Harder-Narasimhan categories

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Abstract

Semistability and Harder-Narasimhan filtration are important notions in algebraic geometry and arithmetic geometry. Althought these notions are associated to mathematical objects of quite different natures in various domains, their definition and the proofs for their existence are quite similar. It is then natural to expect that these notions could be constructed in a general (categorical) setting. We propose in this article a generalization of Quillen's exact category and we discuss conditions on such categories under which one can establish the notion of Harder-Narasimhan filtrations and Harder-Narsimhan polygons. We then prove in our setting the functoriality of Harder-Narasimhan filtrations which was suggested by Faltings.

Key words: algebraic geometry, Arakelov geometry, Harder-Narasimhan filtration, category
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1 Introduction

The notion of *Harder-Narasimhan flag*[1] (or *canonical flag*) of a vector bundle on a smooth projective curve over a field was firstly introduced by Harder and Narasimhan [11] to study the cohomology groups of moduli spaces of vector bundles on curves. Let C be a smooth projective curve on a field k and E be a non-zero locally free $\mathcal{O}_C$-module (i.e. vector bundle) of finite type. Harder and Narasimhan proved that there exists a flag

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_n = E$$

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[1] In most literature this notion is known as "Harder-Narasimhan filtration". However, the so-called "Harder-Narasimhan filtration" is indexed by a finite set, therefore is in fact a flag of the vector bundle. Here we would like to reserve the term "Harder-Narasimhan filtration" for filtration indexed by $\mathbb{R}$, which we shall define later in this article.

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of E such that

1) each sub-quotient E_i/E_{i-1} ($i = 1, \cdots, n$) is semistable[2] in the sense of Mumford,
2) we have the inequality of successive slopes

$$\mu_{\max}(E) := \mu(E_1/E_0) > \mu(E_2/E_1) > \cdots > \mu(E_n/E_{n-1}) =: \mu_{\min}(E).$$

The Harder-Narasimhan polygon of E is the concave function on $[0, \operatorname{rk} E]$, the graph of which is the convex hull of points of coordinate $(\operatorname{rk} F, \deg(F))$, where F runs over all coherent sub-$\mathcal{O}_C$-modules of E. Its vertexes are of coordinate $(\operatorname{rk} E_i, \deg(E_i))$. The avatar of the above constructions in Arakelov geometry was introduced by Stuhler [20] and Grayson [10]. Similar constructions exist also in the theory of filtered isocristals. In [7], Faltings suggested the functoriality of Harder-Narasimhan flags, but it is somehow not so evident to state such a functoriality. Notice that already the length of canonical flag varies when the vector bundle E changes.

Recently, the categorical approach for studying semistability problems has been developed in various context by different authors, among whom we would like to cite Bridgeland [3], Lafforgue [13], Rudakov [18] and Gorodentsev-Kuleshov [9]. In particular, in [9] the authors discussed stability and canonical filtrations on a triangulated category and they proved the functoriality of Harder-Narasimhan filtrations in such context.

The category of vector bundles on a projective variety is exact in the sense of Quillen [17]. However, it is not the case for the category of Hermitian vector bundles on a projective arithmetic variety. We shall propose a new notion — arithmetic exact category — which generalizes simultaneously the three cases above. Furthermore we shall discuss the conditions on such categories under which we can establish the notion of semistability and furthermore the existence of Harder-Narasimhan filtrations. These filtration are indexed by $\mathbb{R}$, the domain in which the slope function takes value. In fact, we take into account the successive slopes of the Harder-Narasimhan flag. This construction permits us to easily state the functoriality of Harder-Narasimhan filtrations and to prove it. We also show how to associate to such a filtration a Borel probability measure on $\mathbb{R}$ which is a linear combination of Dirac measures. This construction is an important tool to study Harder-Narasimhan polygons in the author's forthcoming work [5].

This article is organized as follows. We introduce in the second section the formalism of filtrations in an arbitrary category, which should be considered

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[2] We say that a non-zero locally free $\mathcal{O}_C$-module of finite type F is semistable if for any non-zero sub-module F_0 of F we have $\mu(F_0) \leq \mu(F)$, where the *slope* μ is by definition the quotient of the degree by the rank.

as a preliminary for the study of Harder-Narasimhan filtrations. In the third section, we present the arithmetic exact categories which generalizes the notion of exact categories in the sense of Quillen. We also give several examples. The fourth section is devoted to the formalism of Harder and Narasimhan on an arithmetic exact category equipped with degree and rank functions, subject to certain conditions which we shall precise (such category will be called Harder-Narasimhan category in this article). In the fifth section, we associate to each arithmetic object in a Harder-Narasimhan category a filtration indexed by $\mathbb{R}$, and we establish the fonctoriality of this construction. Our idea is similar to that of [9] where the authors considered directly functors from the set of slopes to a triangulated category. We also explain how to apply this construction to the study of Harder-Narasimhan polygons. As an application, we give a criterion of Harder-Narasimhan categories when the underlying exact category is an Abelian category. The last section contains several examples of Harder-Narasimhan categories where the arithmetic objects are classical in p-adic representation theory, algebraic geometry and Arakelov geometry respectively.

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2 Filtrations in a category

In this section we shall introduce the notion of filtrations in a general category and their functorial properties. Here we are rather interested in left continuous filtrations. However, for the sake of completeness, and for possible applications elsewhere, we shall also discuss the right continuous counterpart, which is **not** dual to the left continuous case.

2.1 Definitions

We fix throughout this section a non-empty totally ordered set I. Let I^* be the extension of I by adding a minimal element $-\infty$. The new totally ordered set I^* can be viewed as a small category. Namely, for any pair (i,j) of objects in I^*, $\mathrm{Hom}(i,j)$ is a one point set $\{u_{ij}\}$ if $i \geq j$, and is the empty set otherwise. The composition of morphisms is defined in an obvious way. Notice that $-\infty$ is the final object of I^*. The subset I of I^* can be viewed as a full subcategory of I^*.

If $i \leq j$ are two elements in I^*, we shall use the expression $[i,j]$ $\Big($resp. $]i,j[$,

$[i,j[$, $]i,j]\big)$ to denote the set $\{k \in I^* \mid i \leq k \leq j\}$ $\big($resp. $\{k \in I^* \mid i < k < j\}$, $\{k \in I^* \mid i \leq k < j\}$, $\{k \in I^* \mid i < k \leq j\}\big)$.

Definition 2.1 Let $\mathcal{C}$ be a category and X be an object of $\mathcal{C}$. We call I-*filtration* of X in $\mathcal{C}$ any functor $\mathcal{F} : I^* \to \mathcal{C}$ such that $\mathcal{F}(-\infty) = X$ and that, for any morphism φ in I^*, $\mathcal{F}(\varphi)$ is a monomorphism.

Let $\mathcal{F}$ and $\mathcal{G}$ be two filtrations in $\mathcal{C}$. We call *morphism of filtrations* from $\mathcal{F}$ to $\mathcal{G}$ any natural transformation from $\mathcal{F}$ to $\mathcal{G}$. All filtrations in $\mathcal{C}$ and all morphisms of filtrations form a category, denoted by $\mathbf{Fil}^I(\mathcal{C})$. It's a full subcategory of the category of functors from I^* to $\mathcal{C}$.

Let (X, Y) be a pair of objects in $\mathcal{C}$, $\mathcal{F}$ be an I-filtration of X and $\mathcal{G}$ be an I-filtration of Y. We say that a morphism $f : X \to Y$ is *compatible* with the filtrations $(\mathcal{F}, \mathcal{G})$ if there exists a morphism of filtrations $F : \mathcal{F} \to \mathcal{G}$ such that $F(-\infty) = f$. If such morphism F exists, it is unique since all canonical morphisms $\mathcal{G}(i) \to Y$ are monomorphic.

We say that a filtration $\mathcal{F}$ is *exhaustive* if $\varinjlim \mathcal{F}|_I$ exists and if the morphism $\varinjlim \mathcal{F}|_I \to X$ defined by the system $(\mathcal{F}(u_{i,-\infty}) : X_i \to X)_{i \in I}$ is an isomorphism. We say that $\mathcal{F}$ is *separated* if $\varprojlim \mathcal{F}$ exists and is an initial object in $\mathcal{C}$.

If i is an element in I, we denote by $I_{<i}$ (resp. $I_{>i}$) the subset of I consisting of all elements strictly smaller (resp. strictly greater) than i. We say that I is *left dense* (resp. *right dense*) at i if $I_{<i}$ (resp. $I_{>i}$) is non-empty and if $\sup I_{<i} = i$ (resp. $\inf I_{>i} = i$). The subsets $I_{<i}$ and $I_{>i}$ can also be viewed as full subcategories of I^*.

The following two easy propositions give criteria for I to be dense (left and right respectively) at a point i in I.

Proposition 2.2 *Let i be an element of I. The following conditions are equivalents:*

1) I is left dense at i;
2) $I_{<i}$ is non-empty and the set $]j,i[$ is non-empty for any $j < i$;
3) $I_{<i}$ is non-empty and the set $]j,i[$ is infinite for any $j < i$.

Proposition 2.3 *Let i be an element of I. The following conditions are equivalents:*

1) I is right dense at i;
2) $I_{>i}$ is non-empty and the set $]i,j[$ is non-empty for any $j > i$;
3) $I_{>i}$ is non-empty and the set $]i,j[$ is infinite for any $i > j$.

We say that a filtration $\mathcal{F}$ is *left continuous* at $i \in I$ if I is *not* left dense at i or if the projective limit of the restriction of $\mathcal{F}$ on $I_{<i}$ exists and the morphism $\mathcal{F}(i) \to \varprojlim \mathcal{F}|_{I_{<i}}$ defined by the system $(\mathcal{F}(u_{ij}) : \mathcal{F}(i) \to \mathcal{F}(j))_{j<i}$ is an isomorphism. Similarly, we say that $\mathcal{F}$ is *right continuous* at $i \in I$ if I is *not* right dense at i or if the inductive limit of the restriction of $\mathcal{F}$ on $I_{>i}$ exists and the morphism $\varinjlim \mathcal{F}|_{I_{>i}} \to \mathcal{F}(i)$ defined by the system $(\mathcal{F}(u_{ji}) : \mathcal{F}(j) \to \mathcal{F}(i))_{j>i}$ is an isomorphism. We say that a filtration $\mathcal{F}$ is *left continuous* (resp. *right continuous*) if it is left continuous (resp. right continuous) at every element of I. We denote by $\mathbf{Fil}^{I,l}(\mathcal{C})$ (resp. $\mathbf{Fil}^{I,r}(\mathcal{C})$) the full subcategory of $\mathbf{Fil}^{I}(\mathcal{C})$ formed by all left continuous (resp. right continuous) filtrations in $\mathcal{C}$.

Given an arbitrary filtration $\mathcal{F}$, we want to construct a left continuous filtration which is "closest" to the original one. The best candidate is of course the filtration $\mathcal{F}^l$ such that

$$\mathcal{F}^l(i) = \begin{cases} \varprojlim_{k<i} \mathcal{F}(k), & I \text{ is left dense at } i, \\ \mathcal{F}(i), & \text{otherwise.} \end{cases}$$

However, this filtration is well defined only when all projective limits $\varprojlim_{k<i} \mathcal{F}(k)$ exist for any $i \in I$ where I is left dense. Therefore, under the following supplementary condition $(\mathbf{M})$ for the category $\mathcal{C}$:

> *any non-empty totally ordered system of monomorphisms in $\mathcal{C}$ has a projective limit,*

for any filtration $\mathcal{F}$ in $\mathcal{C}$, the filtration $\mathcal{F}^l$ exists. Furthermore, $\mathcal{F} \longmapsto \mathcal{F}^l$ is a functor, which is left adjoint to the forgetful functor from $\mathbf{Fil}^{I,l}(\mathcal{C})$ to $\mathbf{Fil}^{I}(\mathcal{C})$.

Similarly, given an arbitrary filtration $\mathcal{F}$ of an object in $\mathcal{C}$, if for any $i \in I$ where I is right dense, the inductive limit of the system $(\mathcal{F}(j))_{j>i}$ exists, and the canonical morphism $\varinjlim_{j>i} \mathcal{F}(j) \to X$ defined by the system $(\mathcal{F}(u_{j,-\infty}) : \mathcal{F}(j) \to X)_{j>i}$ is monomorphic, then the filtration $\mathcal{F}^r$ such that

$$\mathcal{F}^r(i) = \begin{cases} \varinjlim_{j>i} \mathcal{F}(j), & I \text{ is right dense at } i, \\ \mathcal{F}(i), & \text{otherwise,} \end{cases}$$

is right continuous. Therefore, if the following condition $(\mathbf{M}^*)$ is fulfilled for the category $\mathcal{C}$:

> *any non-empty totally ordered system $(X_i \xrightarrow{\alpha_i} X)_{i\in J}$ of subobjects of an object X in $\mathcal{C}$ has an inductive limit, and the canonical morphism $\varinjlim X_i \to X$ induced by $(\alpha_i)_{i\in J}$ is monomorphic,*

then for any filtration $\mathcal{F}$ in $\mathcal{C}$, the filtration $\mathcal{F}^r$ exists, and $\mathcal{F} \longmapsto \mathcal{F}^r$ is a functor, which is right adjoint to the forgetful functor from $\mathbf{Fil}^{I,r}(\mathcal{C})$ to $\mathbf{Fil}^{I}(\mathcal{C})$.

Let X be an object in $\mathcal{C}$. All I-filtrations of X and all morphisms of filtrations equalling to Id_X at $-\infty$ form a category, denoted by $\mathbf{Fil}_X^I$. We denote by $\mathbf{Fil}_X^{I,l}$ (resp. $\mathbf{Fil}_X^{I,r}$) the full subcategory of $\mathbf{Fil}_X^I$ consisting of all left continuous (resp. right continuous) filtrations of X. The category $\mathbf{Fil}_X^I$ has a final object C_X which sends all $i \in I^*$ to X and all morphisms in I^* to Id_X. We call it the *trivial* filtration of X. If the condition $(\mathbf{M})$ is verified for the category $\mathcal{C}$, the restriction of the functor $\mathcal{F} \longmapsto \mathcal{F}^l$ on $\mathbf{Fil}_X^I$ is a functor from $\mathbf{Fil}_X^I$ to $\mathbf{Fil}_X^{I,l}$, which is left adjoint to the forgetful functor $\mathbf{Fil}_X^{I,l} \to \mathbf{Fil}_X^I$. Similarly, if the condition $(\mathbf{M}^*)$ is verified for the category $\mathcal{C}$, the restriction of the functor $\mathcal{F} \longmapsto \mathcal{F}^r$ on $\mathbf{Fil}_X^I$ gives a functor from $\mathbf{Fil}_X^I$ to $\mathbf{Fil}_X^{I,r}$, which is right adjoint to the forgetful functor $\mathbf{Fil}_X^{I,r} \to \mathbf{Fil}_X^I$.

