum_{j=1}^N R_j^3 I^t R_{j\,x}) - 3(R^4 - \sum_{j=1}^N R_j^4)\right\|_{L^2} \le Cte^{-\frac{\sigma\sqrt{\sigma_0}}{4}t}. \tag{60}$$

• Second :

$$u_n^4 - R^4 = (w_n + U(t)V)^4 + (w_n + U(t)V)R \cdot \left(\sum_{k=1}^3 C_4^k (w_n + U(t)V)^{k-1} R^{3-k}\right).$$

From this we get (using (59) on the second term)

$$\begin{aligned} |(D^\alpha(u_n^4 - R^4), D^\alpha I^t w_n)| &\le |(D^\alpha(w_n + U(t)V)^4, D^\alpha I^t w_n)| \\ &\quad + \|(w + U(t)V)\|_{L^2(1-\psi_0(t))}(\|D^\alpha I^t w_n\|_{L^2} + \|DI^t w_n\|_{L^2}). \end{aligned}$$

Now, thanks to the first estimate of Lemma 2 with $g = w_n + U(t)V$, we get

$$\|D^\alpha(w_n + U(t)V)^4\|_{L^2} \le \|g\|_{L^6}^3 \left(\|g_x g\|_{L^\infty}^{1/2} + \|g\|_{L^\infty}^{3\gamma}\|g_x g\|_{L^\infty}^{(1-3\gamma)/2}\right)$$

$$\leq \frac{C}{t^{1-\frac{1}{6}}}\left(\frac{1}{t^{\frac{1}{2}}}+\frac{1}{t^{\gamma}}\cdot\frac{1}{t^{\frac{1-3\gamma}{2}}}\right)\leq \frac{C}{t^{4/3-\gamma/2}}.$$

So that

$$\begin{aligned}|(D^{\alpha}(u_n^4-R^4),D^{\alpha}I^t w_n)| &\leq \frac{C}{t^{1/3}}\|w\|_{H^1(1-\psi_0(t))}\\ &\quad+\left(\frac{C}{t^{4/3-\gamma/2}}+\|w\|_{H^1(1-\psi_0(t))}\right)\|D^{\alpha}I^t w_n\|_{L^2}. \quad (61)\end{aligned}$$

- And for the last remaining term (the first in the expression of $LI^t w_n$),

$$\begin{aligned}&(u_n^3 I u_{nx}-R^3 I^t R_x)\\ &\quad=(w_n+U(t)V)^3 I^t(w_n+U(t)V)_x+(w_n+U(t)V)^3 I^t R_x\\ &\qquad+Ru_n^2 I^t(w_n+U(t)V)_x+R(w_n+U(t)V)(w_n+U(t)V+2R)I^t R_x.\end{aligned}$$

Consider the fist term of the right hand side. Using the second estimate of Lemma 2 in an analogous way as for (61), with $g=w_n+U(t)V$ and $h=I^t w_n+U(t)V$, we have

$$\begin{aligned}&\|D^{\alpha}(w_n+U(t)V)^3 I^t(w_n+U(t)V)_x\|_{L^2}\\ &\qquad\leq C\left(\frac{1}{t^{1/3}}\cdot\frac{1}{t}+\frac{1}{t^{(1-2\gamma)/3}}\cdot\frac{1}{t}+\frac{1}{t^{(1+2\gamma)/3}}\cdot\frac{1}{t^{1-\gamma}}\right)\leq\frac{C}{t^{4/3-2\gamma/3}}.\end{aligned}$$

For all the other terms, we use (59), so that we are looking for an L^2 control.

$$\|(w_n+U(t)V)^3 I^t R_x\|_{L^2}\leq Ct\|w+U(t)V\|_{L^2(1-\psi_0(t))},$$

$$\begin{aligned}&\left\|R\cdot\left(\sum_{k=0}^{2}C_2^k(w_n+U(t)V)^k R^{2-k}\right)\cdot I^t(w_n+U(t)V)_x\right\|_{L^2}\\ &\qquad\leq C\|w_n\|_{H^3(1-\psi_0(t))}+C\|U(t)xV_x\|_{L^2(1-\psi_0(t))},\end{aligned}$$

$$\begin{aligned}&\left\|R(w_n+U(t)V)\cdot\left(\sum_{k=1}^{2}C_2^k(w_n+U(t)V)^{k-1}R^{2-k}\right)\cdot I^t R_x.\right\|_{L^2}\\ &\qquad\leq Ct\|(w+U(t)V\|_{L^2(1-\psi_0(t))}.\end{aligned}$$

And for this last term, we get (using 57))

$$\begin{aligned}&|(D^{\alpha}(u_n^3 I u_{nx}-R^3 I^t R_x),D^{\alpha}I^t w_{nn})|\\ &\leq\left(\frac{1}{t^{4/3-2\gamma/3}}+t\|w_n\|_{H^1(1-\psi_0(t)}+\|w_n\|_{H^3(1-\psi_0(t))}+\|U(t)xV_x\|_{L^2(1-\psi_0(t))}\right)\\ &\quad\times C\left(\frac{1}{t^{1/3}}+\|D^{\alpha}I^t w_{nn}\|_{L^2}\right). \quad (62)\end{aligned}$$

Step 3. We can now sum up the results of (60), (61) and (62), and obtain

$$\begin{aligned}\left|\frac{d}{dt}\|D^{\alpha}I^t w_n\|_{L^2}^2\right|&\leq C\left(\frac{1}{t^{4/3-2\gamma/3}}+t\|w_n\|_{H^1(1-\psi_0(t))}+\|w_n\|_{H^3(1-\psi_0(t))}\right.\\ &\quad\left.+\|U(t)xV_x\|_{L^2(1-\psi_0(t))}\right)\left(\frac{1}{t^{1/3}}+\|D^{\alpha}I^t w_n\|_{L^2}\right).\end{aligned}$$

So that after integration in time between t and S_n, we get

$$\|D^\alpha I^t w_n\|_{L^2} \leq \frac{C}{t^{(1-2\gamma)/3}} = \frac{C}{t^\delta}, \tag{63}$$

as soon as

$$t\|w_n\|_{H^3(1-\psi_0(t))} + t\|U(t)V\|_{H^2(1-\psi_0(t))} + \|U(t)(xV_x)\|_{H^1(1-\psi_0(t))} \leq \frac{C}{t^{p/3}}.$$

(condition for both estimates (57) and (63)).

6.2 $J^t w_n$ estimates

We only need to go from our previous estimates (63) and (57) to estimates on $J^t w_n$. First remind that $I^t f(x) - J^t f(x) = 3t\int_{-\infty}^x Lf$. Thus

$$\|D^\alpha J^t w_n\|_{L^2} + \|DJ^t w_n\|_{L^2} \leq \|D^\alpha I^t w_n\|_{L^2} + \|I^t w_{nx}\|_{L^2} + t\|D^\alpha u_n^4 - D^\alpha R^4\|_{L^2}$$
$$+ t\|Du_n^4 - DR^4\|_{L^2} + t\left\|D^\alpha\left(R^4 - \sum_{j=1}^N R_j^4\right)\right\|_{L^2} + t\left\|D\left(R^4 - \sum_{j=1}^N R_j^4\right)\right\|_{L^2}.$$

From (57) and (63), we have

$$\|D^\alpha I^t w_n\|_{L^2} + \|I^t w_{nx}\|_{L^2} \leq Ct^{-\delta}.$$

Obviously, we also have

$$t\left\|D^\alpha\left(R^4 - \sum_{j=1}^N R_j^4\right)\right\|_{L^2} + t\left\|D\left(R^4 - \sum_{j=1}^N R_j^4\right)\right\|_{L^2} \leq Cte^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}.$$

Now consider

$$t\|D^\alpha u_n^4 - D^\alpha R^4\|_{L^2} + t\|Du_n^4 - DR^4\|_{L^2} \leq t\|u_n^4 - (w_n + U(t)V)^4 - R^4\|_{H^1}$$
$$+ t\|D^\alpha (w_n + U(t)V)^4\|_{L^2} + 4t\|(w_n + U(t)V)_x (w_n + U(t)V)^3\|_{L^2}.$$

Using again the first estimate of Lemma 2 with $g = w_n + U(t)V$ (see (61)) :

$$t\|D^\alpha (w_n + U(t)V)^4\|_{L^2} \leq t\frac{C}{t^{4/3-\gamma/2}} \leq \frac{C}{t^{1/3-\gamma/2}} \leq \frac{C}{t^\delta}.$$