2.2 Fuctorial constructions

Given a morphism $f : X \to Y$ in a category $\mathcal{C}$ and a filtration of X or Y, we shall explain how to construct a natural filtration of the other.

Suppose that $f : X \to Y$ is a morphism in $\mathcal{C}$ and $\mathcal{G}$ is an I-filtration of Y. If the fiber product in the functor category $\mathbf{Fun}(I^*, \mathcal{C})$, defined by $f^*\mathcal{G} := \mathcal{G} \times_{C_Y} C_X$, exists, where C_X (resp. C_Y) is the trivial filtration of X (resp. Y), then the functor $f^*\mathcal{G}$ is a filtration of X. We call it the *inverse image* of $\mathcal{G}$ by the morphism f. The canonical projection P from $f^*\mathcal{G}$ to $\mathcal{G}$ gives a morphism of filtrations in $\mathbf{Fil}^I(\mathcal{C})$ such that $P(-\infty) = f$. In other words, the morphism f is compatible with the filtrations $(f^*\mathcal{G}, \mathcal{G})$. Since the fiber product commutes to projective limits, if $\mathcal{G}$ is left continuous at a point $i \in I$, then also is $f^*\mathcal{G}$.

If in the category $\mathcal{C}$, all fiber products exist[3], then for any morphism $f : X \to Y$ in $\mathcal{C}$ and any filtration $\mathcal{G}$ of Y, the inverse image of $\mathcal{G}$ by f exists, and f^* is a functor from $\mathbf{Fil}_Y^I$ to $\mathbf{Fil}_X^I$ which sends $\mathbf{Fil}_Y^{I,l}$ to $\mathbf{Fil}_X^{I,l}$.

Let $\mathcal{C}$ be a category and $f : X \to Y$ be a morphism in $\mathcal{C}$. We call *admissible decomposition* of f any triplet (Z, u, v) such that:

1) Z is an object of $\mathcal{C}$,
2) $u : X \to Z$ is a morphism in $\mathcal{C}$ and $v : Z \to Y$ is a monomorphism in $\mathcal{C}$ such that $f = vu$.

If (Z, u, v) and (Z', u', v') are two admissible decompositions of f, we call *morphism of admissible decompositions* from (Z, u, v) to (Z', u', v') any morphism

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[3] In this case, for any small category $\mathcal{D}$, the category of functors from $\mathcal{D}$ to $\mathcal{C}$ supports fiber products. In particular, all fiber products in the category $\mathbf{Fil}^I(\mathcal{C})$ exist.

$\varphi : Z \to Z'$ such that $\varphi u = u'$ and that $v = v'\varphi$.

$$\begin{tikzcd}
& Z \arrow[rd, "v"] \arrow[dd, "\varphi"] & \\
X \arrow[ru, "u"] \arrow[rr, "f"] \arrow[rd, "u'"'] & & Y \\
& Z' \arrow[ru, "v'"'] &
\end{tikzcd}$$

All admissible decompositions and their morphisms form a category, denoted by $\mathrm{Dec}(f)$. If the category $\mathrm{Dec}(f)$ has an initial object (Z_0, u_0, v_0), we say that f has an *image*. The monomorphism $v_0 : Z_0 \to Y$ is called an *image* of f, or an *image* of X in Y by the morphism f, denoted by $\operatorname{Im} f$.

Suppose that $f : X \to Y$ is a morphism in $\mathcal{C}$ and that $\mathcal{F}$ is a filtration of X. If for any $i \in I$, the morphism $f \circ \mathcal{F}(u_{i,-\infty}) : \mathcal{F}(i) \to Y$ has an image, then we can define a filtration $f_\flat \mathcal{F}$ of Y, which associates to each $i \in I$ the subobject $\operatorname{Im}(f \circ \mathcal{F}(u_{i,-\infty}))$ of Y. This filtration is called the *weak direct image* of $\mathcal{F}$ by the morphism f. If furthermore the filtration $f_*\mathcal{F} := (f_\flat \mathcal{F})^l$ is well defined, we called it the *strong direct image* by f. Notice that for any filtration $\mathcal{F}$ of X, the morphism f is compatible with filtrations $(\mathcal{F}, f_\flat\mathcal{F})$ and $(\mathcal{F}, f_*\mathcal{F})$ (if $f_\flat\mathcal{F}$ and $f_*\mathcal{F}$ are well defined). Moreover, if any morphism in $\mathcal{C}$ has an image, then $f_\flat$ is a functor from $\mathbf{Fil}_X^I$ to $\mathbf{Fil}_Y^I$. If in addition the condition (**M**) is fulfilled for the category $\mathcal{C}$, f_* is a functor from $\mathbf{Fil}_X^I$ to $\mathbf{Fil}_Y^{I,l}$.

Proposition 2.4 *Let $\mathcal{C}$ be a category which supports fiber products and such that any morphism in it admits an image. If $f : X \to Y$ is a morphism in $\mathcal{C}$, then the functor $f^* : \mathbf{Fil}_Y^I \to \mathbf{Fil}_X^I$ is right adjoint to the functor $f_\flat$.*

Proof. Let $\mathcal{F}$ be a filtration of Y, $\mathcal{G}$ be a filtration of X and $\tau : \mathcal{G} \to f^*\mathcal{F}$ be a morphism. For any $i \in I$ let $\varphi_i : \mathcal{F}(i) \to Y$ and $\psi_i : \mathcal{G}(i) \to X$ be canonical morphisms, and let $(f_\flat\mathcal{G}(i), u_i, v_i)$ be an image of $\mathcal{G}(i)$ by the morphism $f\psi_i$. Since the morphism $\varphi_i : \mathcal{F}(i) \to Y$ is monomorphic, there exists a unique morphism η_i from $f_\flat\mathcal{G}(i)$ to $\mathcal{F}(i)$ such that $\varphi_i\eta_i = v_i$ and that $\eta_i u_i = \mathrm{pr}_1\, \tau(i)$.

$$\begin{tikzcd}
& f^*\mathcal{F}(i) \arrow[rr, "\mathrm{pr}_1"] \arrow[dd] & & \mathcal{F}(i) \arrow[dd, "\varphi_i"] \\
\mathcal{G}(i) \arrow[ru, "\tau(i)"] \arrow[rr, "u_i"] \arrow[rd, "\psi_i"'] & & f_\flat\mathcal{G}(i) \arrow[ru, "\eta_i"] \arrow[rd, "v_i"] & \\
& X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

Hence we have a functorial bijection $\mathrm{Hom}_{\mathbf{Fil}_X^I}(\mathcal{G}, f^*\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Fil}_Y^I}(f_\flat\mathcal{G}, \mathcal{F})$. $\square$

Corollary 2.5 *With the notations of the previous proposition, if we suppose in addition that the condition* (**M**) *is verified for the category* $\mathcal{C}$*, then for any morphism* $f : X \to Y$ *in* $\mathcal{C}$*, the functor* $f^* : \mathbf{Fil}_Y^{I,l} \to \mathbf{Fil}_X^I$ *is right adjoint to the functor* f_**.*

Proof. For any filtration $\mathcal{F}$ of X and any left continuous filtration $\mathcal{G}$ of Y, we have the following functorial bijections

$$\mathrm{Hom}_{\mathbf{Fil}_X^I}(\mathcal{F}, f^*\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Fil}_Y^I}(f_\flat\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Fil}_Y^{I,l}}(f_*\mathcal{F}, \mathcal{G}).$$

□

2.3 Filtrations of finite length

Let $\mathcal{C}$ be a category. We say that a filtration $\mathcal{F}$ of $X \in \mathrm{obj}\,\mathcal{C}$ is *of finite length* if there exists a finite subset I_0 of I such that, for any $i > j$ satisfying $I_0 \cap [j, i] = \emptyset$, the morphism $\mathcal{F}(u_{ij})$ is isomorphic. The subset I_0 of I is called a *jumping set* of $\mathcal{F}$. We may have different choices of jumping set. In fact, if I_1 is an arbitrary finite subset of I and if I_0 is a jumping set of $\mathcal{F}$, then $I_0 \cup I_1$ is also a jumping set of $\mathcal{F}$. However, the intersection of all jumping sets of $\mathcal{F}$ is itself a jumping set, called the *minimal jumping set* of $\mathcal{F}$.

Suppose that $f : X \to Y$ is a morphism in $\mathcal{C}$. If $\mathcal{G}$ is a filtration of finite length of Y such that $f^*\mathcal{G}$ is well defined, then the filtration $f^*\mathcal{G}$ is also of finite length since the fibre product preserves isomorphisms.

Let $\mathcal{C}$ be a category, X be an object in $\mathcal{C}$ and $\mathcal{F}$ be an I-filtration of X. We say that $\mathcal{F}$ is *left locally constant* at $i \in I$ if I is not left dense at i or if there exists $j < i$ such that $\mathcal{F}(u_{ij})$ is an isomorphism, or equivalently $\mathcal{F}(u_{ik})$ is an isomorphism for any $k \in [j, i[$. Similarly, we say that $\mathcal{F}$ is *right locally constant* at i if I is not right dense at i or if there exists $j > i$ such that $\mathcal{F}(u_{ji})$ is an isomorphism, or equivalently, $\mathcal{F}(u_{ki})$ is an isomorphism for any $k \in]i, j]$. We say that the filtration $\mathcal{F}$ is *left locally constant* (resp. *right locally constant*) if it is left locally constant (resp. right locally constant) at any point $i \in I$.

Proposition 2.6 *Let* $\mathcal{C}$ *be a category,* X *be an object in* $\mathcal{C}$*,* $\mathcal{F}$ *be a filtration of finite length of* X*, and* I_0 *be a jumping set of* $\mathcal{F}$*. For any* $i \in I \setminus I_0$*, the filtration* $\mathcal{F}$ *is left and right locally constant at* i*.*

Proof. Let $i \in I \setminus I_0$ be an element where I is left dense. Since I_0 is a finite set, also is $I_{<i} \cap I_0$. Let $j_0 = \max(I_{<i} \cap I_0)$. We have $j_0 < i$, therefore the set $]j_0, i[$ is non-empty since I is left dense at i. Choose an arbitrary element

$j \in]j_0, i[$. We have $[j, i] \cap I_0 = \emptyset$, so $\mathcal{F}(u_{i,j})$ is an isomorphism. Therefore, $\mathcal{F}$ is left locally constant at i. The proof for the fact that $\mathcal{F}$ is right locally constant at i is similar. □

Proposition 2.7 *If the filtration $\mathcal{F}$ is left locally constant (resp. right locally constant), then it is left continuous (resp. right continuous). The converse is true when the filtration $\mathcal{F}$ is of finite length.*

Proof. "$\Longrightarrow$" is trivial.

"$\Longleftarrow$": Suppose that $\mathcal{F}$ is a left continuous filtration of finite length. Let I_0 be a jumping set of $\mathcal{F}$. If I is left dense at i, there then exists an element $j < i$ in I such that $[j, i[\cap I_0 = \emptyset$. Since $\mathcal{F}$ is left continuous at i, $\mathcal{F}(i)$ is the projective limit of a totally ordered system of isomorphisms. Therefore $\mathcal{F}(u_{ij})$ is an isomorphism. The proof of the other assertion is the same. □

Corollary 2.8 *Let $\mathcal{C}$ be a category and $\mathcal{F}$ be a filtration of finite length in $\mathcal{C}$. If $\mathcal{F}^l$ (resp. $\mathcal{F}^r$) is well defined, then it is also of finite length.*

Proof. Let I_0 be a jumping set of $\mathcal{F}$. We know that the filtration $\mathcal{F}$ is left continuous outside I_0, hence for any $i \in I \backslash I_0$ we have $\mathcal{F}^l(i) = \mathcal{F}(i)$, and if $[j, i] \subset I$ doesn't encounter I_0, then $\mathcal{F}^l(u_{ij}) = \mathcal{F}(u_{ij})$. Therefore, $\mathcal{F}^l$ is of finite length, and I_0 is a jumping set of $\mathcal{F}^l$. The proof for the other assertion is similar. □

Corollary 2.9 *Let $\mathcal{C}$ be a category, $f : X \to Y$ be a morphism in $\mathcal{C}$ and $\mathcal{F}$ be a filtration of finite length of X. If $f_\flat(\mathcal{F})$ (resp. $f_*(\mathcal{F})$) is well defined, then it is also of finite length.*

Let $\mathcal{C}$ be a category and $\mathcal{F}$ be an I-filtration of an object X in $\mathcal{C}$ which is of finite length. The filtration $\mathcal{F}$ is exhaustive if and only if there exists $i_1 \in I$ such that $\mathcal{F}(u_{ji})$ is isomorphism for all $i \leq j \leq i_1$ in I^*. Suppose that $\mathcal{C}$ has an initial object, then the filtration $\mathcal{F}$ is separated if and only if there exists $i_2 \in I$ such that $\mathcal{F}(i)$ is an initial object for any $i \geq i_2$.

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F : \mathcal{C} \to \mathcal{D}$ be a functor. If F sends any monomorphism in $\mathcal{C}$ to a monomorphism in $\mathcal{D}$, then F induces a functor $\widetilde{F} : \mathbf{Fil}^I(\mathcal{C}) \to \mathbf{Fil}^I(\mathcal{D})$. If X is an object of $\mathcal{C}$ and if $\mathcal{F}$ is a filtration of X, $\widetilde{F}(\mathcal{F})$ is called the filtration *induced* from $\mathcal{F}$ by the functor F. The following assertions can be deduced immediately from definition:

i) If $\mathcal{F}$ is left locally constant (resp. right locally constant, of finite length), then $\widetilde{F}(\mathcal{F})$ is left locally constant (resp. right locally constant, of finite length).

ii) If $\mathcal{F}$ is of finite length and exhaustive, then also is $\widetilde{F}(\mathcal{F})$.

iii) Suppose that $\mathcal{C}$ and $\mathcal{D}$ have initial objects and that F preserves initial objects. If $\mathcal{F}$ is separated and of finite length, the also is $\widetilde{F}(\mathcal{F})$.