And also,

$$t\|(w_n + U(t)V)_x (w_n + U(t)V)^3\|_{L^2} \leq Ct\|(w_n + U(t)V)_x (w_n + U(t)V)\|_{L^\infty}$$
$$\times \|w_n + U(t)V\|_{L^\infty} \|w_n + U(t)V\|_{L^2} \leq \frac{C}{t^{1/3}}.$$

Finally

$$u_n^4 - (w_n + U(t)V)^4 - R^4 = (w_n + U(t)V)R \cdot \left(\sum_{k=1}^3 C_4^k (w_n + U(t)V)^{k-1} R^{3-k}\right)$$
$$= (w_n + U(t)V)RA,$$

where $\|A\|_{H^1} \leq C$. As H^1 is an algebra,

$$\|u_n^4 - (w_n + U(t)V)^4 - R^4\|_{L^2} \leq \|(w_n + U(t)V)R\|_{H^1}\|A\|_{H^1} \leq C\|w_n\|_{H^1(1-\psi_0(t))}.$$

And we are done as soon as $\|w_n\|_{H^1(1-\psi_0(t))} \leq Ct^{-4/3}$.

Finally we obtained

$$\|D^\alpha J^t w_n\|_{L^2} + \|DJ^t w_n\|_{L^2} \leq Ct^{-\delta}.$$

This concludes the proof of Proposition 1', and thus of Proposition 1.

Appendix. H^3 and H^4 uniform decay estimates on $w_n(t)$

We complete the proof of 3, by giving the detailed proof of the H^3 and H^4 estimates.

Proof of Proposition 3, H^3 and H^4 cases. $\dot{H}^3$ estimate.

Step 1 : deriving the H^3 almost conservation law. Let us differentiate (11) three times :

$$w_{nxxxt} + w_{nxxxxxx} + \left(u_n^4 - \sum_{j=1}^N R_j^4\right)_{xxxx} = 0.$$

We multiply it by w_{nxxx}, and do an integration by parts, to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int w_{nxxx}^2 &= \int \left(u_n^4 - \sum_{j=1}^N R_j^4\right)_{xxx} w_{nxxxx} \\ &= \int \left(u_n^4 - R^4\right)_{xxx} w_{nxxxx} + \int \left(R^4 - \frac{C}{t^a}\sum_{j=1}^N R_j^4\right)_{xxxx} w_{nxxx}.\end{aligned}$$

The second integral is harmless. Let us develop the first term :

$$\begin{aligned}\left(u_n^4 - R^4\right)_{xxx} &= 4(u_{nxxx}u_n^3 - R_{xxx}R^3) + 36(u_{nxx}u_{nx}u_n^2 - R_{xx}R_xR^2) \\ &\quad + 24(u_{nx}^3 u_n - R_x^3 R) \\ &= 4w_{nxxx}u_n^3 + 4((U(t)V + R)_{xx}u_n^3 - R_{xx}R^3) \\ &\quad + 36(u_{nxx}u_{nx}u_n^2 - R_{xx}R_xR^2) + 24(u_{nx}^3 u_n - R_x^3 R).\end{aligned}$$

We try to get rid of the w_{nxxxx} terms, by integration by parts.

$$\begin{aligned}&\int \left(u_n^4 - R^4\right)_{xxx} w_{nxxxx} \\ &= -6\int w_{nxxx}^2 u_{nx}u_n^2 - 4\int w_{nxxx}((U(t)V + R)_{xxxx}u_n^3 - R_{xxxx}R^3) \\ &\quad - 12\int w_{nxxx}((U(t)V + R)_{xxx}u_{nx}u_n^2 - R_{xxx}R_xR^2) - 36\int w_{nxxx}^2 u_{nx}u_n^2 \\ &\quad - 36\int w_{nxxx}((U(t)V + R)_{xxx}u_{nx}u_n^2 - R_{xxx}R_xR^2) \\ &\quad - 36\int w_{nxxx}(u_{nxx}^2 u_n^2 - R_{xx}^2 R^2) - 144\int w_{nxxx}(u_{nxx}u_{nx}^2 u_n - R_{xx}R_x^2 R)\end{aligned}$$

$$- 24 \int w_{nxxx}({u_n}_x^4 - R_x^4).$$

We now get the troublesome term $-42 \int {w_n}_{xxx}^2 u_{nx} u_n^2$. We thus introduce

$$\begin{aligned}
\frac{d}{dt} \int {w_n}_{xx}^2 u_n^3 &= 2 \int w_{nxxt} w_{nxx} u_n^3 + 3 \int {w_n}_{xx}^2 u_{nt} u_n^2 \\
&= -2 \int w_{nxxxxx} w_{nxx} u_n^3 - \int \left(u_n^4 - \sum_{j=1}^n R_j^4 \right)_{xxx} w_{nxx} u_n^3 \\
&\quad - 3 \int {w_n}_{xx}^2 u_{nxxx} u_n^2 - 12 \int {w_n}_{xx}^2 u_{nx} u_n^5.
\end{aligned}$$

First :

$$\begin{aligned}
&- \int \left(u_n^4 - \sum_{j=1}^n R_j^4 \right)_{xxx} w_{nxx} u_n^3 \\
&\qquad = - \int (u_n^4 - R^4)_{xxx} w_{nxx} u_n^3 - \int \left(R^4 - \sum_{j=1}^n R_j^4 \right)_{xxx} w_{nxx} u_n^3,
\end{aligned}$$

where the second integral will be treated as usual. Now we rearrange the term with high derivatives (more than 3) through integrations by parts.

$$\begin{aligned}
&-2 \int w_{nxxxxx} w_{nxx} u_n^3 \\
&= 2 \int w_{nxxxx} w_{nxxx} u_n^3 + 6 \int w_{nxxxx} w_{nxx} u_{nx} u_n^2 \\
&= -9 \int {w_n}_{xxx}^2 u_{nx} u_n^2 - 6 \int w_{nxxx} w_{nxx} u_{nxx} u_n^2 - 12 \int w_{nxxx} w_{nxx} {u_n}_x^2 u_n.
\end{aligned}$$

So that we get

$$\begin{aligned}
&\frac{d}{dt} \int {w_n}_x^2 u_n^3 \\
&\quad = -9 \int {w_n}_{xx}^2 u_{nx} u_n^2 - 24 \int w_{nxx} w_{nx} {u_n}_x^2 u_n - \int (u_n^4 - R^4)_{xx} w_{nx} u_n^3 \\
&\qquad - \int \left(R^4 - \sum_{j=1}^n R_j^4 \right)_{xx} w_{nx} u_n^3 - 12 \int {w_n}_x^2 u_{nx} u_n^5.
\end{aligned}$$

We derived the desired relation on w_{nn} at level $\dot{H}^3$:

$$\begin{aligned}
&\frac{d}{dt} \left(\frac{1}{2} \int {w_n}_{xxx}^2 - \frac{28}{3} \int {w_n}_{xx}^2 u_n^3 \right) \\
&= -4 \int w_{nxxx}((U(t)V + R)_{xxxx} u_n^3 - R_{xxxx} R^3) \\
&\quad - 48 \int w_{nxxx}((U(t)V + R)_{xxx} u_{nx} u_n^2 - R_{xxx} R_x R^2) \\
&\quad - 36 \int w_{nxxx}({u_n}_{xx}^2 u_n^2 - R_{xx}^2 R^2) - 144 \int w_{nxxx}(u_{nxx} {u_n}_x^2 u_n - R_{xx} R_x^2 R)
\end{aligned}$$

$$
\begin{aligned}
&- 24\int w_{nxxx}({u_n}_x^4 - R_x^4) + 52\int w_{nxxx}w_{nxx}u_{nxx}u_n^2\\
&+ 104\int w_{nxxx}w_{nxx}{u_n}_x^2 u_n + 28\int {w_n}_{xx}^2 u_{nxxx}u_n^2\\
&- 112\int {w_n}_{xx}^2 u_{nx}u_n^5 + \frac{28}{3}\int (u_n^4 - R^4)_{xxx}w_{nxx}u_n^3\\
&- \frac{28}{3}\int \left(R^4 - \sum_{j=1}^{n} R_j^4\right)_{xxx} w_{nxx}u_n^3 + \int \left(R^4 - \sum_{j=1}^{N} R_j^4\right)_{xxxx} w_{nxxx}. \quad (64)
\end{aligned}
$$

Step 2. Estimating terms in (64). There are 10 lines to consider. From now on, A_i, A_i', A_i'', ... will denote a polynomial in w_n, $U(t)V$, R and their derivatives (involved for the term on line i), defining a function whose properties are given right after we introduced it.

- $\int w_{nxxx}((U(t)V+R)_{xxx}u_n^3 - R_{xxx}R^3)$.

$$
\begin{aligned}
\|(U(t)V+R)_{xxxx}u_n^3 - R_{xxxx}R^3\|_{L^2} \le{}& \|U(t)V_{xxxx}(w_n+U(t)V)^3\|_{L^2}\\
&+ \|U(t)V_{xxxx}RA_1\|_{L^2} + \|R_{xxxx}(w_n+U(t)V)A_1'\|_{L^2}.
\end{aligned}
$$

with $\|A_1\|_{L^\infty} + \|A_1'\|_{L^\infty} \le C$. Using that $V \in H^{4,1}$, that is $V_{xxx} \in H^{1,1}$, we get

$$
\begin{aligned}
&\left|\int w_{nxxx}((U(t)V+R)_{xxxx}u_n^3 - R_{xxxx}R^3)\right|\\
&\qquad \le C\left(\frac{1}{t^{4/3}} + \|U(t)V\|_{H^4(1-\psi_0(t))} + \|w_n\|_{L^2(1-\psi_0(t))}\right)\|w_{nxxx}\|_{L^2}. \quad (65)
\end{aligned}
$$

- $\int w_{nxxx}((U(t)V+R)_{xxx}u_{nx}u_n^2 - R_{xxx}R_xR^2)$.

$$
\begin{aligned}
&(U(t)V+R)_{xxx}u_{nx}u_n^2 - R_{xxx}R_xR^2 =\\
&\quad U(t)V_{xxx}(w_n+U(t)V)_x(w_n+U(t)V)^2 + U(t)V_{xxx}(w_n+U(t)V)_xRA_2\\
&\qquad + U(t)V_{xxx}R_xu_n^2 + R_{xxx}(w_n+U(t)V)_xu_n^2 + R_{xxx}R_x(w_n+U(t)V)A_2',
\end{aligned}
$$

with $\|A_1\|_{L^\infty} + \|A_1'\|_{L^\infty} \le C$. For the "linear", we bound $U(t)V_{xxx}$ in L^2 and the rest using the point wise estimates of lemma 1, and obtain

$$
\begin{aligned}
&\left|\int w_{nxxx}((U(t)V+R)_{xxx}u_{nx}u_n^2 - R_{xxx}R_xR^2\right|\\
&\qquad \le C\left(\frac{1}{t^{4/3}} + \|U(t)V\|_{H^3(1-\psi_0(t))} + \|w_n\|_{H^1(1-\psi_0(t))}\right)\|w_{nxxx}\|_{L^2}. \quad (66)
\end{aligned}
$$

- $\int w_{nxxx}({u_n}_{xx}^2u_n^2 - R_{xx}^2R^2)$.

$$
\begin{aligned}
{u_n}_{xx}^2 u_n^2 - R_{xx}^2 R^2 &= (w_n + U(t)V)_{xx}^2 (w_n + U(t)V)^2 \\
&\quad + (w_n + U(t)V)_{xx}^2 R A_3 + 2(w_n + U(t)V)_{xx} R_{xx} u_n^2 + R_{xx}^2 (w_n + U(t)V) A_3',
\end{aligned}
$$

with $\|A_3\|_{L^\infty} + \|A_3'\|_{L^\infty} \leq C$. The second line is bounded in L^2 norm by $\|w_n + U(t)V\|_{H^2(1-\psi_0(t))}$. The first term needs some attention, and the use of the estimate $\|w_{nxx}\|_{L^2} \leq Ct^{-1/3}$ obtained earlier.

$$
\begin{aligned}
&\left| \int w_{nxxx} (w_n + U(t)V)_{xx}^2 (w_n + U(t)V)^2 \right| \\
&\quad \leq C \|w_{nxxx}\|_{L^2} \|w_{xx}\|_{L^4}^2 \|w_n + U(t)V\|_{L^\infty}^2 \\
&\qquad + C \|w_{nxxx}\|_{L^2} \|U(t)V_{xx}\|_{L^2} \|U(t)V_{xx}(w_n + U(t)V)\|_{L^\infty} \|w_n + U(t)V\|_{L^\infty} \\
&\quad \leq C \|w_{nxxx}\|_{L^2} \left(\|w_{nxx}\|_{L^2}^{3/2} \|w_{xxx}\|_{L^2}^{1/2} \frac{1}{t^{2/3}} + \frac{1}{t^{4/3}} \right) \\
&\quad \leq \frac{C}{t^{4/3}} \|w_{nxxx}\|_{L^2} + \frac{C}{t^{7/6}} \|w_{nxxx}\|_{L^2}^{3/2}.
\end{aligned}
$$

(we used $V \in H^{3,1}$). And for this term :

$$
\begin{aligned}
&\left| \int w_{nxxx} ({u_n}_{xx}^2 u_n^2 - R_{xx}^2 R^2) \right| \\
&\leq C \left(\frac{1}{t^{4/3}} + \|w_n + U(t)V\|_{H^2(1-\psi_0(t))} \right) \|w_{nxxx}\|_{L^2} + \frac{C}{t^{7/6}} \|w_{nxxx}\|_{L^2}^{3/2}. \qquad (67)
\end{aligned}
$$

- $\int w_{nxxx} (u_{nxx} {u_n}_x^2 u_n - R_{xx} R_x^2 R)$.

$$
\begin{aligned}
&u_{nxx} {u_n}_x^2 u_n - R_{xx} R_x^2 R \\
&= (w_n + U(t)V)_{xx} (w_n + U(t)V)_x^2 (w_n + U(t)V) \\
&\quad + (w_n + U(t)V)_{xx} (w_n + U(t)V)_x^2 R A_4 \\
&\quad + (w_n + U(t)V)_{xx} R_x (2(w_n + U(t)V)_x + R_x) u_n + R_{xx} R_x^2 (w_n + U(t)V) A_4' \\
&\quad + R_{xx} (w_n + U(t)V)_x (2(w_n + U(t)V)_x + R_x) u_n,
\end{aligned}
$$

with $\|A_4\|_{L^\infty} + \|A_4'\|_{L^\infty} \leq C$. Let aside the first term, all the others are bounded in L^2 norm by $\|w_n + U(t)V)_x\|_{H^2(1-\psi_0(t))}$. Now for the remaining first term

$$
\begin{aligned}
&\|(w_n + U(t)V)_{xx} (w_n + U(t)V)_x^2 (w_n + U(t)V)\|_{L^2} \\
&\leq \|(w + U(t)V)_{xx}\|_{L^2} \|(w_n + U(t)V)_x\|_{L^\infty} \|(w_n + U(t)V)_x (w_n + U(t)V)\|_{L^\infty}.
\end{aligned}
$$

Now $\|w_{nx}\|_{L^\infty} \leq Ct^{1/3}$ by interpolation, and as $V \in H^{2,2}$, $V_x \in L^1$ so that $\|U(t)V_x\|_{L^\infty} \leq Ct^{-1/3}$. So that our term bounded by

$$
Ct^{-1/3} t^{-1} \leq Ct^{-4/3},
$$

and we get

$$
\left| \int w_{nxxx} (u_{nxx} {u_n}_x^2 u_n - R_{xx} R_x^2 R) \right|
$$

$$\leq C\left(\frac{1}{t^{4/3}} + \|w_n + U(t)V\|_{H^2(1-\psi_0(t))}\right)\|w_{nxxx}\|_{L^2}. \quad (68)$$

- $\displaystyle\int w_{nxxx}({u_n}_x^4 - R_x^4)$.