3 Arithmetic exact categories

The notion of exact categories is defined by Quillen [17]. It is a generalization of Abelian categories. For example, the category of all locally free sheaves on a smooth projective curve is an exact category, but it is not an Abelian category. Furthermore, there are natural categories which fail to be exact, but look alike. Any object in such a category can be described as an object in an exact category equipped with certain additional structure. A typical example is the category of Hermitian vector spaces and linear applications of norm ≤ 1.

In the following we shall formalize the above observation by a new notion — *arithmetic exact category* — by proposing some axioms and we shall provide several examples. Let us begin by recalling the exact categories in the sense of Quillen. Let $\mathcal{C}$ be an essentially small category and let $\mathcal{E}$ be a class of diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r] & X \arrow[r] & X'' \arrow[r] & 0 . \end{tikzcd}$$

If $\begin{tikzcd} 0 \arrow[r] & X' \arrow[r, "f"] & X \arrow[r, "g"] & X'' \arrow[r] & 0 \end{tikzcd}$ is a diagram in $\mathcal{E}$, we say that f is an *admissible monomorphism* and that g is an *admissible epimorphism*. We shall use the symbol "$\rightarrowtail$" to denote an admissible monomorphism, and "$\twoheadrightarrow$" for an admissible epimorphism.

If $\mathcal{F}:\ \begin{tikzcd} 0 \arrow[r] & X' \arrow[r] & X \arrow[r] & X'' \arrow[r] & 0 \end{tikzcd}$ and $\mathcal{G}:\ \begin{tikzcd} 0 \arrow[r] & Y' \arrow[r] & Y \arrow[r] & Y'' \arrow[r] & 0 \end{tikzcd}$ are two diagrams of morphisms in $\mathcal{C}$, we call *morphism* from $\mathcal{F}$ to $\mathcal{G}$ any commutative diagram

$$(\Phi):\quad \begin{tikzcd} 0 \arrow[r] & X' \arrow[r] \arrow[d, "\varphi'"'] & X \arrow[r] \arrow[d, "\varphi"'] & X'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\ 0 \arrow[r] & Y' \arrow[r] & Y \arrow[r] & Y'' \arrow[r] & 0. \end{tikzcd}$$

We say that (Φ) is an *isomorphism* if φ', φ and φ'' are all isomorphisms in $\mathcal{C}$.

Definition 3.1 (Quillen) We say that $(\mathcal{C},\mathcal{E})$ is an *exact category* if the following axioms are verified:

(**Ex**1) For ay diagram

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r, "\varphi"] & X \arrow[r, "\psi"] & X'' \arrow[r] & 0 \end{tikzcd}$$

in $\mathcal{E}$, φ is a *kernel* of ψ and ψ is a *cokernel* of φ.

(**Ex**2) If X and Y are two objects in $\mathcal{C}$, then the diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "(\mathrm{Id},0)"] & X \amalg Y \arrow[r, "\mathrm{pr}_2"] & Y \arrow[r] & 0 \end{tikzcd}$$

is in $\mathcal{E}$.

(**Ex**3) Any diagram which is isomorphe to a diagram in $\mathcal{E}$ lies also in $\mathcal{E}$.

(**Ex**4) If $f : X \to Y$ and $g : Y \to Z$ are two admissible monomorphisms (resp. admissible epimorphismes), then also is gf.

(**Ex**5) For any admissible monomorphism $f : X' \rightarrowtail X$ and any morphism $u : X' \to Y$ in $\mathcal{C}$, the fiber coproduct of f and u exists. Furthermore, if the diagram

$$\begin{tikzcd} X' \arrow[r, tail, "f"] \arrow[d, "u"'] & X \arrow[d, "v"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

is cocartesian, then g is an admissible monomorphism.

(**Ex**6) For any admissible epimorphism $f : X \twoheadrightarrow X''$ and any morphism $u : Y \to X''$ in $\mathcal{C}$, the fiber product of f and u exists. Furthermore, if the diagram

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "v"'] & Y \arrow[d, "u"] \\ X \arrow[r, two heads, "f"'] & X'' \end{tikzcd}$$

is cartesian, then g is an admissible epimorphism.

Keller [12] has shown that the axioms of exact category implies the following property which was initially an axiom in Quillen's definition:

(**Ex**7) For any morphism $f : X \to Y$ in $\mathcal{C}$ having a kernel (resp. cokernel), if there exists an morphism $g : Z \to X$ (resp. $g : Y \to Z$) such that fg (resp. gf) is an admissible epimorphism (resp. admissible monomorphism), then also is f itself.

If $f : X \to Y$ is an admissible monomorphism, by the axiom (**Ex**1), the morphism f admits a cokernel which we shall note Y/X. The pair (X, f) is called an *admissible subobject* of Y.

After [17], if $\mathcal{C}$ is an Abelian category and if $\mathcal{E}$ is the class of all exact sequences in $\mathcal{C}$, then $(\mathcal{C}, \mathcal{E})$ is an exact category. Furthermore, any exact category can be naturally embedded (through the additive version of Yoneda's functor) into an Abelian category.

The following result is important for the axiom (**A**7) in Definition 3.3 below.

Proposition 3.2 *Let $(\mathcal{C}, \mathcal{E})$ be an exact category and $f : X \to Y$ be a morphism in $\mathcal{C}$.*

1) The diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "(\mathrm{Id}_X, f)"] & X \amalg Y \arrow[r, "f \circ \mathrm{pr}_1 - \mathrm{pr}_2"] & Y \arrow[r] & 0 \end{tikzcd}$$

is in $\mathcal{E}$.

2) The morphism f factorizes as $f = \mathrm{pr}_2 \circ (\mathrm{Id}_X, f)$. Furthermore, the second projection $\mathrm{pr}_2 : X \amalg Y \to Y$ is an admissible epimorphism and $(\mathrm{Id}_X, f) : X \to X \amalg Y$ is an admissible monomorphism.

Proof. Let $Z = X \amalg Y$. Consider the morphisms $u = (\mathrm{Id}_X, f) : X \to Z$ and $v = \mathrm{pr}_2 : Z \to Y$. Clearly we have $vu = f$. Moreover, after the axiom (**Ex**2), v is an admissible epimorphism. Therefore it suffices to verify that u is an admissible monomorphism. Consider the morphism $w = f \circ \mathrm{pr}_1 - \mathrm{pr}_2 : Z \to Y$. We shall prove that w is the cokernel of u. First we have $wu = 0$. Furthermore, any morphism $\alpha : Z \to S$ can be written in the form $\alpha = \alpha_1 \circ \mathrm{pr}_1 - \alpha_2 \circ \mathrm{pr}_2$, where $\alpha_1 \in \mathrm{Hom}(X, S)$ and $\alpha_2 \in \mathrm{Hom}(Y, S)$. If $\alpha u = 0$, we have $\alpha_2 f = \alpha_1$, i.e., the diagram

$$\begin{tikzcd} X \arrow[r, "u"] & X \amalg Y \arrow[r, "w"] \arrow[dr, "\alpha"'] & Y \arrow[d, dotted, "\alpha_2"] \\ & & S \end{tikzcd}$$

is commutative. Finally, if $\beta : Y \to S$ satisfies $\beta w = \beta f - \beta = \alpha$, then $\beta = \alpha_2$. Therefore, we have proved that u has a cokernel.

Since the composition of moprhisms $X \xrightarrow{u} Z \xrightarrow{\mathrm{pr}_1} X$ is the identity morphism Id_X, which is an admissible monomorphism, we obtain, thanks to (**Ex**7), that u is also an admissible monomorphism. $\square$

We now introduce the notion of arithmetic exact categories. As explained above, an arithmetic exact category is an exact category where each object is equipped with a set of "arithmetic structures", subject to some compatibility conditions (axioms (**A**1) — (**A**6)). Finally, the axiom (**A**7) shall be used to describe morphisms compatible with arithmetic structures.

Definition 3.3 Let $(\mathcal{C}, \mathcal{E})$ be an exact category. We call *arithmetic structure* on $(\mathcal{C}, \mathcal{E})$ the following data:

1) a mapping A from $\mathrm{obj}\, \mathcal{C}$ to the class of sets,
2) for any admissible monomorphism $f : X \to Y$, a mapping $f^* : A(Y) \to A(X)$,
3) for any admissible epimorphism $g : X \to Y$, a mapping $g_* : A(X) \to A(Y)$,

subject to the following axioms:

(**A**1) $A(0)$ is a one-point set,

(**A**2) if $X \overset{i}{\rightarrowtail} Y \overset{j}{\rightarrowtail} Z$ are admissible monomorphisme, we have $(ji)^* = i^*j^*$,

(**A**3) if $X \overset{p}{\twoheadrightarrow} Y \overset{q}{\twoheadrightarrow} Z$ are admissible epimorphisms, we have $(qp)_* = q_*p_*$,

(**A**4) for any object X of $\mathcal{C}$, $\mathrm{Id}_X^* = \mathrm{Id}_{X*} = \mathrm{Id}_{A(X)}$,

(**A**5) if $f : X \to Y$ is an isomorphism, we have $f^*f_* = \mathrm{Id}_{A(X)}$ and $f_*f^* = \mathrm{Id}_{A(Y)}$,

(**A**6) for any cartesian or[4] cocartesian square

$$\begin{tikzcd} X \arrow[r, tail, "u"] \arrow[d, two heads, "p"'] & Y \arrow[d, two heads, "q"] \\ Z \arrow[r, tail, "v"'] & W \end{tikzcd} \tag{1}$$

in $\mathcal{C}$, where u and v (resp. p and q) are admissible monomorphisms (resp. admissible epimorphisms), we have $v^*q_* = p_*u^*$,

(**A**7) if $X \overset{u}{\twoheadrightarrow} Y \overset{v}{\rightarrowtail} Z$ is a diagram in $\mathcal{C}$ where u (resp. v) is an admissible epimorphism (resp. admissible monomorphism) and if $(h_X, h_Z) \in A(X) \times A(Z)$ satisfies $u_*(h_X) = v^*(h_Z)$, then there exists $h \in A(X \amalg Z)$ such that $(\mathrm{Id}, vu)^*(h) = h_X$ and that $\mathrm{pr}_{2*}(h) = h_Z$.

The triplet $(\mathcal{C}, \mathcal{E}, A)$ is called an *arithmetic exact category.* For any object X of $\mathcal{C}$, we call *arithmetic structure* on X any element h in $A(X)$. The pair (X, h) is called an *arithmetic object* in $(\mathcal{C}, \mathcal{E}, A)$. If $p : X \to Z$ is an admissible epimorphism, $p_*(h)$ is called the *quotient arithmetic structure* on Z. If $i : Y \to X$ is an admissible monomorphism, i^*h is called the *induced arithmetic structure* on Y. $(Z, p_*(h))$ is called an *arithmetic quotient* and $(Y, i^*(h)$ is called an *arithmetic subobject* of (X, h).

Let $(\mathcal{C}, \mathcal{E})$ be an exact category. If for any object X of $\mathcal{C}$, we denote by $A(X)$ a one point set, and we define induced and quotient arithmetic structure in the obvious way, then $(\mathcal{C}, \mathcal{E}, A)$ becomes an arithmetic exact category. The arithmetic structure A is called the *trivial arithmetic structure* on the exact category $(\mathcal{C}, \mathcal{E})$. Therefore, exact categories can be viewed as *trivial* arithmetic exact categories.

Let $(\mathcal{C}, \mathcal{E}, A)$ be an arithmetic exact category. If (X', h') and (X'', h'') are two arithmetic objects in $(\mathcal{C}, \mathcal{E}, A)$, we say that a morphism $f : X' \to X''$ in $\mathcal{C}$ is *compatible* with arithmetic structures if there exists an arithmetic object (X, h), an admissible monomorphism $u : X' \to X$ and an admissible epimorphism $v : X \to X''$ such that $h' = u^*(h)$ and that $h'' = v_*(h)$.

---

[4] Here we can prove that the square is actually cartesian **and** cocartesian.

From the definition of morphisms compatible with arithmetic structures, we obtain the following assertions:

1) If (X_1, h_1) and (X_2, h_2) are two arithmetic objects and if $f : X_1 \to X_2$ is an admissible monomorphism (resp. admissible epimorphism) such that $f^*h_2 = h_1$ (resp. $f_*h_1 = h_2$), then f is compatible with arithmetic structures.
2) If (X_1, h_1) and (X_2, h_2) are two arithmetic objects and if $f : X_1 \to X_2$ is the zero morphism, then f is compatible with arithmetic structures.
3) The composition of two morphisms compatible with arithmetic structure is also compatible with arithmetic structure. This is a consequence of the axiom (**A**7).

Let $(\mathcal{C}, \mathcal{E}, A)$ be an arithmetic exact category. After the argument 3) above, all arithmetic objects in $(\mathcal{C}, \mathcal{E}, A)$ and morphisms compatible with arithmetic structures form a category which we shall denote by $\mathcal{C}_A$. In the following, in order to simplify the notations, we shall use the symbol $\overline{X}$ to denote an arithmetic object (X, h) if there is no ambiguity on the arithmetic structure h.

Let $(\mathcal{C}, \mathcal{E})$ be an exact category. Suppose that $(A_i)_{i\in I}$ is a family of arithmetic structure on $(\mathcal{C}, \mathcal{E})$. For any object X in $\mathcal{C}$, let $A(X) = \prod_{i\in I} A_i(X)$. Suppose that $h = (h_i)_{i\in I}$ is an element in $A(X)$. For any admissible monomorphism $u : Y \to X$, we define $u^*h := (u^*h_i)_{i\in I} \in A(Y)$; for any admissible epimorphism $\pi : X \to Z$, we define $\pi_*h = (\pi_*h_i)_{i\in I} \in A(Z)$. Then it is not hard to show that A is an arithmetic structure on $(\mathcal{C}, \mathcal{E})$. We say that A is the *product arithmetic structure* of $(A_i)_{i\in I}$, denoted by $\prod_{i\in I} A_i$.

We now give some examples of arithmetic exact categories.