$$ {u_n}_x^4 - R_x^4 = (w_n + U(t)V)_x^4 + (w_n + U(t)V)_x^4 R_x A_5,$$

where A_5 has factors with 1 derivative. As $\|(w_n + U(t)V)_x\|_{L^\infty} \leq \|w_n + U(t)V\|_{H^2} \leq C$, $\|A_5\|_{L^\infty} \leq C$. With the same estimate, we get that the last two terms are bounded in L^2 norm by $\|w_n + U(t)V\|_{H^1(1-\psi_0(t))}$. For the very first term, notice that

$$\|(w_n + U(t)V)_x^4\|_{L^2} \leq C\|w_{nx}\|_{L^8}^4 + \|U(t)V_x\|_{L^2}\|U(t)V_x\|_{L^\infty}^3 \leq \frac{C}{t^{4/3}} + \frac{C}{t^{\frac{3}{2}}}.$$

Indeed, we interpolate $\|w_{nx}\|_{L^8}$ between $\|w_{nx}\|_{L^2}$ and $\|w_{nxx}\|_{L^2}$, which both get decay rate of $Ct^{-1/3}$, so that $\|w_{nx}\|_{L^8} \leq Ct^{-1/3}$. Furthermore,

$$\|U(t)V_x^2\|_{L^\infty} \leq \frac{C}{t} M_0^t(U(t)V) M_0^t(U(t)V_x) \leq \frac{C}{t}\|V\|_{H^{2,2}},$$

hence the second estimate. And we have

$$\left|\int w_{nxxx}({u_n}_x^4 - R_x^4)\right| \leq C\left(\frac{1}{t^{4/3}} + \|w_n + U(t)V\|_{H^2(1-\psi_0(t))}\right)\|w_{nxxx}\|_{L^2}. \quad (69)$$

- $\displaystyle\int w_{nxxx} w_{nxx} u_{nxx} u_n^2$.

$$\begin{aligned} u_{nxx}u_n^2 = {} & w_{nxx}(w_n + U(t)V)^2 + U(t)V_{xx}(w_n + U(t)V)^2 \\ & + (w_n + U(t)V)_{xx} R A_6 + R_{xx} u_n^2, \end{aligned}$$

with $\|A_6\|_{L^\infty} \leq C$. Then we compute :

$$\begin{aligned} \left|\int w_{nxxx} w_{nxx}^2 (w_n + U(t)V)^2\right| &\leq \|w_{nxxx}\|_{L^2}\|w_{nxx}\|_{L^4}^2\|w_n + U(t)V\|_{L^\infty}^2 \\ &\leq \frac{C}{t^{1/3\cdot 3/2 + 2/3}}\|w_{nxxx}\|_{L^2}^{3/2} \leq \frac{C}{t^{7/6}}\|w_{nxxx}\|_{L^2}^{3/2}. \end{aligned}$$

($\|w_{nxx}\|_{L^4} \leq \|w_{nxx}\|_{L^2}^{3/4}\|w_{nxxx}\|_{L^2}^{1/4}$). For the second term, as $V_x \in H^{1,1}$,

$$\begin{aligned} &\left|\int w_{nxxx} w_{nxx} U(t)V_{xx}(w_n + U(t)V)^2\right| \\ &\leq \|w_{nxxx}\|_{L^2}\|w_{nxx}\|_{L^2}\|U(t)V_{xx}(w_n + U(t)V)\|_{L^\infty}\|w_n + U(t)V\|_{L^\infty} \\ &\leq \frac{C}{t^{5/3}}\|w_{nxxx}\|_{L^2}. \end{aligned}$$

And for the last two terms, as $\|w_{nxx}\|_{L^\infty} \leq \|w_{nxx}\|_{L^2}^{1/2}\|w_{nxxx}\|_{L^2}^{1/2}$,

$$\left|\int w_{nxxx} w_{nxx} (w_n + U(t)V)_{xx} R A_6\right| + \left|\int w_{nxxx} w_{nxx} R_{xx} u_n^2\right|$$

$$\leq \|w_{nxxx}\|_{L^2}\left(\|w_{nxx}\|_{L^\infty}\|w_n+U(t)V\|_{H^2(1-\psi_0(t))}+\|_{L^2}\|w_{nxx}\|_{H^2(1-\psi_0(t))}\right)$$
$$\leq \|w_n+U(t)V\|_{H^2(1-\psi_0(t))}\|w_{nxxx}\|_{L^2}^{3/2}+\|w_{nxx}\|_{L^2(1-\psi_0(t))}\|w_{nxxx}\|_{L^2}.$$

Therefore, for the whole term :

$$\left|\int w_{nxxx}w_{nxx}u_{nxx}u_n^2\right|\leq C\left(\frac{1}{t^{(5/3}}+\|w_{nxx}\|_{L^2(1-\psi_0(t))}\right)\|w_{nxxx}\|_{L^2}$$
$$+C\left(\frac{1}{t^{7/6}}+\|w_n+U(t)V\|_{H^2(1-\psi_0(t))}\right)\|w_{nxxx}\|_{L^2}^{3/2}. \quad (70)$$

- $\displaystyle\int w_{nxxx}w_{nxx}{u_{n}}_x^2u_n.$

$${u_n}_x^2u_n=(w_n+U(t)V)_x^2(w_n+U(t)V)$$
$$+(w_n+U(t)V)_x^2RA_7+(w_n+U(t)V)_xR_xu_n+R_x^2u_n,$$

with $\|A_7\|_{L^\infty}\leq C$. As $\|(w_n+U(t)V)_x\|_{L^\infty}\leq C$ we get

$$\left|\int w_{nxxx}w_{nxx}(w_n+U(t)V)_x^2(w_n+U(t)V)\right|$$
$$\leq \|w_{nxxx}\|_{L^2}\|w_{nxx}\|_{L^2}\|(w_n+U(t)V)_x\|_{L^\infty}\|(w_n+U(t)V)_x(w_n+U(t)V)\|_{L^\infty}$$
$$\leq \frac{C}{t^{4/3}}\|w_{nxxx}\|_{L^2},$$

and for the remaining terms, we clearly have

$$\left|\int w_{nxxx}w_{nxx}(w_n+U(t)V)_x^2RA_7+(w_n+U(t)V)_xR_xu_n+R_x^2u_n\right|$$
$$\leq \|w_{nxxx}\|_{L^2}\|w_{nxx}\|_{L^2(1-\psi_0(t))}.$$

So that

$$\left|\int w_{nxxx}w_{nxx}{u_n}_x^2u_n\right|\leq C\left(\frac{1}{t^{4/3}}+\|w_{nxx}\|_{L^2(1-\psi_0(t))}\right)\|w_{nxxx}\|_{L^2}. \quad (71)$$

- $\displaystyle\int {w_n}_{xx}^2u_{nx}u_n^5+\int(u_n^4-R^4)_{xxx}w_{nxx}u_n^3.$

$$u_{nx}u_n^5=(w_n+U(t)V)_x(w_n+U(t)V)^5+(w_n+U(t)V)_xRA_9+R_xu_n^5,$$

with $\|A_8\|_{L^\infty}\leq C$. So that we get directly

$$\left|\int {w_n}_{xx}^2u_{nx}u_n^5\right|\leq\frac{C}{t^{2/3+1+4/3}}+\|w_{nxx}\|_{L^2(1-\psi_0(t))}\|w_n+U(t)V\|_{H^1(1-\psi_0(t))}$$
$$+C\|w_{nxx}\|_{L^2(1-\psi_0(t))}^2.$$

Now for the right term

$$\int (u_n^4 - R^4)_{xxx} w_{nxx} u_n^3 = -\int (u_n^4 - R^4)_{xx} w_{nxxx} u_n^3 \\ - 3\int (u_n^4 - R^4)_{xx} w_{nxx} u_{nx} u_n^2.$$

As
$$(u_n^4 - R^4)_{xx} = 4(u_{nxx}u_n^3 - R_{xx}R^2) + 12({u_{nx}}^2 u_n^2 - R_x^2 R^2),$$
we get that :

$$\begin{aligned}&(u_n^4 - R^4)_{xx} u_n^3 = \Big(4\big(w_{nxx}(w_n + U(t)V)^3 + U(t)V_{xx}(w_n + U(t)V)^3 \\ &+ (w_n + U(t)V)_{xx} R A_8' + R_{xx}(w_n + U(t)V)A_8''\big) \\ &+ 12\big((w_n + U(t)V)_x^2 (w_n + U(t)V)^2 + (w_n + U(t)V)_x^2 R A_8''' \\ &+ 2(w_n + U(t)V)_x R_x u_n^3 + R_{xx}^2 (w_n + U(t)V) A_8''''\big)\Big)(A_8''''' R + (w_n + U(t)V)^3),\end{aligned}$$

where all the $A_8'^{\cdots\prime}$ are bounded in L^∞. Now when developing carefully, we get that :

$$\left|\int (u_n^4 - R^4)_{xxx} w_{nxx} u_n^3\right| \le \frac{C}{t^{8/3}} + C\|w_n\|^2_{H^2(1-\psi_0(t))} + C\|U(t)V\|^2_{H^2(1-\psi_0(t))} \tag{72}$$

- $\displaystyle\int \left(R^4 - \sum_{j=1}^n R_j^4\right)_{xxx} w_{nxx} u_n^3$ and $\displaystyle\int \left(R^4 - \sum_{j=1}^N R_j^4\right)_{xxxx} w_{nxxx}$.

  We obviously have exponential decay :

$$\left|\int \left(R^4 - \sum_{j=1}^n R_j^4\right)_{xxx} w_{nxx} u_n^3\right| + \left|\int \left(R^4 - \sum_{j=1}^N R_j^4\right)_{xxxx} w_{nxxx}\right| \\ \le Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|w_n\|_{H^3}. \tag{73}$$

And finally :

- $\displaystyle\int {w_{nxx}}^2 u_n^3$.

  As $u_n^3 = (w_n + U(t)V)^3 + RA_{10}$, with $\|A_{10}\|_{L^\infty} \le C$, we have :

$$\left|\int {w_{nxx}}^2 u_n^3\right| \le \frac{C}{t^{2/3+1}} + C\|w_{nxx}\|^2_{L^2(1-\psi_0(t))} \le \frac{C}{t^{5/3}} + C\|w_{nxx}\|^2_{L^2(1-\psi_0(t))}. \tag{74}$$

Step 3. We can now conclude our estimate of 64. Let us sum all our estimates (65)-(73). Then let us integrate in time between t and S_n, and plug in (74). We get

$$\begin{aligned}&\|w_{nxxx}\|^2_{L^2} \\ &\le \frac{C}{t^{5/3}} + C\|w_{nxx}\|^2_{L^2(1-\psi_0(t))} + C\int_t^{S_n} \frac{\|w_{nxxx}(\tau)\|^{3/2}_{L^2}}{t^{7/6}} d\tau \\ &\quad + C\int_t^{S_n} \left(\frac{1}{t^{4/3}} + \|w_n\|_{H^2(1-\psi_0(t))} + \|U(t)V\|_{H^3(1-\psi_0(t))}\right)\|w_{nxxx}(\tau)\|_{L^2} d\tau.\end{aligned}$$

Now, $\left(\frac{7}{6}-1\right)\cdot\frac{1}{1-3/4}=\frac{2}{3}$, so that from Lemma 4, we get

$$\|w_{nxxx}\|_{L^2}\leq\frac{C}{t^{1/3}},$$

as soon as $V\in H^{4,1}\cap H^{2,2}$ and

$$\|w_n\|_{H^2(1-\psi_0(t))}+\|U(t)V\|_{H^4(1-\psi_0(t))}\leq\frac{C}{t^{4/3}}.$$

This conclude the $\dot{H}^3$ estimate.

$\dot{H}^4$ estimate

Let us summarize what we obtained until now. We dispose of the global estimates

$$\|w_n(t)\|_{H^4}+M_0^t(w_n(t))\leq\varepsilon_0,\quad\text{and}\quad\|w_n(t)\|_{H^3}\leq\frac{C}{t^{-1/3}},$$

along with the following decay on the right estimates (from Corollary 1) :

$$t\|w_n\|_{H^3(1-\psi_0(t))}+\|U(t)V\|_{H^5(1-\psi_0(t))}\leq\frac{C}{t^{4/3}}.$$

Step 1 : deriving the H^4 conservation law. Let us differentiate (11) four times :

$$w_{nxxxxt}+w_{nxxxxxxx}+\left(u_n^4-\sum_{j=1}^N R_j^4\right)_{xxxxx}=0.$$

We multiply it by w_{nxxxx}, and do an integration by parts, to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int w_{nxxxx}^2&=\int\left(u_n^4-\sum_{j=1}^N R_j^4\right)_{xxxx}w_{nxxxxx}\\&=\int\left(u_n^4-R^4\right)_{xxxx}w_{nxxxxx}+\int\left(R^4-\sum_{j=1}^N R_j^4\right)_{xxxxx}w_{nxxxx}.\end{aligned}$$

The second integral is harmless. Let us develop the first term

$$\begin{aligned}\left(u_n^4-R^4\right)_{xxxx}&=4(u_{nxxxx}u_n^3-R_{xxxx}R^3)+48(u_{nxxx}u_{nx}u_n^2-R_{xxx}R_xR^2)\\&\quad+36(u_{nxx}^2u_n^2-R_{xx}^2R^2)+144(u_{nxx}u_{nx}^2u_n-R_{xx}R_x^2R)\\&\quad+24(u_{nx}^4-R_x^4)\\&=4w_{nxxxx}u_n^3+4((U(t)V+R)_{xxx}u_n^3-R_{xxx}R^3)\\&\quad+48(u_{nxxx}u_{nx}u_n^2-R_{xxx}R_xR^2)+36(u_{nxx}^2u_n^2-R_{xx}^2R^2)\\&\quad+144(u_{nxx}u_{nx}^2u_n-R_{xx}R_x^2R)+24(u_{nx}^4u_n-R_x^4R).\end{aligned}$$

We try to get rid of the w_{nxxxx} terms, by integration by parts.

$$\int\left(u_n^4-R^4\right)_{xxxx}w_{nxxxxx}$$

$$
\begin{aligned}
&= -6\int {w_n}_{xxxx}^2 u_{nx}u_n^2 - 4\int w_{nxxxx}((U(t)V+R)_{xxxxx}u_n^3 - R_{xxxxx}R^3)\\
&\quad - 60\int w_{nxxxx}((U(t)V+R)_{xxxx}u_{nx}u_n^2 - R_{xxxx}R_xR^2)\\
&\quad - 48\int {w_n}_{xxxx}^2 u_{nx}u_n^2 - 120\int w_{nxxxx}(u_{nxxx}u_{nxx}u_n^2 - R_{xxx}R_{xx}R^2)\\
&\quad - 240\int w_{nxxxx}(u_{nxxx}{u_n}_x^2 u_n - R_{xxx}R_x^2R)\\
&\quad - 360\int w_{nxxxx}({u_n}_{xx}^2 u_{nx}u_n - R_{xx}^2R_xR)\\
&\quad - 240\int w_{nxxxx}(u_{nxx}{u_n}_x^3 - R_{xx}R_x^3)
\end{aligned}
$$

We now want to get rid of the troublesome term $-54\int {w_n}_{xxx}^2 u_{nx}u_n^2$. We thus introduce

$$
\begin{aligned}
\frac{d}{dt}\int {w_n}_{xxx}^2 u_n^3 &= 2\int w_{nxxxt}w_{nxxx}u_n^3 + 3\int {w_n}_{xxx}^2 u_{nt}u_n^2\\
&= -2\int w_{nxxxxxx}w_{nxxx}u_n^3 - \int\left(u_n^4 - \sum_{j=1}^n R_j^4\right)_{xxxx} w_{nxxx}u_n^3\\
&\quad - 3\int {w_n}_{xxx}^2 u_{nxxx}u_n^2 - 12\int {w_n}_{xxx}^2 u_{nx}u_n^5.
\end{aligned}
$$

First :

$$
\begin{aligned}
&-\int\left(u_n^4 - \sum_{j=1}^n R_j^4\right)_{xxxx} w_{nxxx}u_n^3\\
&\qquad = -\int (u_n^4 - R^4)_{xxxx}w_{nxxx}u_n^3 - \int\left(R^4 - \sum_{j=1}^n R_j^4\right)_{xxxx} w_{nxxx}u_n^3\\
&\qquad = \int (u_n^4 - R^4)_{xxx}w_{nxxxx}u_n^3 + 3\int (u_n^4 - R^4)_{xxx}w_{nxxx}u_{nx}u_n^2\\
&\qquad\quad - \int\left(R^4 - \sum_{j=1}^n R_j^4\right)_{xxxx} w_{nxxx}u_n^3,
\end{aligned}
$$

where the third integral is immediately controlled. Now we rearrange the term with high derivatives (more than 3) through integrations by parts.