3.1 Hermitian spaces

Let $\mathbf{Vec}_{\mathbb{C}}$ be the category of finite dimensional vector spaces over $\mathbb{C}$. It is an Abelian category. Let $\mathcal{E}$ be the class of short exact sequences of finite dimensional vector spaces. For any finite dimensional $\mathbb{C}$-vector space E over $\mathbb{C}$, let $A(E)$ be the set of all Hermitian metrics on E. Suppose that h is a Hermitian metric on E. If $i : E_0 \to E$ is a subspace of E, we denote by $i^*(h)$ the induced metric on E_0. If $\pi : E \to F$ is a quotient space of E, we denote by $\pi_*(h)$ the quotient metric on F. Then $(\mathbf{Vec}_{\mathbb{C}}, \mathcal{E}, A)$ is an arithmetic exact category. In fact, the axioms (**A**1) — (**A**6) are easily verified. The verification of the axiom (**A**7) relies on the following proposition.

Proposition 3.4 *Let E, F_0 and F be Hermitian spaces such that F_0 is a quotient Hermitian space of E and a Hermitian subspace of F. We denote by*

$\pi : E \to F_0$ the projection of E onto F_0 and by $i : F_0 \to F$ the inclusion and we note $\varphi = i\pi$. Then there exists a Hermitian metric on $E \oplus F$ such that in the diagram

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, "(\mathrm{Id},\varphi)"] & E \oplus F \arrow[r, "\mathrm{pr}_2"] & F \arrow[r] & 0 \end{tikzcd},$$

$(\mathrm{Id}, \varphi) : E \to E \oplus F$ is an inclusion and $\mathrm{pr}_2 : E \oplus F \to F$ is a projection of Hermitian spaces.

Proof. Suppose that $E \oplus F$ is equipped with the Hermitian metric $\|\cdot\|$ such that, for any $(x, y) \in E \oplus F$,

$$\|(x, y)\|^2 = \|x - \varphi^* y\|_E^2 + \|y\|_F^2$$

where $\|\cdot\|_E$ and $\|\cdot\|_F$ are Hermitian metrics on E and on F respectively. Clearly with this metric, $\mathrm{pr}_2 : E \oplus F \to F$ is a projection of Hermitian spaces. Moreover, π^* is the identification of F_0 to $(\operatorname{Ker}\pi)^\perp$, i^* is the orthogonal projection of F onto F_0. Therefore, $\varphi^*\varphi : E \to E$ is the orthogonal projection of E onto $(\operatorname{Ker}\pi)^\perp$. Hence, for any vector $x \in E$, we have

$$\|(x, \varphi(x))\|^2 = \|x - \varphi^*\varphi(x)\|_E^2 + \|\varphi(x)\|_F^2 = \|x\|_E^2.$$

□

The assertion above works also in Hilbert spaces case with the same choice of metric. Furthermore, it can be generalized to the family case. Suppose that X is a space ringed in $\mathbb{R}$-algebra (resp. smooth manifold) and E, F_0 and F are locally free $\mathcal{O}_{X,\mathbb{C}}$-modules of finite rank such that F_0 is a quotient of E and a submodule of F. We denote by φ the canoical homomorphism defined by the composition of the projection from E to F_0 and the inclusion of F_0 into F. If E and F are equipped with continuous (resp. smooth) Hermitian metrics such that for any point $x \in X$, the quotient metric on $F_{0,x}$ by the projection of E_x coincides with the metric induced from that of F_x, then there exists a continuous (resp. smooth) Hermitian metric on $E \oplus F$ such that for any point $x \in X$, the graph of φ_x defines an inclusion of Hermitian spaces and the second projection $E_x \oplus F_x \to F_x$ is a projection of Hermitian spaces.

The arithmetic objects in $(\mathbf{Vec}_{\mathbb{C}}, \mathcal{E}, A)$ are nothing other than Hermitian spaces. From definition we see immediately that if a linear mapping $\varphi : E \to F$ of Hermitian spaces is compatible with arithmetic structure, then the norm of φ must be smaller or equal to 1. The following proposition shows that the converse is also true.

Proposition 3.5 *Let $\varphi : E \to F$ be a linear map of Hermitian spaces. If $\|\varphi\| \leq 1$, then there exists a Hermitian metric on $E \oplus F$ such that in the decomposition $E \xrightarrow{(\mathrm{Id},\varphi)} E \oplus F \xrightarrow{\mathrm{pr}_2} F$ of φ, (Id, φ) is an inclusion of Hermitian spaces and pr_2 is a projection of Hermitian spaces.*

Proof. Since $\|\varphi\| \leq 1$, we have $\|\varphi^*\| \leq 1$. Therefore, we obtain the inequalities $\|\varphi^*\varphi\| \leq 1$ and $\|\varphi\varphi^*\| \leq 1$. Hence $\mathrm{Id}_E - \varphi^*\varphi$ and $\mathrm{Id}_F - \varphi\varphi^*$ are Hermitian endomorphisms with positive eigenvalues. So there exist two Hermitian endomorphisms with positive eigenvalues P and Q of E and F respectively such that $P^2 = \mathrm{Id}_E - \varphi^*\varphi$ and $Q^2 = \mathrm{Id}_F - \varphi\varphi^*$.

If x is an eigenvector of $\varphi\varphi^*$ associated to the eigenvalue λ, then $\varphi^* x$ is an eigenvector of $\varphi^*\varphi$ associated to the same eigenvalue. Therefore $\varphi^* Q x = \sqrt{1-\lambda}\varphi^* x = P\varphi^* x$. As F is generated by eigenvectors of $\varphi\varphi^*$, we have $\varphi^* Q = P\varphi^*$. For the same reason we have $Q\varphi = \varphi P$. Let $R = \begin{pmatrix} P & \varphi^* \\ \varphi & -Q \end{pmatrix}$. As R is clearly Hermitian, and verifies

$$R^2 = \begin{pmatrix} P^2 + \varphi^*\varphi & P\varphi^* - \varphi^* Q \\ \varphi P - Q\varphi & \varphi\varphi^* + Q^2 \end{pmatrix} = \mathrm{Id}_{E\oplus F},$$

it is an isometry for the orthogonal sum metric on $E \oplus F$. Let $u : E \to E \oplus F$ be the mapping which sends x to $\begin{pmatrix} x \\ 0 \end{pmatrix}$. The diagram

$$\begin{tikzcd}
E \arrow[r, "\varphi"] \arrow[d, "u"'] & F \\
E \oplus F \arrow[r, "R"'] & E \oplus F \arrow[u, "\mathrm{pr}_2"']
\end{tikzcd}$$

is commutative. The endomorphism $\varphi^*\varphi$ is auto-adjoint, there exists therefore an orthonormal base $(x_i)_{1\leq i\leq n}$ of E such that $\varphi^*\varphi x_i = \lambda_i x_i$. Suppose that $0 \leq \lambda_j < 1$ for any $1 \leq j \leq k$ and that $\lambda_j = 1$ for any $k < j \leq n$. Let $B : E \to E$ be the $\mathbb{C}$-linear mapping such that $B(x_j) = \sqrt{1-\lambda_j} x_j$ for $1 \leq j \leq k$ and that $B(x_j) = x_j$ for $j > k$. Define $S = \begin{pmatrix} B & \varphi^* \\ 0 & \mathrm{Id}_F \end{pmatrix} : E \oplus F \to E \oplus F$. Since $Ru = \begin{pmatrix} P \\ \varphi \end{pmatrix}$ and since

$$(BP + \varphi^*\varphi)(x_i) = \sqrt{1-\lambda_i} B x_i + \lambda_i x_i = \begin{cases} (1-\lambda_i)x_i + \lambda_i x_i = x_i, & 1 \leq i \leq k, \\ 0 B x_i + x_i = x_i, & k < i \leq n, \end{cases}$$

the diagram

$$\begin{tikzcd}
 & E \oplus F \arrow[dd, "S"] \arrow[dr, "\mathrm{pr}_2"] & \\
E \arrow[ur, "Ru"] \arrow[dr, "\tau"'] & & F \\
 & E \oplus F \arrow[ur, "\mathrm{pr}_2"'] &
\end{tikzcd}$$

is commutative, where $\tau = \begin{pmatrix} \mathrm{Id}_E \\ \varphi \end{pmatrix}$. We equip $E \oplus F$ with the Hermitian product $\langle \cdot, \cdot \rangle_0$ such that, for any $(\alpha, \beta) \in (E \oplus F)^2$, we have

$$\langle \alpha, \beta \rangle_0 = \left\langle S^{-1}\alpha, S^{-1}\beta \right\rangle,$$

where $\langle \cdot, \cdot \rangle$ is the orthogonal direct sum of Hermitian products on E and on F. Then for any $(x, y) \in E \times E$,

$$\langle \tau(x), \tau(y) \rangle_0 = \langle SRu(x), SRu(y) \rangle_0 = \langle Ru(x), Ru(y) \rangle = \langle u(x), u(y) \rangle = \langle x, y \rangle .$$

Finally, the kernel of pr_2 is stable by the action of S, so the projections of $\langle \cdot, \cdot \rangle_0$ and of $\langle \cdot, \cdot \rangle$ by pr_2 are the same. □

From the proof of Proposition 3.5, we see that a weaker form (the case where $\|\varphi\| < 1$) can be generalized to the family case, no matter the family of Hermitian metrics is continuous or smooth.

3.2 Ultranormed space

Let k be a field equipped with a non-Archimedean absolute value $|\cdot|$ under which k is complete. We denote by $\mathbf{Vec}_k$ the category of finite dimensional vector spaces over k, which is clearly an Abelian category. Let $\mathcal{E}$ be the class of short exact sequence in $\mathbf{Vec}_k$. For any finite dimension vector space E over k, we denote by $A(E)$ the set of all ultranorms (see [2] for definition) on E. Suppose that h is an ultranorm on E. If $i : E_0 \to E$ is a subspace of E, we denote by $i^*(h)$ the induced ultranorm on E_0. If $\pi : E \to F$ is a quotient space of E, we denote by $\pi_*(h)$ the quotient ultranorm on F. Then $(\mathbf{Vec}_k, \mathcal{E}, A)$ is an arithmetic exact category. In particular, the axiom (**A7**) is justified by the following proposition, which can be generalized without any difficulty to Banach space case or family case.

Proposition 3.6 *Let $\varphi : E \to F$ be a linear map of vector spaces over k. Suppose that E and F are equipped respectively with the ultranorms h_E and h_F such that $\|\varphi\| \leq 1$. If we equip $E \oplus F$ with the ultranorm h such that, for any $(x, y) \in E \oplus F$, $h(x, y) = \max(h_E(x), h_F(y))$, then in the decomposition $E \xrightarrow{(\mathrm{Id}, \varphi)} E \oplus F \xrightarrow{\mathrm{pr}_2} F$ of φ, we have $(\mathrm{Id}, \varphi)^*(h) = h_E$ and $\mathrm{pr}_{2*}(h) = h_F$.*

Proof. In fact, for any element $x \in E$, $h(x, \varphi(x)) = \max(h_E(x), h_F(\varphi(x))) = h_E(x)$ since $h_F(\varphi(x)) \leq \|\varphi\| h_E(x) \leq h_E(x)$. Furthermore, by definition it is clear that $h_F = \mathrm{pr}_{2*}(h)$. Therefore the proposition is true. $\square$

3.3 Hermitian vector bundles

Let K be a number field and $\mathcal{O}_K$ be its integer ring. For any scheme $\mathscr{X}$ of finite type and flat over $\operatorname{Spec} \mathcal{O}_K$ such that $\mathscr{X}_K$ is smooth, we denote by $\mathbf{Vec}(\mathscr{X})$ the category of locally free modules of finite rank on $\mathscr{X}$. If we denote by $\mathcal{E}$ the class of all short exact sequence of coherent sheaves in $\mathbf{Vec}(\mathscr{X})$, then $(\mathbf{Vec}(\mathscr{X}), \mathcal{E})$ is an exact category. Let Σ_∞ be the set of all embeddings of K in $\mathbb{C}$. The space $\mathscr{X}(\mathbb{C})$ of complex points of $\mathscr{X}$, which is a complex analytic manifold, can be written as a disjoint union

$$\mathscr{X}(\mathbb{C}) = \coprod_{\sigma \in \Sigma_\infty} \mathscr{X}_\sigma(\mathbb{C}),$$

where $\mathscr{X}_\sigma(\mathbb{C})$ is the space of complex points in $\mathscr{X} \times_{\mathcal{O}_K, \sigma} \operatorname{Spec} \mathbb{C}$. Notice that the complex conjugation of $\mathbb{C}$ induces an involution $F_\infty : \mathscr{X}(\mathbb{C}) \to \mathscr{X}(\mathbb{C})$ which sends $\mathscr{X}_\sigma(\mathbb{C})$ onto $\mathscr{X}_{\overline{\sigma}}(\mathbb{C})$.

We call *Hermitian vector bundle* on $\mathscr{X}$ any pair (E, h) where E is an object in $\mathbf{Vec}(\mathscr{X})$ and $h = (h_\sigma)_{\sigma \in \Sigma_\infty}$ is a collection such that, for any $\sigma \in \Sigma_\infty$, h_σ is a continuous Hermitian metric on $E_\sigma(\mathbb{C})$, E_σ being $E \otimes_{\mathcal{O}_K, \sigma} \mathbb{C}$, subject to the condition that the collection $h = (h_\sigma)_{\sigma \in \Sigma_\infty}$ should be invariant under the action of F_∞. The collection of Hermitian metrics h is called a *Hermitian structure* on E. One can consult for example [1] and [4] for details. If $i : E_0 \to E$ is an injective homomorphism of $\mathcal{O}_{\mathscr{X}}$-modules in $\mathbf{Vec}(\mathscr{X})$, we denote by $i^*(h)$ the collection of induced metrics on $(E_{0,\sigma}(\mathbb{C}))_{\sigma \in \Sigma_\infty}$; if $\pi : E \to F$ is a surjective homomorphism of $\mathcal{O}_{\mathscr{X}}$-modules in $\mathbf{Vec}(\mathscr{X})$, we denote by $\pi_*(h)$ the collection of quotient metric on $(F_\sigma(\mathbb{C}))_{\sigma \in \Sigma_\infty}$. For any object E in $\mathbf{Vec}(\mathscr{X})$, let $A(E)$ be the set of all Hermitian structures on E. The family version of Proposition 3.4 implies that $(\mathbf{Vec}(\mathscr{X}), \mathcal{E}, A)$ is an arithmetic exact category. The family version of Proposition 3.5 implies that, if (E, h_E) and (F, h_F) are two Hermitian vector bundles over $\mathscr{X}$ and if $\varphi : E \to F$ is a homomorphism of $\mathcal{O}_{\mathscr{X}}$-modules in $\mathbf{Vec}(\mathscr{X})$ such that, for any $x \in \mathscr{X}(\mathbb{C})$, $\|\varphi_x\| < 1$, then φ is compatible with arithmetic structures.