$$
\begin{aligned}
&-2\int w_{nxxxxxx}w_{nxxx}u_n^3\\
&\qquad = 2\int w_{nxxxxx}w_{nxxxx}u_n^3 + 6\int w_{nxxxxx}w_{nxxx}u_{nx}u_n^2\\
&\qquad = -9\int {w_n}_{xxxx}^2 u_{nx}u_n^2 - 6\int w_{nxxxx}w_{nxxx}u_{nxx}u_n^2\\
&\qquad\quad - 12\int w_{nxxxx}w_{nxxx}{u_n}_x^2 u_n.
\end{aligned}
$$

So that we get :

$$
\frac{d}{dt}\int {w_n}_{xxx}^2 u_n^3
$$

$$
\begin{aligned}
&= -9\int {w_n}_{xxxx}^2 u_{nx}u_n^2 - 6\int w_{nxxxx}w_{nxxx}u_{nxx}u_n^2\\
&\quad - 12\int w_{nxxxx}w_{nxxx}{u_n}_x^2 u_n - 3\int {w_n}_{xxx}^2 u_{nxxx}u_n^2 - 12\int {w_n}_{xxx}^2 u_{nx}u_n^5\\
&\quad + \int (u_n^4 - R^4)_{xxx} w_{nxxxx}u_n^3 + 3\int (u_n^4 - R^4)_{xxx} w_{nxxx}u_{nx}u_n^2\\
&\quad - \int \left(R^4 - \sum_{j=1}^n R_j^4\right)_{xxxx} w_{nxxx}u_n^3
\end{aligned}
$$

And we obtain the following (and last) relation, at level $\dot{H}^4$:

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\int {w_n}_{xxxx}^2 - 12\int {w_n}_{xxx}^2 u_n^3\right) =\\
&- 4\int w_{nxxxx}((U(t)V+R)_{xxxxx}u_n^3 - R_{xxxxx}R^3)\\
&- 60\int w_{nxxxx}((U(t)V+R)_{xxxx}u_{nx}u_n^2 - R_{xxxx}R_xR^2)\\
&- 120\int w_{nxxxx}(u_{nxxx}u_{nxx}u_n^2 - R_{xxx}R_{xx}R^2)\\
&- 240\int w_{nxxxx}(u_{nxxx}{u_n}_x^2 u_n - R_{xxx}R_x^2R)\\
&- 360\int w_{nxxxx}({u_n}_{xx}^2 u_{nx}u_n - R_{xx}^2R_xR) - 240\int w_{nxxxx}(u_{nxx}{u_n}_x^3 - R_{xx}R_x^3R)\\
&+ 72\int w_{nxxxx}w_{nxxx}u_{nxx}u_n^2 + 144\int w_{nxxxx}w_{nxxx}{u_n}_x^2 u_n\\
&+ 36\int {w_n}_{xxx}^2 u_{nxxx}u_n^2 + 144\int {w_n}_{xxx}^2 u_{nx}u_n^5 - 12\int (u_n^4 - R^4)_{xxx}w_{nxxxx}u_n^3\\
&- 36\int (u_n^4 - R^4)_{xxx}w_{nxxx}u_{nx}u_n^2 + 12\int \left(R^4 - \sum_{j=1}^n R_j^4\right)_{xxxx} w_{nxxx}u_n^3\\
&+ \int \left(R^4 - \sum_{j=1}^N R_j^4\right)_{xxxxx} w_{nxxxx}. \qquad (75)
\end{aligned}
$$

Step 2. Estimating terms in (75). There are 13 lines to consider, and as for the H^3 norm, we will do them one by one. We now note B_i in place of A_i in the previous lemma : all B_i are bounded in L^∞. For the lower order (ie L^2 or H^1) estimates on the right, we will systematically bound it by $\|w_n\|_{H^1(1-\psi_0(t))}$ as $\|U(t)V\|_{H^1(1-\psi_0(t))} \le \|w_n\|_{H^1(1-\psi_0(t))}$.

- $\int w_{nxxxx}((U(t)V+R)_{xxxxx}u_n^3 - R_{xxxxx}R^3)$.

$$
\begin{aligned}
(U(t)V+R)_{xxxxx}u_n^3 - R_{xxxxx}R^3 &= U(t)V_{xxxxx}(w_n + U(t)V)^3\\
&\quad + U(t)V_{xxxxx}RB_1 + R_{xxxxx}(w_n + U(t)V)B_1'.
\end{aligned}
$$

So that as $V \in H^{4,1}$, we obtain :

$$\left|\int w_{nxxxxx}((U(t)V+R)_{xxxxx}u_n^3 - R_{xxxxx}R^3\right|$$
$$\leq C\bigg(\frac{1}{t^{4/3}} + \|w_n\|_{H^1(1-\psi_0(t))} + \|U(t)V\|_{H^5(1-\psi_0(t))}\bigg)\|w_{nxxxx}\|_{L^2}. \quad (76)$$

- $\displaystyle\int w_{nxxxx}((U(t)V+R)_{xxxx}u_{nx}u_n^2 - R_{xxxx}R_xR^2)$.

$$\begin{aligned}
&(U(t)V+R)_{xxxx}u_{nx}u_n^2 - R_{xxxx}R_xR^2 \\
&= U(t)V_{xxxx}(w_n+U(t)V)_x(w_n+U(t)V)^2 + U(t)V_{xxxx}(w_n+U(t)V)_xRB_2 \\
&\quad + U(t)V_{xxxx}R_xu_n^2 + R_{xxxx}(w_n+U(t)V)_xu_n^2 \\
&\quad + R_{xxxx}R_x(w_n+U(t)V)B_2'.
\end{aligned}$$

And as $V \in H^4$, we simply get :

$$\left|\int w_{nxxxx}((U(t)V+R)_{xxxx}u_{nx}u_n^2 - R_{xxxx}R_xR^2)\right|$$
$$\leq C\bigg(\frac{1}{t^{4/3}} + \|U(t)V\|_{H^4(1-\psi_0(t))} + \|w_n\|_{H^1(1-\psi_0(t))}\bigg)\|w_{nxxxx}\|_{L^2}. \quad (77)$$

- $\displaystyle\int w_{nxxxx}(u_{nxxx}u_{nxx}u_n^2 - R_{xxx}R_{xx}R^2)$.

$$\begin{aligned}
&u_{nxxx}u_{nxx}u_n^2 - R_{xxx}R_{xx}R^2 \\
&= (w_n+U(t)V)_{xxx}(w_n+U(t)V)_{xx}(w_n+U(t)V)^2 \\
&\quad + (w_n+U(t)V)_{xxx}(w_n+U(t)V)_{xx}RB_3 + (w_n+U(t)V)_{xxx}R_{xx}u_n^2 \\
&\quad + R_{xxx}(w_n+U(t)V)_{xx}u_n^2 + R_{xxx}R_{xx}(w_n+U(t)V)B_3'.
\end{aligned}$$

Then only considering the first term :

$$\begin{aligned}
&\left|\int w_{nxxxx}(w_n+U(t)V)_{xxx}(w_n+U(t)V)_{xx}(w_n+U(t)V)^2\right| \\
&\quad \leq C\|w_{nxxxx}\|_{L^2}\Big(\|w_{nxxx}\|_{L^2}\|w_{nxx}\|_{L^\infty}\|w_n+U(t)V\|_{L^\infty} \\
&\qquad + \|w_{nxxx}\|_{L^2}\|U(t)V)_{xx}(w_n+U(t)V)\|_{L^\infty} \\
&\qquad + \|U(t)V)_{xxx}(w_n+U(t)V)\|_{L^\infty}\|w_n+U(t)V\|_{H^2}\Big)\|w_n+U(t)V\|_{L^\infty}^2 \\
&\quad \leq \left(\frac{C}{t^{4/3}} + \frac{C}{t^{5/3}} + \frac{C}{t^{4/3}}\right)\|w_{nxxxx}\|_{L^2}.
\end{aligned}$$

(where we used $V_{xx} \in H^{1,1}$). So that for this term :

$$\left|\int w_{nxxxx}(u_{nxxx}u_{nxx}u_n^2 - R_{xxx}R_{xx}R^2)\right|$$
$$\leq C\bigg(\frac{1}{t^{4/3}} + \|w_n+U(t)V\|_{H^3(1-\psi_0(t))}\bigg)\|w_{nxxxx}\|_{L^2}. \quad (78)$$