We say that a Hermitian structure $h = (h_\sigma)_{\sigma \in \Sigma_\infty}$ on a vector bundle E on $\mathscr{X}$ is *smooth* if for any $\sigma \in \Sigma_\infty$, h_σ is a smooth Hermitian metric. For any vector bundle E on $\mathscr{X}$, let $A_0(E)$ be the set of all smooth Hermitian structures on E. Then $(\mathbf{Vec}(\mathscr{X}), \mathcal{E}, A_0)$ is also an arithmetic exact category. If (E, h_E) and (F, h_F) are two smooth Hermitian vector bundles over $\mathscr{X}$, then any homomorphism $\varphi : E \to F$ which has norm < 1 at every complex point of $\mathscr{X}$ is compatible with arithmetic structures.

3.4 Filtrations in an Abelian category

Let $\mathcal{C}$ be an essentially small Abelian category and $\mathcal{E}$ be the class of short exact sequences in $\mathcal{C}$. It is well known that any finite projective limit (in particular any fiber product) exists in $\mathcal{C}$. Furthermore, any morphism in $\mathcal{C}$ has an image, which is isomorphic to the cokernel of its kernel, or the kernel of its cokernel. For any object X in $\mathcal{C}$, we denote by $A(X)$ the set[5] of isomorphism classes of left continuous I-filtrations of X, where I is a totally ordered set, as explained in the beginning of the second section. For any left continuous I-filtration $\mathcal{F}$ of X, we denote by $[\mathcal{F}]$ the isomorphism class of $\mathcal{F}$. If $u : X_0 \to X$ is a monomorphism, we define $u^*[\mathcal{F}]$ to be the class of the inverse image $u^*\mathcal{F}$. If $\pi : X \to Y$ is an epimorphism, we define $\pi_*[\mathcal{F}]$ to be the class of the strong direct image $\pi_*\mathcal{F}$.

We assert that $(\mathcal{C}, \mathcal{E}, A)$ is an arithmetic exact category. In fact, the axioms (**A**1) — (**A**5) are clearly satisfied. We now verify the axiom (**A**6). Consider the diagram (1) in Definition 3.3, which is the right sagittal square of the following diagram (2). Suppose given an I-filtration $\mathcal{F}$ of Y. For any $i \in I$, we note $Y_i = \mathcal{F}(i)$ and we denote by $b_i : Y_i \to Y$ the canonical monomorphism.

$$\begin{tikzcd}
& Z_i \arrow[rr, tail, "c_i"] \arrow[dd, tail, "v_i"] & & Z \arrow[dd, tail, "v"] \\
X_i \arrow[ur, dotted, two heads, "p_i"] \arrow[rr, tail, "a_i"'] \arrow[dd, tail, "u_i"'] & & X \arrow[ur, two heads, "p"'] \arrow[dd, tail, "u"] & \\
& W_i \arrow[rr, tail, "d_i"] & & W \\
Y_i \arrow[ur, two heads, "q_i"] \arrow[rr, tail, "b_i"'] & & Y \arrow[ur, two heads, "q"'] &
\end{tikzcd} \tag{2}$$

Let $d_i : W_i \to W$ be the image of qb_i in W and $q_i : Y_i \to W_i$ be the canonical epimorphism. Let (Z_i, c_i, v_i) be the fiber product of v and d_i, and (X_i, a_i, u_i) be the fiber product of u and b_i. Therefore, in the diagram (2), the two coronal square and the right sagittal square are cartesian, the inferior square is commutative. As $vpa_i = qua_i = qb_iu_i = d_iq_iu_i$, there exists a unique morphism $p_i : X_i \to Z_i$ such that $c_ip_i = pa_i$ and that $v_ip_i = q_iu_i$. It is then not hard to verify that the left sagittal square is cartesian, therefore p_i is an epimorphism, so Z_i is the image of pa_i. The axiom (**A**6) is therefore verified. Finally, the verification of the axiom (**A**7) follows from the following proposition.

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[5] This is a set because $\mathcal{C}$ is essentially small.

Proposition 3.7 *Let X and Y be two objects in $\mathcal{C}$ and let $\mathcal{F}$ (resp. $\mathcal{G}$) be an I-filtration of X (resp. Y). If $f : X \to Y$ is a morphism which is compatible with the filtrations $(\mathcal{F}, \mathcal{G})$, then there exists a filtration $\mathcal{H}$ on $X \oplus Y$ such that $\Gamma_f^* \mathcal{H} = \mathcal{F}$ and $\mathrm{pr}_{2*} \mathcal{H} = \mathcal{G}$, where $\Gamma_f = (\mathrm{Id}, f) : X \to X \oplus Y$ is the graph of f and $\mathrm{pr}_2 : X \oplus Y \to Y$ is the projection onto the second factor.*

Proof. Let $\mathcal{H}$ be the filtration such that $\mathcal{H}(i) = \mathcal{F}(i) \oplus \mathcal{G}(i)$. Clearly it is left continuous, and $\mathrm{pr}_{2\flat} \mathcal{H} = \mathcal{G}$. Therefore $\mathrm{pr}_{2*} \mathcal{H} = \mathcal{G}^l = \mathcal{G}$. Moreover, for any $i \in I$, consider the square

$$\begin{tikzcd}
\mathcal{F}(i) \arrow[r, "\phi_i"] \arrow[d, "(\mathrm{Id}, f_i)"'] & X \arrow[d, "(\mathrm{Id}, f)"] \\
\mathcal{F}(i) \oplus \mathcal{G}(i) \arrow[r, "\Phi_i"'] & X \oplus Y
\end{tikzcd} \tag{3}$$

where $\phi_i : \mathcal{F}(i) \to X$ and $\Phi_i = \phi_i \oplus \psi_i : \mathcal{F}(i) \oplus \mathcal{G}(i) \to X \oplus Y$ are canonical inclusions, $f_i : \mathcal{F}(i) \to \mathcal{G}(i)$ is the morphism through which the restriction of f on $\mathcal{F}(i)$ (i.e., $f\phi_i$) factorizes. Then the square (3) is commutative. Suppose that $\alpha : Z \to X$ and $\beta = (\beta_1, \beta_2) : Z \to \mathcal{F}(i) \oplus \mathcal{G}(i)$ are two morphisms such that $(\mathrm{Id}, f)\alpha = \Phi_i \beta$.

$$\begin{tikzcd}
Z \arrow[rrd, bend left, "\alpha"] \arrow[rd, dotted, "\beta_1"] \arrow[rdd, bend right, "\beta"'] & & \\
& \mathcal{F}(i) \arrow[r, "\phi_i"] \arrow[d, "(\mathrm{Id}, f_i)"] & X \arrow[d, "(\mathrm{Id}, f)"] \\
& \mathcal{F}(i) \oplus \mathcal{G}(i) \arrow[r, "\Phi_i"'] & X \oplus Y
\end{tikzcd} \tag{4}$$

Then we have $\alpha = \phi_i \beta_1$ and $f\alpha = \psi_i \beta_2$. So

$$\psi_i \beta_2 = f\alpha = f\phi_i \beta_1 = \psi_i f_i \beta_1.$$

As ψ_i is a monomorphism, we obtain that $f_i \beta_1 = \beta_2$. So $\beta_1 : Z \to \mathcal{F}(i)$ is the only morphism such that the diagram (4) commutes. Hence we get $\mathcal{F} = (\mathrm{Id}, f)^* \mathcal{H}$. □

Notice that the category of arithmetic objects $\mathcal{C}_A$ is equivalent to the category $\mathbf{Fil}^{I,l}(\mathcal{C})$ of left continuous filtrations. Moreover, there exist some variants of $(\mathcal{C}, \mathcal{E}, A)$. For example, if for any object X in $\mathcal{C}$, we denote by $A_0(X)$ the set of isomorphism classes of I-filtrations which are separated, exhaustive, left continuous and of finite length. Then $(\mathcal{C}, \mathcal{E}, A_0)$ is also an arithmetic exact category. Furthermore, the category $\mathcal{C}_{A_0}$ is equivalent to $\mathbf{Fil}^{I,\mathrm{self}}(\mathcal{C})$, the full subcategory of $\mathbf{Fil}^{I,l}(\mathcal{C})$ consisting of filtrations which are **s**eparated, **e**xhaustive, **l**eft continous and of **f**inite length.

4 Harder-Narasimhan categories

In this section we introduce the formalism of Harder-Narasimhan filtrations (indexed by $\mathbb{R}$) on arithmetic exact categories.

4.1 Degree function and rank function on an arithmetic exact category

Let $(\mathcal{C},\mathcal{E},A)$ be an arithmetic exact category. We say that an arithmetic object (X,h) is *non-zero* if X is non-zero in $\mathcal{C}$. Since $\mathcal{C}$ is essentially small, the isomorphism classes of objects in $\mathcal{C}_A$ form a set.

We denote by $\mathcal{E}_A$ the class of diagrams of the form

$$\begin{tikzcd} 0 \arrow[r] & (X',h') \arrow[r, "i"] & (X,h) \arrow[r, "p"] & (X'',h'') \arrow[r] & 0 \end{tikzcd}$$

where (X',h'), (X,h) and (X'',h'') are arithmetic objects and

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r, "i"] & X \arrow[r, "p"] & X'' \arrow[r] & 0 \end{tikzcd}$$

is a diagram in $\mathcal{E}$ such that $h'=i^*(h)$ and $h''=p_*(h)$.

Let $K_0(\mathcal{C},\mathcal{E},A)$ be the free Abelian group generated by isomorphism classes in $\mathcal{C}_A$, modulo the subgroup generated by elements of the form $[(X,h)]-[(X',h')]-[(X'',h'')]$, where

$$\begin{tikzcd} 0 \arrow[r] & (X',h') \arrow[r, "i"] & (X,h) \arrow[r, "p"] & (X'',h'') \arrow[r] & 0 \end{tikzcd}$$

is a diagram in $\mathcal{E}_A$, in other words, $\begin{tikzcd} 0 \arrow[r] & X' \arrow[r, "i"] & X \arrow[r, "p"] & X'' \arrow[r] & 0 \end{tikzcd}$, and $i^*(h)=h'$, $p_*(h)=h''$. The group $K_0(\mathcal{C},\mathcal{E},A)$ is called the *Grothendieck group* of the arithmetic exact category $(\mathcal{C},\mathcal{E},A)$. We have a "*forgetful*" homomorphism from $K_0(\mathcal{C},\mathcal{E},A)$ to $K_0(\mathcal{C},\mathcal{E})$, the Grothendieck group[6] of the exact category $(\mathcal{C},\mathcal{E})$, which sends $[(X,h)]$ to $[X]$.

In order to establish the semi-stability of arithmetic objects and furthermore the Harder-Narasimhan formalism, we need two auxiliary homomorphisms of groups. The first one, from $K_0(\mathcal{C},\mathcal{E},A)$ to $\mathbb{R}$, is called a *degree function* on $(\mathcal{C},\mathcal{E},A)$; and the second one, from $K_0(\mathcal{C},\mathcal{E})$ to $\mathbb{Z}$, which takes strictly positive values on elements of the form $[X]$ with X non-zero, is called a *rank function* on $(\mathcal{C},\mathcal{E})$.

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[6] Which is, by definition, the free Abelian group generated by isomorphism classes in $\mathcal{C}$, modulo the sub-group generated by elements of the form $[X]-[X']-[X'']$, where $\begin{tikzcd} 0 \arrow[r] & X' \arrow[r] & X \arrow[r] & X'' \arrow[r] & 0 \end{tikzcd}$ is a diagram in $\mathcal{E}$.

Now let $\widehat{\deg} : K_0(\mathcal{C}, \mathcal{E}, A) \to \mathbb{R}$ be a degree function on $(\mathcal{C}, \mathcal{E}, A)$ and $\operatorname{rk} : K_0(\mathcal{C}, \mathcal{E}) \to \mathbb{Z}$ be a rank function on $(\mathcal{C}, \mathcal{E})$. For any arithmetic object (X, h) in $(\mathcal{C}, \mathcal{E}, A)$, we shall use the expressions $\widehat{\deg}(X, h)$ and $\operatorname{rk}(X)$ to denote $\widehat{\deg}([(X, h)])$ and $\operatorname{rk}([X])$, and call them the *arithmetic degree* and the *rank* of (X, h) respectively. If (X, h) is non-zero, the quotient $\widehat{\mu}(X, h) = \widehat{\deg}(X, h)/\operatorname{rk}(X)$ is called the *arithmetic slope* of (X, h). We say that a non-zero arithmetic object (X, h) is *semistable* if for any non-zero arithmetic subobject (X', h') of (X, h), we have $\widehat{\mu}(X', h') \le \widehat{\mu}(X, h)$.

The following proposition provides some basic properties of arithmetic degrees and of arithmetic slopes.

Proposition 4.1 *Let us keep the notations above.*

1) If $0 \longrightarrow (X', h') \longrightarrow (X, h) \longrightarrow (X'', h'') \longrightarrow 0$ *is a diagram in* $\mathcal{E}_A$*, then*

$$\widehat{\deg}(X, h) = \widehat{\deg}(X', h') + \widehat{\deg}(X'', h'').$$

2) If (X, h) *is an arithmetic object of rank* 1*, then it is semistable.*
3) Any non-zero arithmetic object (X, h) *is semistable if and only if for any non-trivial arithmetic quotient* (X'', h'') *(i.e.,* X'' *does not reduce to zero and is not canonically isomorphic to* X*), we have* $\widehat{\mu}(X, h) \le \widehat{\mu}(X'', h'')$*.*

Proof. Since $\widehat{\deg}$ is a homomorphism from $K_0(\mathcal{C}, \mathcal{E}, A)$ to $\mathbb{R}$, 1) is clear.

2) If (X', h') is an arithmetic subobject of (X, h), then it fits into a diagram

$$\begin{tikzcd} 0 \arrow[r] & (X', h') \arrow[r, "f"] & (X, h) \arrow[r] & (X'', h'') \arrow[r] & 0 \end{tikzcd}$$

in $\mathcal{C}_A$. Since X' is non-zero, $\operatorname{rk}(X') \ge 1$. Therefore $\operatorname{rk}(X'') = 0$ and hence $X'' = 0$. In other words, f is an isomorphism. So we have $\widehat{\mu}(X', h') = \widehat{\mu}(X, h)$.

3) For any diagram

$$0 \longrightarrow (X', h') \longrightarrow (X, h) \longrightarrow (X'', h'') \longrightarrow 0$$

in $\mathcal{E}_A$, (X'', h'') is non-trivial if and only if (X', h') is non-trivial. If (X', h') and (X'', h'') are both non-trivial, we have the following equality

$$\widehat{\mu}(X, h) = \frac{\operatorname{rk}(X')}{\operatorname{rk}(X)} \widehat{\mu}(X', h') + \frac{\operatorname{rk}(X'')}{\operatorname{rk}(X)} \widehat{\mu}(X'', h'').$$

Therefore $\widehat{\mu}(X', h') \le \widehat{\mu}(X, h) \iff \widehat{\mu}(X'', h'') \ge \widehat{\mu}(X, h)$. □

We are now able to introduce conditions ensuring the existence and the uniqueness of Harder-Narasimhan "flag". The conditions will be proposed as axioms in the coming definition, and in the theorem which follows, we shall prove the existence and the uniqueness of Harder-Narasimhan "flag".

Definition 4.2 Let $(\mathcal{C},\mathcal{E},A)$ ba an arithmetic exact category, $\widehat{\deg}:K_0(\mathcal{C},\mathcal{E},A)\rightarrow\mathbb{R}$ be a degree function and $\mathrm{rk}:K_0(\mathcal{C},\mathcal{E})\rightarrow\mathbb{Z}$ be a rank function. We say that $(\mathcal{C},\mathcal{E},A,\widehat{\deg},\mathrm{rk})$ is a *Harder-Narasimhan category* if the following two axioms are verified:

(**HN**1) For any non-zero arithmetic object (X,h), there exists an arithmetic subobject $(X_{\mathrm{des}},h_{\mathrm{des}})$ of (X,h) such that

$$\widehat{\mu}(X_{\mathrm{des}},h_{\mathrm{des}})=\sup\{\widehat{\mu}(X',h')\mid (X',h')\text{ is a non-zero arithmetic subobject of }(X,h)\}.$$

Furthermore, for any non-zero arithmetic subobject (X_0,h_0) of (X,h) such that $\widehat{\mu}(X_0,h_0)=\widehat{\mu}(X_{\mathrm{des}},h_{\mathrm{des}})$, there exists an admissible monomorphism $f:X_0\rightarrow X_{\mathrm{des}}$ such that the diagram

$$\begin{tikzcd} X_0 \arrow[r, "f"] \arrow[dr, "j"'] & X_{\mathrm{des}} \arrow[d, "i"] \\ & X \end{tikzcd}$$

is commutative and that $f^*(h_{\mathrm{des}})=h_0$, where i and j are canonical admissible monomorphisms.

(**HN**2) If (X_1,h_1) and (X_2,h_2) are two semistable arithmetic objects such that $\widehat{\mu}(X_1,h_1)>\widehat{\mu}(X_2,h_2)$, there exists **no** non-zero morphism from X_1 to X_2 which is compatible with arithmetic structures.

With the notations of Definition 4.2, if (X,h) is a non-zero arithmetic object, then $(X_{\mathrm{des}},h_{\mathrm{des}})$ is a semistable arithmetic object. If in addition (X,h) is not semistable, we say that $(X_{\mathrm{des}},h_{\mathrm{des}})$ is the arithmetic subobject which *destabilizes* (X,h).

Theorem 4.3 *Let $(\mathcal{C},\mathcal{E},A,\widehat{\deg},\mathrm{rk})$ be a Harder-Narasimhan category. If (X,h) is a non-zero arithmetic object, then there exists a sequence of admissible monomorphisms in $\mathcal{C}$:*

$$0=X_0\longrightarrow X_1\longrightarrow\cdots\longrightarrow X_{n-1}\longrightarrow X_n=X\,, \tag{5}$$

unique up to a unique isomorphism, such that, if for any integer $0\leq i\leq n$, we denote by h_i the induced arithmetic structure (from h) on X_i and if we equip, *for any integer* $1 \leq j \leq n$, X_j/X_{j-1} *with the quotient arithmetic structure (of* h_j*), then*

1) *for any integer* $1 \leq j \leq n$, *the arithmetic object* $\overline{X_j/X_{j-1}}$ *defined above is semistable;*
2) *we have the inequalities* $\widehat{\mu}(\overline{X_1/X_0}) > \widehat{\mu}(\overline{X_2/X_1}) > \cdots > \widehat{\mu}(\overline{X_n/X_{n-1}})$.

Proof. First we prove the existence by induction on the rank r of X. The case where (X, h) is semistable is trivial, and *a fortiori* the existence is true for $r = 1$. Now we consider the case where (X, h) isn't semistable. Let $(X_1, h_1) = (X_{\mathrm{des}}, h_{\mathrm{des}})$. It's a semistable arithmetic object, and $X' = X/X_1$ is non-zero. The rank of X' being strictly smaller than r, we can therefore apply the induction hypothesis on (X', h'), where h' is the quotient arithmetic structure. We then obtain a sequence of admissible monomorphisms

$$\begin{tikzcd} 0 = X'_1 \arrow[r, "f'_1"] & X'_2 \arrow[r] & \cdots \arrow[r] & X'_{n-1} \arrow[r, "f'_{n-1}"] & X'_n = X' \end{tikzcd}$$

verifying the desired condition.

Since the canonical morphism from X to X' is an admissible epimorphism, for any $1 \leq i \leq n$, if we note $X_i = X \times_{X'} X'_i$, then by the axiom (**Ex**6), the projection $\pi_i : X_i \to X'_i$ is an admissible epimorphism. For any integer $1 \leq i < n$, we have a canonical morphism from X_i to X_{i+1} and the square

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, "\pi_i"'] & X_{i+1} \arrow[d, "\pi_{i+1}"] \\ X'_i \arrow[r, "f'_i"'] & X'_{i+1} \end{tikzcd} \tag{6}$$

is cartesian. Since f'_i is an monomorphism, also is f_i (cf. [14] V. 7). On the other hand, since the square (6) is cartesian, f_i is the kernel of the composed morphism

$$\begin{tikzcd} X_{i+1} \arrow[r, "\pi_{i+1}"] & X'_{i+1} \arrow[r, "p_i"] & X'_{i+1}/X'_i \end{tikzcd},$$

where p_i is the canonical morphism. Since π_{i+1} and p_i are admissible epimorphisms, also is $p_i\pi_{i+1}$ (see axiom (**Ex**4)). Therefore f_i is an admissible monomorphism. Hence we obtain a commutative diagram

$$\begin{tikzcd} 0 = X_0 \arrow[r] & X_1 \arrow[r, "f_1"] \arrow[d, "\pi_1"'] & X_2 \arrow[r] \arrow[d, "\pi_2"'] & \cdots \arrow[r] & X_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "\pi_{n-1}"] & X_n = X \arrow[d, "\pi"] \\ & 0 = X'_1 \arrow[r, "f'_1"'] & X'_2 \arrow[r] & \cdots \arrow[r] & X'_{n-1} \arrow[r, "f'_{n-1}"'] & X'_n = X' \end{tikzcd}$$

where the horizontal morphisms in the lines are admissible monomorphisms and the vertical morphisms are admissible epimorphisms. Furthermore, for any integer $1 \leq i \leq n-1$, we have a natural isomorphism φ_i from X_{i+1}/X_i

to X'_{i+1}/X'_i. We denote by g_i (resp. g'_i) the canonical morphism from X_i (resp. X'_i) to X (resp. X'). Let $h_i = g_i^*(h)$ (resp. $h'_i = g_i'^*(h')$) be the induced arithmetic structure on X_i (resp. X'_i). After the axiom (**A**6), $\pi_{i*}(h_i) = \pi_{i*}f_i^*(h) = f_i'^*\pi_*(h) = h'_i$. Therefore φ_{i*} sends the quotient arithmetic structure on X_{i+1}/X_i to that on X'_{i+1}/X'_i. Hence the arithmetic object $\overline{X_{i+1}/X_i}$ is semistable and we have the equality $\widehat{\mu}(\overline{X_{i+1}/X_i}) = \widehat{\mu}(\overline{X'_{i+1}/X'_i})$. Finally, since $\overline{X}_1 = \overline{X}_{\mathrm{des}}$, we have

$$\widehat{\mu}(\overline{X_2/X_1}) = \frac{\mathrm{rk}(X_2)\widehat{\mu}(\overline{X}_2) - \mathrm{rk}(X_1)\widehat{\mu}(\overline{X}_1)}{\mathrm{rk}(X_2) - \mathrm{rk}(X_1)} < \widehat{\mu}(\overline{X}_1).$$

Therefore the sequence $0 = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n = X$ satisfies the desired conditions.

We then prove the uniqueness of the sequence (5). By induction we only need to prove that $\overline{X}_1 \cong \overline{X}_{\mathrm{des}}$. Let i be the first index such that the canonical morphism $X_{\mathrm{des}} \to X$ factorizes through X_{i+1}. The composed morphism $X_{\mathrm{des}} \to X_{i+1} \to X_{i+1}/X_i$ is then non-zero. Since $\overline{X}_{\mathrm{des}}$ and $\overline{X_{i+1}/X_i}$ are semistable, we have $\widehat{\mu}(\overline{X}_{\mathrm{des}}) \leq \widehat{\mu}(\overline{X_{i+1}/X_i})$. This implies $i = 0$ and $\widehat{\mu}(\overline{X}_{\mathrm{des}}) = \widehat{\mu}(\overline{X}_1)$. Therefore the morphism $X_1 \to X$ factorizes through X_{des}. So we have $X_{\mathrm{des}} \cong X_1$. □

From the proof above we see that the axiom (**HN**1) suffices for the existence. It is the axiom (**HN**2) which ensures the uniqueness.

Definition 4.4 With the notations of Theorem 4.3, the sequence (5) is called the *Harder-Narasimhan sequence* of the (non-zero) arithmetic object (X, h). Sometimes we write instead

$$0 = \overline{X}_0 \longrightarrow \overline{X}_1 \longrightarrow \cdots \longrightarrow \overline{X}_{n-1} \longrightarrow \overline{X}_n = \overline{X}$$

for underlining the arithmetic structures. The real numbers $\widehat{\mu}(\overline{X}_1)$ and $\widehat{\mu}(\overline{X/X_{n-1}})$ are called respectively the *maximal slope* and the *minimal slope* of $\overline{X}$, denoted by $\widehat{\mu}_{\max}(\overline{X})$ and $\widehat{\mu}_{\min}(\overline{X})$. We point out that for any integer $1 \leq i \leq n$,

$$0 = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{i-1} \longrightarrow X_i$$

is the Harder-Narasimhan sequence of $\overline{X}_i$. Therefore we have $\widehat{\mu}_{\min}(\overline{X}_i) = \widehat{\mu}(\overline{X_i/X_{i-1}})$. Finally, we define by convention $\widehat{\mu}_{\max}(0) = -\infty$ and $\widehat{\mu}_{\min}(0) = +\infty$.

Corollary 4.5 *Let $(\mathcal{C}, \mathcal{E}, A, \widehat{\deg}, \mathrm{rk})$ be a Harder-Narasimhan category and $\overline{X}$ be a non-zero arithmetic object.*

1) For any non-zero arithmetic subobject $\overline{Y}$ of $\overline{X}$, we have $\widehat{\mu}_{\max}(\overline{Y}) \leq \widehat{\mu}_{\max}(\overline{X})$.

2) For any non-zero arithmetic quotient $\overline{Z}$ of $\overline{X}$, we have $\widehat{\mu}_{\min}(\overline{Z}) \geq \widehat{\mu}_{\min}(\overline{X})$.
3) We have the inequalities $\widehat{\mu}_{\min}(\overline{X}) \leq \widehat{\mu}(\overline{X}) \leq \widehat{\mu}_{\max}(\overline{X})$.

Proof. Let $0 = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n = X$ be the Harder-Narasimhan sequence of $\overline{X}$.

1) After replacing $\overline{Y}$ by $\overline{Y}_{\mathrm{des}}$ we may suppose that $\overline{Y}$ is semistable. Let i be the first index such that the canonical morphism $Y \to X$ factorizes through X_{i+1}. The composed morphism $Y \to X_{i+1} \to X_{i+1}/X_i$ is non-zero and compatible with arithmetic structures. Therefore

$$\widehat{\mu}(Y) \leq \widehat{\mu}(X_{i+1}/X_i) \leq \widehat{\mu}_{\max}(X).$$

2) After replacing $\overline{Z}$ by a semistable quotient we may suppose that $\overline{Z}$ is itself semistable. Let $f : X \to Z$ be the canonical morphism. It is an admissible epimorphism. Let i be the smallest index such that the composed morphism $X_{i+1} \to X \xrightarrow{f} Z$ is non-zero. Since the composed morphism $X_i \to X \xrightarrow{f} Z$ is zero, we obtain a non-zero morphism from X_{i+1}/X_i to Z which is compatible with arithmetic structures after Axiom (**A**6).

$$\begin{tikzcd}
X_{i+1} \arrow[r, tail] \arrow[d, two heads] & X \arrow[d, two heads] & \\
X_{i+1}/X_i \arrow[r, tail] & X/X_i \arrow[r, two heads] & Z
\end{tikzcd}$$

Therefore $\widehat{\mu}(\overline{Z}) \geq \widehat{\mu}(\overline{X_{i+1}/X_i}) \geq \widehat{\mu}_{\min}(\overline{X})$.

3) We have $\widehat{\deg}(\overline{X}) = \sum_{i=1}^{n} \widehat{\deg}(\overline{X_i/X_{i-1}})$. Therefore

$$\widehat{\mu}(X) = \sum_{i=1}^{n} \frac{\mathrm{rk}(X_i/X_{i-1})}{\mathrm{rk}(X)} \widehat{\mu}(\overline{X_i/X_{i-1}}) \in \Big[\widehat{\mu}_{\min}(\overline{X}), \widehat{\mu}_{\max}(\overline{X})\Big].$$

□

It is well known that if E and F are two vector bundles on a smooth projective curve C such that $\mu_{\min}(E) > \mu_{\max}(F)$, then there isn't any non-zero homomorphism from E to F. The following result (Proposition 4.7) generalizes this fact to Harder-Narasimhan categories.