• $\displaystyle\int w_{nxxxx}(u_{nxxx}{u_{n}}_x^2u_n - R_{xxx}R_x^2R)$.

$$\begin{aligned}
&u_{nxxx}{u_{n}}_x^2u_n - R_{xxx}R_x^2R\\
&= (w_n+U(t)V)_{xxx}(w_n+U(t)V)_x^2(w_n+U(t)V)\\
&\quad + (w_n+U(t)V)_{xxx}(w_n+U(t)V)_x^2RB_4 + (w_n+U(t)V)_{xxx}R_xB_4'u_n\\
&\quad + R_{xxx}R_x^2(w_n+U(t)V)B_4'' + R_{xxx}R_x(w+U(t)V)_xB_4'''u_n.
\end{aligned}$$

Now, we have :

$$\|w_{nx}+U(t)V_x\|_{L^\infty} \leq \|w_n\|_{H^2} + \|U(t)V_x\|_{L^\infty} \leq \frac{C}{t^{1/3}},$$

as $V_x \in L^1$. So that :

$$\|(w_n+U(t)V)_{xxx}(w_n+U(t)V)_x^2(w_n+U(t)V)\|_{L^2} \leq \frac{C}{t^{1/3}}\cdot\frac{C}{t} \leq \frac{C}{t^{4/3}}.$$

And we get :

$$\begin{aligned}
&\left|\int w_{nxxxx}(u_{nxxx}{u_{n}}_x^2u_n - R_{xxx}R_x^2R)\right|\\
&\qquad\qquad \leq C\left(\frac{1}{t^{4/3}} + \|w_n+U(t)V\|_{H^3(1-\psi_0(t))}\right)\|w_{nxxxx}\|_{L^2}. \qquad (79)
\end{aligned}$$

• $\displaystyle\int w_{nxxxx}({u_{n}}_{xx}^2u_{nx}u_n - R_{xx}^2R_xR)$.

$$\begin{aligned}
&{u_{n}}_{xx}^2u_{nx}u_n - R_{xx}^2R_xR\\
&\quad= (w_n+U(t)V)_{xx}^2(w_n+U(t)V)_x(w_n+U(t)V)\\
&\qquad + (w_n+U(t)V)_{xx}^2(w_n+U(t)V)_xRB_5 + (w_n+U(t)V)_{xx}^2R_xu_n\\
&\qquad + (w_n+U(t)V)_{xx}R_xB_5'u_{nx}u_n + R_{xx}^2(w_n+U(t)V)_xu_n\\
&\qquad + R_{xx}^2R_x(w_n+U(t)V)B_5''.
\end{aligned}$$

As previously as $V_{xx} \in H^{1,1}$:

$$\|(w_n+U(t)V)_{xx}\|_{L^\infty} \leq \|w_n\|_{H^3} + \|U(t)V_{xx}\|_{L^\infty} \leq \frac{C}{t^{1/3}}.$$

So that

$$\begin{aligned}
&\|(w_n+U(t)V)_{xx}^2(w_n+U(t)V)_x(w_n+U(t)V)\|_{L^2}\\
&\qquad\qquad \leq C\|(w_n+U(t)V)_{xx}\|_{L^2}\cdot\frac{1}{t^{\frac{1}{3}+1}} \leq \frac{C}{t^{4/3}}.
\end{aligned}$$

And we get

$$\left|\int w_{nxxxx}({u_{n}}_{xx}^2u_{nx}u_n - R_{xx}^2R_xR)\right|$$

$$\leq C\bigg(\frac{1}{t^{4/3}} + \|w_n + U(t)V\|_{H^3(1-\psi_0(t))}\bigg)\|w_{nxxxx}\|_{L^2}. \quad (80)$$

- $\displaystyle\int w_{nxxxx}(u_{nxx}{u_{nx}}^3 - R_{xx}R_x^3).$

$$\begin{aligned} u_{nxx}{u_{nx}}^3 - R_{xx}R_x^3 =& w_{nxx}(w_n + U(t)V)_x^3 + U(t)V_{xx}w_{nx}B_6 + U(t)V_{xx}U(t)V_x^3 \\ &+ (w_n + U(t)V)_{xx}R_xB_6' + R_{xx}(w_n + U(t)V)_xB_6'', \end{aligned}$$

where $\|B_6\|_{L^\infty} \leq Ct^{-2/3}$ (it is a homogeneous polynomial of degree 2 in w_{nx} and $U(t)V_x$), and B_6', B_6'' are bounded in L^∞. Now (the L^2 norm goes to a w_n-type term when possible, and $V_x \in H^{1,1}$) :

$$\begin{aligned} \|w_{nxx}(w_n + U(t)V)_x^3\|_{L^2} &\leq \frac{C}{t^{1/3}} \cdot \frac{C}{t}, \\ \|U(t)V_{xx}w_{nx}B_6\|_{L^2} &\leq \frac{C}{t^{1/3}} \cdot \frac{C}{t^{1/3}} \cdot \frac{C}{t^{2/3}}, \\ \|U(t)V_{xx}U(t)V_x^3\|_{L^2} &\leq \frac{C}{t} \cdot \frac{C}{t^{1/3}}. \end{aligned}$$

So that the “linear term” is bounded by $Ct^{-4/3}$, and we have

$$\begin{aligned} &\left|\int w_{nxxxx}(u_{nxx}{u_{nx}}^3 - R_{xx}R_x^3)\right| \\ &\qquad\leq C\bigg(\frac{1}{t^{4/3}} + \|w_n + U(t)V\|_{H^2(1-\psi_0(t))}\bigg)\|w_{nxxxx}\|_{L^2}. \quad (81) \end{aligned}$$

- $\displaystyle\int w_{nxxxx}w_{nxxx}u_{nxx}u_n^2.$

$$u_{nxx}u_n^2 = (w_n + U(t)V)_{xx}(w_n + U(t)V)^2 + (w_n + U(t)V)_{xx}RB_8 + R_{xx}u_n^2.$$

No as $\|w_{nxxx}\|_{L^2} \leq Ct^{-1/3}$ and :

$$\|(w_n + U(t)V)_{xx}(w_n + U(t)V)^2\|_{L^\infty} \leq \frac{C}{t},$$

we get

$$\begin{aligned} &\left|\int w_{nxxxx}w_{nxxx}u_{nxx}u_n^2\right| \\ &\quad\leq C\bigg(\frac{1}{t^{5/3}} + \|w_n\|_{H^3(1-\psi_0(t))} + \|U(t)V\|_{H^2(1-\psi_0(t))}\bigg)\|w_{nxxxx}\|_{L^2}. \quad (82) \end{aligned}$$

- $\displaystyle\int w_{nxxxx}w_{nxxx}{u_{nx}}^2u_n.$
  It is almost like the previous one.

$$ {u_{nx}}^2u_n = (w_n + U(t)V)_x^2(w_n + U(t)V) + (w_n + U(t)V)_x^2RB_9 + RB_9'u_n.$$

Now along with $\|w_{nxxx}\|_{L^2} \leq Ct^{-1/3}$, we have :

$$\|(w_n + U(t)V)_x^2(w_n + U(t)V)\|_{L^\infty} \leq \frac{C}{t^{1+1/3}} = \frac{C}{t^{4/3}}.$$

So that

$$\left| \int w_{nxxxx} w_{nxxx} u_{nx}^2 u_n \right| \leq C\left(\frac{1}{t^{5/3}} + \|w_n\|_{H^3(1-\psi_0(t))} \right) \|w_{nxxxx}\|_{L^2}. \tag{83}$$

- $\int w_{nxxx}^2 u_{nxxx} u_n^2$.

$$\begin{aligned} u_{nxxx} u_n^2 = w_{nxxx}(w_n + U(t)V)^2 + U(t)V_{xxx}(w_n + U(t)V)^2 \\ + (w_n + U(t)V)_{xxx} RB_{10} + R_{xxx} u_n^2. \end{aligned}$$

Now as $V \in H^{3,1}$, we have

$$\|U(t)V_{xxx}(w_n + U(t)V)^2\|_{L^\infty} \leq \frac{C}{t^{4/3}}.$$

Then, of course

$$\left| \int w_{nxxx}^2((w_n + U(t)V)_{xxx} RB_{10} + R_{xxx} u_n^2) \right| \leq \|w_n\|^2_{H^3(1-\psi_0(t))}.$$