Lemma 4.6 *Let $(\mathcal{C}, \mathcal{E}, A)$ be an arithmetic exact category. Suppose that any epimorphism in $\mathcal{C}$ has a kernel. Let (X, h_X) and (Z, h_Z) be two arithmetic objects, (Y, h_Y) be an arithmetic quotient of (X, h_X), and $f : Y \to Z$ be a morphism in $\mathcal{C}$. Denote by $\pi : X \to Y$ the canonical admissible epimorphism.*

The morphism f is compatible with arithmetic structures if and only if it is the case for $f\pi$.

Proof. Since π is compatible with arithmetic structures, the compatibility of f with arithmetic structures implies that of $f\pi$. It then suffices to verify the converse assertion. By definition there exists an arithmetic object (W, h_W) and a decomposition $X \overset{i}{\rightarrowtail} W \overset{p}{\twoheadrightarrow} Z$ of $f\pi$ such that $i^*h_W = h_X$ and $p_*h_W = h_Z$. Let T be the fiber coproduct of i and π and let $j : Y \rightarrow T$ and $q : W \rightarrow T$ be canonical morphisms. After Axiom (**Ex**5), j is an admissible monomorphism. Let $\tau : U \rightarrowtail X$ be the kernel of π. We assert that $q = \operatorname{Coker}(i\tau)$. On one hand, we have $qi\tau = j\pi\tau = 0$. On the other hand, if $\alpha : W \rightarrow V$ is a morphism in $\mathcal{C}$ such that $\alpha i\tau = 0$, then there exists a unique morphism $\beta : Y \rightarrow V$ such that $\beta\pi = \alpha i$ since π is a cokernel of τ. Therefore, there exits a unique morphism $\gamma : T \rightarrow V$ such that $\gamma q = \alpha$. So q is a cokernel of $i\tau$, hence an admissible epimorphism. The morphisms $p : W \twoheadrightarrow Z$ and $f : Y \rightarrow Z$ induce a morphism $g : T \rightarrow Z$:

$$\begin{tikzcd}
U \arrow[d, tail, "\tau"'] & & \\
X \arrow[r, tail, "i"] \arrow[d, two heads, "\pi"'] & W \arrow[d, two heads, "q"'] \arrow[dr, two heads, "p"] & \\
Y \arrow[r, tail, "j"'] & T \arrow[r, two heads, "g"'] & Z
\end{tikzcd}$$

Since g is an epimorphism, by hypothesis it has a kernel. After Axiom (**Ex**7), it is an admissible epimorphism. Finally if we denote by h_T the arithmetic structure q_*h_W on T, we have $g_*(h_T) = p_*(h_W) = h_Z$ and $j^*(h_T) = \pi_*(i^*h_W) = \pi_*(h_X) = h_Y$. □

Proposition 4.7 *Let $(\mathcal{C}, \mathcal{E}, A, \widehat{\deg}, \operatorname{rk})$ be a Harder-Narasimhan category. Suppose that any epimorphism in $\mathcal{C}$ has a kernel. If $\overline{X}$ and $\overline{Y}$ are two arithmetic objects and if $f : \overline{X} \rightarrow \overline{Y}$ is a non-zero morphism compatible with arithmetic structures, then $\widehat{\mu}_{\min}(\overline{X}) \leq \widehat{\mu}_{\max}(\overline{Y})$.*

Proof. Let $0 = \overline{X}_0 \longrightarrow \overline{X}_1 \longrightarrow \cdots \longrightarrow \overline{X}_{n-1} \longrightarrow \overline{X}_n = \overline{X}$ be the Harder-Narasimhan sequence of $\overline{X}$. For any integer $0 \leq i \leq n$, let $h_i : \overline{X}_i \rightarrow \overline{X}$ be the canonical monomorphism. Let $1 \leq j \leq n$ be the first index such that fh_j is non-zero. Since $fh_{j-1} = 0$, the morphism fh_j factorizes through X_j/X_{j-1}, so we get a non-zero morphism g from X_j/X_{j-1} to Y. After Lemma 4.6, g is compatible with arithmetic structures. Let

$$0 = \overline{Y}_0 \longrightarrow \overline{Y}_1 \longrightarrow \cdots \longrightarrow \overline{Y}_{m-1} \longrightarrow \overline{Y}_m$$

be the Harder-Narasimhan sequence of $\overline{Y}$. Let $1 \leq k \leq n$ be the first index such that g factorizes through Y_k. If $\pi : Y_k \to Y_k/Y_{k-1}$ is the canonical morphism, then πg is non-zero since g doesn't factorize through Y_{k-1}. Furthermore, it is compatible with arithmetic structures. Therefore, we have

$$\widehat{\mu}_{\min}(\overline{X}) \leq \widehat{\mu}(\overline{X_j/X_{j-1}}) \leq \widehat{\mu}(\overline{Y_k/Y_{k-1}}) \leq \widehat{\mu}_{\max}(\overline{Y}).$$

□

Corollary 4.8 *Keep the notations and the hypothesis of Proposition 4.7.*

1) If in addition f is monomorphic, then $\widehat{\mu}_{\max}(\overline{X}) \leq \widehat{\mu}_{\max}(\overline{Y})$.
2) If in addition f is epimorphic, then $\widehat{\mu}_{\min}(\overline{X}) \leq \widehat{\mu}_{\min}(\overline{Y})$.

Proof. Suppose that f is monomorphic. Let $i : X_{\mathrm{des}} \to X$ be the canonical morphism. Then the composed morphism $fi : \overline{X}_{\mathrm{des}} \to \overline{Y}$ is non-zero and compatible with arithmetic structures. Therefore $\widehat{\mu}_{\max}(\overline{X}) = \widehat{\mu}_{\min}(\overline{X}_{\mathrm{des}}) \leq \widehat{\mu}_{\max}(\overline{Y})$. The proof of the other assertion is similar. □

If the arithmetic structure A is trivial, then any morphism in $\mathcal{C}$ is compatible with arithmetic structures. Therefore in this case we may remove the hypothesis on the existence of kernels in Proposition 4.7 and in Corollary 4.8. However, we don't know whether in general case we can remove the hypothesis that any epimorphism in $\mathcal{C}$ has a kernel, although this condition is fulfilled for all examples that we have discussed in the previous section.

In the following, we give an example of Harder-Narasimhan category, which will play an important role in the next section. Let $\mathcal{C}$ be an Abelian category and $\mathcal{E}$ be the class of all short exact sequences in $\mathcal{C}$. We suppose given a rank function $\mathrm{rk} : K_0(\mathcal{C}) \to \mathbb{Z}$. In this example we take the totally ordered set I as a subset of $\mathbb{R}$ (with the induced order). For any object X in $\mathcal{C}$, let $A_0(X)$ be the set of isomorphism classes in $\mathbf{Fil}_X^{I,\mathrm{self}}$. We have shown in the previous section that $(\mathcal{C}, \mathcal{E}, A_0)$ is an arithmetic exact category. Any arithmetic object $\overline{X} = (X, h)$ of this arithmetic exact category may be considered, after choosing a representative in h, as an object X in $\mathcal{C}$ equipped with an $\mathbb{R}$-filtration $(X_\lambda)_{\lambda \in I}$ which is separated, exhaustive, left continuous and of finite length. We define a real number[7]

$$\widehat{\deg}(\overline{X}) = \sum_{\lambda \in I} \lambda \Big(\mathrm{rk}(X_\lambda) - \sup_{j > \lambda, j \in I} \mathrm{rk}(X_j) \Big).$$

The summation above turns out to be finite since the filtration is of finite length and its value doesn't depend on the choice of the representative in h.

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[7] Here $\sup_{\emptyset} = 0$ by convention.

If $\overline{X}=(X,(X_\lambda)_{\lambda\in I})$ and $\overline{Y}=(Y,(Y_\lambda)_{\lambda\in I})$ are two arithmetic objects and if $f:X\rightarrow Y$ is an isomorphism which is compatible with arithmetic structures, then for any $\lambda\in I$, we have $\mathrm{rk}(X_\lambda)\le\mathrm{rk}(Y_\lambda)$. Therefore we have $\widehat{\deg}(\overline{X})\le\widehat{\deg}(\overline{Y})$ by Abel's summation formula.

We now show that the function $\widehat{\deg}$ defined above extends naturally to a homomorphism from $K_0(\mathcal{C},\mathcal{E},A_0)$ to $\mathbb{R}$. Let

$$\begin{tikzcd}0 \arrow[r] & X' \arrow[r, "u"] & X \arrow[r, "p"] & X'' \arrow[r] & 0\end{tikzcd}$$

be a short exact sequence in $\mathcal{C}$. Suppose that $\mathcal{F}'=(X'_\lambda)_{\lambda\in I}$ (resp. $\mathcal{F}=(X_\lambda)_{\lambda\in I}$, $\mathcal{F}''=(X''_\lambda)_{\lambda\in I}$) is an $\mathbb{R}$-filtration of X' (resp. X, X'') which is separated, exhaustive, left continuous and of finite length, and such that $\mathcal{F}'=u^*(\mathcal{F})$, $\mathcal{F}''=p_*(\mathcal{F})$. Then for any real number $\lambda\in I$ we have a canonical exact sequence

$$\begin{tikzcd}0 \arrow[r] & X'_\lambda \arrow[r] & X_\lambda \arrow[r] & X''_\lambda \arrow[r] & 0\,.\end{tikzcd}$$

Therefore, $\widehat{\deg}(X,[\mathcal{F}])=\widehat{\deg}(X',[\mathcal{F}'])+\widehat{\deg}(X'',[\mathcal{F}''])$. Notice that an non-zero arithmetic object $\overline{X}=(X,[\mathcal{F}])$ is semistable if and only if the filtration $\mathcal{F}$ has a jumping set which reduces to a one point set. If $\overline{X}$ is semistable and if $\{\lambda\}$ is a jumping set of $\mathcal{F}$, then the arithmetic slope of $\overline{X}$ is just λ. Therefore, if $\overline{X}=(X,[\mathcal{F}])$ and $\overline{Y}=(Y,[\mathcal{G}])$ are two semistable arithmetic objects such that $\lambda:=\widehat{\mu}(\overline{X})>\widehat{\mu}(\overline{Y})$, then any morphism $f:X\rightarrow Y$ which is compatible with filtrations sends $\mathcal{F}(\lambda)=X$ into $\mathcal{G}(\lambda)=0$, therefore is the zero morphism.

If $\overline{X}=(X,[\mathcal{F}])$ is a non-zero arithmetic object, we denote by X_{des} the non-zero object in the filtration $\mathcal{F}$ having the maximal index. The existence of X_{des} is justified by the finiteness and the left continuity of $\mathcal{F}$. The arithmetic subobject $\overline{X}_{\mathrm{des}}$ of $\overline{X}$ is semistable. Furthermore, for any non-zero arithmetic subobject $\overline{Y}=(Y,[\mathcal{G}])$ of $\overline{X}$, we have

$$\begin{aligned}\widehat{\mu}(\overline{Y})&=\frac{1}{\mathrm{rk}(Y)}\sum_{\lambda\in I}\lambda\Big(\mathrm{rk}(\mathcal{G}(\lambda))-\sup_{j>\lambda,j\in I}\mathrm{rk}(\mathcal{G}(j))\Big)\\&\le\frac{1}{\mathrm{rk}(Y)}\sum_{\lambda\in I}\widehat{\mu}(\overline{X}_{\mathrm{des}})\Big(\mathrm{rk}(\mathcal{G}(\lambda))-\sup_{j>\lambda,j\in I}\mathrm{rk}(\mathcal{G}(j))\Big)=\widehat{\mu}(\overline{X}_{\mathrm{des}}).\end{aligned}$$

The equality holds if and only if $\overline{Y}$ is semistable and of slope $\widehat{\mu}(\overline{X}_{\mathrm{des}})$, in this case, the canonical morphism from Y to X factorizes through X_{des} since it is compatible with filtrations. Hence we have proved that $(\mathcal{C},\mathcal{E},A_0,\widehat{\deg},\mathrm{rk})$ is a Harder-Narasimhan category.

Suppose that $\overline{X}=(X,[\mathcal{F}])$ is a non-zero arithmetic object, where $\mathcal{F}=(X_\lambda)_{\lambda\in\mathbb{R}}$. If $E=\{\lambda_1>\lambda_2>\cdots>\lambda_n\}$ is the minimal jumping set of $\mathcal{F}$ (i.e. the intersection of all jumping sets of $\mathcal{F}$, which is itself a jumping set of $\mathcal{F}$), then

$$\begin{tikzcd}0 \arrow[r] & X_{\lambda_1} \arrow[r] & X_{\lambda_2} \arrow[r] & \cdots \arrow[r] & X_{\lambda_n}=X\end{tikzcd}$$

is the Harder-Narasimhan sequence of $\overline{X}$. Furthermore, $\widehat{\mu}(\overline{X}_{\lambda_1})=\lambda_1$, and for any $2\le i\le n$, $\widehat{\mu}(\overline{X_{\lambda_i}/X_{\lambda_{i-1}}})=\lambda_i$.

5 Harder-Narasimhan filtrations and polygons

We fix in this section a Harder-Narasimhan category $(\mathcal{C},\mathcal{E},A,\widehat{\deg},\mathrm{rk})$. We shall introduce the notions of Harder-Narasimhan filtrations and Harder-Narasimhan measures for an arithmetic object in $(\mathcal{C},\mathcal{E},A,\widehat{\deg},\mathrm{rk})$. We shall also explain that if $\mathcal{D}$ is an Abelian category equipped with a rank function and if there exists an exact functor $F:\mathcal{C}\to\mathcal{D}$ which preserves rank functions, then for any non-zero arithmetic object $\overline{X}$ in $\mathcal{C}$, the Harder-Narasimhan filtration of $\overline{X}$ induces a filtration of $F(X)$, which defines an arithmetic object $\overline{F(X)}$ of the Harder-Narasimhan category defined by $\mathbb{R}$-filtrations in $\mathcal{D}$ which are separated, exhaustive, left continuous and of finite length. Furthermore, the Harder-Narasimhan polygon (resp. measure) of $\overline{F(X)}$ coincides with that of $\overline{X}$. Therefore, filtered objects in Abelian categories equipped with rank functions can be considered in some sense as models to study Harder-Narasimhan polygons.