And for the first term, we have to be a bit more foxy :

$$\int w_{nxxx}^3 (w_n + U(t)V)^2 = -3 \int w_{nxxxx} w_{nxxx}^2 (w_n + U(t)V)_x (w_n + U(t)V).$$

This last term is bounded by

$$\begin{aligned} \|w_{nxxxx}\|_{L^2} \|w_{nxxx}\|_{L^2} \|w_{nxxx}\|_{L^\infty} \|(w_n + U(t)V)_x (w_n + U(t)V)\|_{L^\infty} \\ \leq \frac{C}{t^{4/3}} \|w_{nxxxx}\|_{L^2}. \end{aligned}$$

And we get

$$\left| \int w_{xxx}^2 u_{nxxx} u_n^2 \right| \leq \frac{C}{t^{5/3}} + \frac{C}{t^{4/3}} \|w_{nxxxx}\|_{L^2} + \|w_n\|^2_{H^3(1-\psi_0(t))}. \tag{84}$$

- $\int w_{nxxx}^2 u_{nx} u_n^5$

$$u_{nx} u_n^5 = (w_n + U(t)V)_x (w_n + U(t)V)^5 + (w_n + U(t)V)_x RB_{11} + R_x u_n^5.$$

We can use directly the usual L^∞ bound for the first term and get a $Ct^{-7/3}$ decay, so that

$$\left| \int \int w_{nxxx}^2 u_{nx} u_n^5 \right| \leq \frac{C}{t^3} + \frac{C\|w_n\|_{H^1(1-\psi_0(t))}}{t^{2/3}}. \tag{85}$$

- $\displaystyle\int (u_n^4 - R^4)_{xxx} w_{nxxxx} u_n^3$.

The only trouble with this 8-power integral is the expression of the differentiated term.

$$
\begin{aligned}
&(u_n^4 - R^4)_{xxx} \\
&= 4(u_{nxxx}u_n^3 - R_{xxx}R^3) + 36(u_{nxx}{u_n}_x^2 R^2 - R_{xx}R_x^2R^2) + 24({u_n}_x^3 u_n - R_x^3 R) \\
&= 4(w_{nxxx}(w_n + U(t)V)^3 + U(t)V_{xxx}(w_n + U(t)V)^3 \\
&\quad + (w_n + U(t)V)_{xxx}RB_{12} + R_{xxx}(w_n + U(t)V)B'_{12}) \\
&\quad + 36((w_n + U(t)V)_{xx}(w_n + U(t)V)_x^2(w_n + U(t)V)^2 \\
&\quad + (w_n + U(t)V)_{xx}(w_n + U(t)V)_x^2 RB''_{12} + (w_n + U(t)V)_{xx}RB'''_{12}u_n^2 \\
&\quad + R_{xx}R_x^2(w_n + U(t)V)B''''_{12} + R_{xx}(w_n + U(t)V)_x B'''''_{12}u_n^2 \\
&\quad + 24((w_n + U(t)V)_x^3(w_n + U(t)V) + (w_n + U(t)V)_x^3 RB''''''_{12} \\
&\quad + (w_n + U(t)V)_x RB'''''''_{12}u_n + R_x^3(w_n + U(t)V)B''''''''_{12}.
\end{aligned} \tag{86}
$$

Now along with $u_n^3 = (w_n + U(t)V)^3 + RB'''''''''_{12}$, we develop the product $(u_n^4 - R^4)_{xxx}u_n^3$. Looking only on terms without R, we have the L^2 bound on these terms :

$$\left(\frac{C}{t^{\frac{1}{3}+1}} + \frac{C}{t^{1+1/3}} + \frac{C}{t^2} + \frac{C}{t^{\frac{1}{3}+1}}\right)\cdot\frac{C}{t^1} \le \frac{C}{t^{7/3}}.$$

On the other side, for any of the terms containing R, we have the following on-the-right bound

$$C\|w_n + U(t)V\|_{H^3(1-\psi_0(t))}.$$

So that finally

$$
\begin{aligned}
&\left|\int (u_n^4 - R^4)_{xxx} w_{nxxxx} u_n^3\right| \\
&\qquad\qquad \le C\left(\frac{1}{t^{7/3}} + \|w_n + U(t)V\|_{H^3(1-\psi_0(t))}\right)\|w_{nxxxx}\|_{L^2}.
\end{aligned} \tag{87}
$$

- $\displaystyle\int (u_n^4 - R^4)_{xxx} w_{nxxx} u_{nx} u_n^2$.

We reuse the development (86), along with

$$u_{nx}u_n^2 = (w_n + U(t)V)_x(w_n + U(t)V)^2 + (w_n + U(t)V)_x RB_{14} + R_x u_n^2,$$

to have L^2 bounds on the product $(u_n^4 - R^4)_{xxx}u_{nx}u_n^2$. For the terms with no R, we get

$$\left(\frac{C}{t^{\frac{1}{3}+1}} + \frac{C}{t^{1+\frac{1}{3}}} + \frac{C}{t^2} + \frac{C}{t^{\frac{1}{3}+1}}\right)\cdot\frac{C}{t^{\frac{4}{3}}} \le \frac{C}{t^{8/3}}.$$

And as for the previous integral, for any of the terms containing R, we have the on-the-right bound :

$$C\|w_n + U(t)V\|_{H^3(1-\psi_0(t))}.$$

Then $\|w_{nxxx}\|_{L^2} \le Ct^{-1/3}$ gives the estimate :

$$\left|\int (u_n^4 - R^4)_{xxx} w_{nxxx} u_{nx} u_n^2\right| \le C\left(\frac{1}{t^3} + \frac{\|w_n + U(t)V\|_{H^3(1-\psi_0(t))}}{t^{1/3}}\right). \tag{88}$$

- $\displaystyle\int \left(R^4 - \sum_{j=1}^{n} R_j^4\right)_{xxxx} w_{nxxx} u_n^3$ and $\displaystyle\int \left(R^4 - \sum_{j=1}^{N} R_j^4\right)_{xxxxx} w_{nxxxx}$.

  We obviously have exponential decay :

$$\left|\int \left(R^4 - \sum_{j=1}^{n} R_j^4\right)_{xxx} w_{nxx} u_n^3\right| + \left|\int \left(R^4 - \sum_{j=1}^{N} R_j^4\right)_{xxxx} w_{nxxx}\right| \\ \leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|w_n\|_{H^4}. \quad (89)$$

And finally :

- $\displaystyle\int w_{nxxx}^2 u_n^3$.

  As $u_n^3 = (w_n + U(t)V)^3 + RB_{15}$, we have

$$\left|\int w_{nxxx}^2 u_n^3\right| \leq \frac{C}{t^{\frac{2}{3}+1}} + C\|w_{nxxx}\|^2_{L^2(1-\psi_0(t))} \leq \frac{C}{t^{5/3}} + C\|w_{nxxx}\|^2_{L^2(1-\psi_0(t))}. \quad (90)$$

Step 3. Let us sum all our estimates (76)-(89) (aside from 86). Then we integrate in time between t and S_n, and plug in (90). We get

$$\|w_{nxxx}\|^2_{L^2} \leq \frac{C}{t^{2/3}} + C\int_t^{S_n} \|w_n(\tau\|^2_{H^3(1-\psi_0(t))}d\tau + C\|w_{nxxx}\|^2_{L^2(1-\psi_0(t))} \\ + C\int_t^{S_n}\left(\frac{1}{t^{4/3}} + \|w_n\|_{H^3(1-\psi_0(t))} + \|U(t)V\|_{H^5(1-\psi_0(t))}\right)\|w_{nxxxx}(\tau)\|_{L^2}d\tau$$

(Notice that we don't have an exponent greater than 1 on $\|w_{nxxxx}(\tau)\|_{L^2}$). So that we obtain

$$\|w_{nxxx}\|_{L^2} \leq \frac{C}{t^{1/3}}.$$

as soon as $V \in H^{5,1} \cap H^{2,2}$ and :

$$\|w_n\|_{H^3(1-\psi_0(t))} + \|U(t)V\|_{H^5(1-\psi_0(t))} \leq \frac{C}{t^{4/3}}.$$

This is follows from Corollary 1, and completes the proof of Proposition 3. □

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