5.1 Construction of Harder-Narasimhan filtrations

Proposition 5.1 *Let $\overline{X}$ be a non-zero arithmetic object and*

$$\begin{tikzcd} 0=X_0^{\mathrm{HN}} \arrow[r] & X_1^{\mathrm{HN}} \arrow[r] & \cdots \arrow[r] & X_{n-1}^{\mathrm{HN}} \arrow[r] & X_n^{\mathrm{HN}}=X \end{tikzcd}$$

be its Harder-Narasimhan sequence. If for any real number λ we denote by[8]

$$i_{\overline{X}}(\lambda)=\max\{1\le i\le n\mid \widehat{\mu}(\overline{X}_i^{\mathrm{HN}}/\overline{X}_{i-1}^{\mathrm{HN}})\ge\lambda\}$$

and $X_\lambda=X^{\mathrm{HN}}_{i_{\overline{X}}(\lambda)}$, then $(X_\lambda)_{\lambda\in\mathbb{R}}$ is an $\mathbb{R}$-filtration of the object X in $\mathcal{C}$. Furthermore, this filtration is separated, exhaustive, left continuous and of finite length.

Proof. If $\lambda>\lambda'$, then $i_{\overline{X}}(\lambda)\le i_{\overline{X}}(\lambda')$, hence $(X_\lambda)_{\lambda\in\mathbb{R}}$ is an $\mathbb{R}$-filtration of X. Moreover, for any $\lambda\in\mathbb{R}$, $X_\lambda\in\{X_0^{\mathrm{HN}},\cdots,X_n^{\mathrm{HN}}\}$, therefore this filtration is of finite length. When $\lambda>\widehat{\mu}_{\max}(\overline{X})$, we have $i_{\overline{X}}(\lambda)=0$, which implies that $\overline{X}_\lambda=\overline{X}_0^{\mathrm{HN}}=0$ is the zero object, so the filtration is separated. When $\lambda<\widehat{\mu}_{\min}(\overline{X})$, $i_{\overline{X}}(\lambda)=n$, so $\overline{X}_\lambda=\overline{X}$, i.e., the filtration is exhaustive. To prove the left continuity of this filtration, it suffices to verify that the function

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[8] By convention $\max\emptyset=0$.

$\lambda \mapsto i_{\overline{X}}(\lambda)$ is left continuous. Actually, this function is left locally constant: if $i_{\overline{X}}(\lambda) = 0$, then for any integer $1 \leq i \leq n$, we have $\widehat{\mu}(\overline{X}_i^{\mathrm{HN}} / \overline{X}_{i-1}^{\mathrm{HN}}) < \lambda$, so there exists $\varepsilon_0 > 0$ such that for any $0 \leq \varepsilon < \varepsilon_0$, we have $\widehat{\mu}(\overline{X}_i^{\mathrm{HN}} / \overline{X}_{i-1}^{\mathrm{HN}}) < \lambda - \varepsilon$, i.e., $i_{\overline{X}}(\lambda - \varepsilon) = 0$; if $i_{\overline{X}}(\lambda) = n$, then for any integer $1 \leq i \leq n$ and any real number $\varepsilon \geq 0$, we have $\widehat{\mu}(\overline{X}_i^{\mathrm{HN}} / \overline{X}_{i-1}^{\mathrm{HN}}) \geq \lambda \geq \lambda - \varepsilon$, so $i_{\overline{X}}(\lambda - \varepsilon) = n$; finally if $1 \leq i_{\overline{X}}(\lambda) \leq n - 1$, then we have $\widehat{\mu}(\overline{X}_{i_{\overline{X}}(\lambda)}^{\mathrm{HN}} / \overline{X}_{i_{\overline{X}}(\lambda)-1}^{\mathrm{HN}}) \geq \lambda$ and $\widehat{\mu}(\overline{X}_{i_{\overline{X}}(\lambda)+1}^{\mathrm{HN}} / \overline{X}_{i_{\overline{X}}(\lambda)}^{\mathrm{HN}}) < \lambda$, hence there exists $\varepsilon_0 > 0$ such that, for any $0 \leq \varepsilon < \varepsilon_0$, we have $\widehat{\mu}(\overline{X}_{i_{\overline{X}}(\lambda)}^{\mathrm{HN}} / \overline{X}_{i_{\overline{X}}(\lambda)-1}^{\mathrm{HN}}) \geq \lambda - \varepsilon$ and $\widehat{\mu}(\overline{X}_{i_{\overline{X}}(\lambda)+1}^{\mathrm{HN}} / \overline{X}_{i_{\overline{X}}(\lambda)}^{\mathrm{HN}}) < \lambda - \varepsilon$, i.e., $i_{\overline{X}}(\lambda - \varepsilon) = i_{\overline{X}}(\lambda)$. □

Definition 5.2 With the notations of Proposition 5.1, the filtration $(X_\lambda)_{\lambda \in \mathbb{R}}$ is called the *Harder-Narasimhan filtration* (or *canonical filtration*) of $\overline{X}$, denoted by $\mathrm{HN}(\overline{X})$. Clearly, $\widehat{\mu}_{\min}(\overline{X}_\lambda) \geq \lambda$ for any $\lambda \in \mathbb{R}$. We define the *Harder-Narasimhan filtration* (or *canonical filtration*) of the zero object to be its only $\mathbb{R}$-filtration which associates to each $\lambda \in \mathbb{R}$ the zero object itself.

5.2 Functoriality of Harder-Narasimhan filtrations

Theorem 5.3 *Keep the notations of Proposition 5.1. Suppose in addition that any epimorphism in $\mathcal{C}$ has a kernel in the case where A is non-trivial. Then any morphism in $\mathcal{C}_A$ is compatible with Harder-Narasimhan filtrations.*

Proof. Let $f : X \to Y$ be a morphism which is compatible with arithmetic structures. The case where X or Y is zero is trivial. We now suppose that X and Y are non-zero. Let

$$\begin{tikzcd}0 = X_0^{\mathrm{HN}} \arrow[r] & X_1^{\mathrm{HN}} \arrow[r] & \cdots \arrow[r] & X_{n-1}^{\mathrm{HN}} \arrow[r] & X_n^{\mathrm{HN}} = X\end{tikzcd}$$

be the Harder-Narasimhan sequence of $\overline{X}$ and

$$\begin{tikzcd}0 = Y_0^{\mathrm{HN}} \arrow[r] & Y_1^{\mathrm{HN}} \arrow[r] & \cdots \arrow[r] & Y_{m-1}^{\mathrm{HN}} \arrow[r] & Y_m^{\mathrm{HN}} = Y\end{tikzcd}$$

be the Harder-Narasimhan sequence of Y. For all integers $0 \leq i < j \leq m$, let $P_{j,i}$ be the canonical morphism from Y_j^{HN} to $Y_j^{\mathrm{HN}}/Y_i^{\mathrm{HN}}$. For any integer $0 \leq i \leq n$, let U_i be the canonical monomorphism from X_i^{HN} to X. Suppose that λ is a real number. If $i_{\overline{X}}(\lambda) = 0$ or if $i_{\overline{Y}}(\lambda) = 0$, we define F_λ as the zero morphism from X_λ to Y_λ; if $i_{\overline{Y}}(\lambda) = m$, we have $\overline{Y}_\lambda = \overline{Y}$ and we define F_λ as the composition $fU_{i_X(\lambda)}$; otherwise we have $\widehat{\mu}(\overline{X}_{i_{\overline{X}}(\lambda)}^{\mathrm{HN}} / \overline{X}_{i_{\overline{X}}(\lambda)-1}^{\mathrm{HN}}) \geq \lambda$ and $\widehat{\mu}(\overline{Y}_{i_{\overline{Y}}(\lambda)}^{\mathrm{HN}} / \overline{Y}_{i_{\overline{Y}}(\lambda)-1}^{\mathrm{HN}}) \geq \lambda$, but $\widehat{\mu}(\overline{Y}_j^{\mathrm{HN}} / \overline{Y}_{j-1}^{\mathrm{HN}}) < \lambda$ for any $j > i_{\overline{Y}}(\lambda)$. We will prove by induction that the morphism $fU_{i_{\overline{X}}(\lambda)}$ factorizes through $Y_{i_{\overline{Y}}(\lambda)}^{\mathrm{HN}}$.

First it is obvious that the morphism $fU_{i_{\overline{X}}(\lambda)}$ factorizes through $Y^{\mathrm{HN}}_m = Y$. If it factorizes through certain $\varphi_j : X^{\mathrm{HN}}_{i_{\overline{X}}(\lambda)} \to Y^{\mathrm{HN}}_j$, where $j > i_{\overline{Y}}(\lambda)$, then the composition $P_{j,j-1}\varphi_j$ must be zero since (see Proposition 4.7 and the remark after its proof)

$$\widehat{\mu}(\overline{Y}^{\mathrm{HN}}_j/\overline{Y}^{\mathrm{HN}}_{j-1}) < \lambda \le \widehat{\mu}(\overline{X}^{\mathrm{HN}}_{i_{\overline{X}}(\lambda)}/\overline{X}^{\mathrm{HN}}_{i_{\overline{X}}(\lambda)-1}) = \widehat{\mu}_{\min}(\overline{X}^{\mathrm{HN}}_{i_{\overline{X}}(\lambda)}).$$

So the morphism $fU_{i_{\overline{X}}(\lambda)}$ factorizes through Y^{HN}_{j-1}. By induction we obtain that $fU_{i_{\overline{X}}(\lambda)}$ factorizes (in unique way) through a morphism $F_\lambda : X_{i_{\overline{X}}(\lambda)} \to Y_{i_{\overline{Y}}(\lambda)}$. The family of morphisms $F = (F_\lambda)_{\lambda\in\mathbb{R}}$ defines a natural transformation such that (F, f) is a morphism of filtrations. Therefore the morphism f is compatible with Harder-Narasimhan filtrations. □

Remark 5.4 Theorem 5.3 implies that HN defines actually a functor from the category $\mathcal{C}_A$ to the full sub-category $\mathbf{Fil}^{\mathbb{R},\mathrm{self}}(\mathcal{C})$ of $\mathbf{Fil}^{\mathbb{R}}(\mathcal{C})$ consisting of $\mathbb{R}$-filtrations which are separated, exhaustive, left continuous and of finite length, which sends an arithmetic object $\overline{X}$ to its Harder-Narasimhan filtration.

Corollary 5.5 *Suppose in the case where A is non-trivial that any epimorphism in $\mathcal{C}$ has a kernel. Let $\overline{X}$ and $\overline{Y}$ be two arithmetic objects and $f : Y \to X$ be a morphism which is compatible with arithmetic structures. If $\widehat{\mu}_{\min}(\overline{Y}) \ge \lambda$, then the morphism f factorizes through X_λ.*

Proof. Since f is compatible with arithmetic structures, it is compatible with Harder-Narasimhan filtrations. So the restriction of f on Y_λ factorizes through X_λ. As $\widehat{\mu}_{\min}(\overline{Y}) \ge \lambda$, we have $Y_\lambda = Y$, therefore f factorizes through X_λ. □

5.3 Harder-Narasimhan polygons and Harder-Narasimhan measures

Let $\overline{X}$ be a non-zero arithmetic object and

$$0 = X^{\mathrm{HN}}_0 \longrightarrow X^{\mathrm{HN}}_1 \longrightarrow \cdots \longrightarrow X^{\mathrm{HN}}_{n-1} \longrightarrow X^{\mathrm{HN}}_n = X$$

be its Harder-Narasimhan sequence. For any integer $0 \le i \le n$, we note $t_i = \operatorname{rk} X^{\mathrm{HN}}_i/\operatorname{rk} X$. For any integer $1 \le i \le n$, we note $\lambda_i = \widehat{\mu}(X^{\mathrm{HN}}_i/X^{\mathrm{HN}}_{i-1})$. Then the function

$$P_{\overline{X}}(t) = \sum_{i=1}^{n} \left(\frac{\widehat{\deg}(\overline{X}^{\mathrm{HN}}_{i-1})}{\operatorname{rk} X} + \lambda_i(t - t_{i-1}) \right) \mathbb{1}_{[t_{i-1},t_i]}(t)$$

is a polygon[9] on $[0,1]$, called the *normalized Harder-Narasimhan polygon* of $\overline{X}$. The function $P_{\overline{X}}$ takes value 0 at the origin, and its first order derivative is given by

$$P'_{\overline{X}}(t)=\sum_{i=1}^{n}\lambda_i\,\mathbb{1}_{[t_{i-1},t_i[}(t).$$

The probability measure

$$\nu_{\overline{X}}:=\sum_{i=1}^{n}\frac{\mathrm{rk}(X_i^{\mathrm{HN}})-\mathrm{rk}(X_{i-1}^{\mathrm{HN}})}{\mathrm{rk}\,X}\delta_{\lambda_i}=\sum_{i=1}^{n}(t_i-t_{i-1})\delta_{\lambda_i}$$

is called the *Harder-Narasimhan measure* of $\overline{X}$. We define the Harder-Narasimhan measure of the zero arithmetic object to be the zero measure on $\mathbb{R}$. After Proposition 5.1, if $\overline{X}$ is a non-zero arithmetic object and if $(X_\lambda)_{\lambda\in\mathbb{R}}$ is the Harder-Narasimhan filtration of $\overline{X}$, then the Harder-Narasimhan measure $\nu_{\overline{X}}$ of $\overline{X}$ is the first order derivative (in distribution sense) of the function $t\longmapsto-\mathrm{rk}(X_t)$. Finally we point out that the Harder-Narasimhan polygon of a non-zero arithmetic object $\overline{X}$ can be uniquely determined in an explicit way from its Harder-Narasimhan measure.

Proposition 5.6 *Suppose in the case where A is non-trivial that any epimorphism in $\mathcal{C}$ has a kernel. If $\overline{X}$ and $\overline{Y}$ are two non-zero arithmetic objects and if $f:X\rightarrow Y$ is an isomorphism which is compatible with arithmetic structures, then $\widehat{\mu}(\overline{X})\leq\widehat{\mu}(\overline{Y})$, and therefore $\widehat{\deg}(\overline{X})\leq\widehat{\deg}(\overline{Y})$.*

Proof. Let $(X_\lambda)_{\lambda\in\mathbb{R}}$ and $(Y_\lambda)_{\lambda\in\mathbb{R}}$ be the Harder-Narasimhan filtrations of $\overline{X}$ and of $\overline{Y}$ respectively. Theorem 5.3 implies that f is compatible with filtrations. Hence $\mathrm{rk}(X_\lambda)\leq\mathrm{rk}(Y_\lambda)$ for any $\lambda\in\mathbb{R}$. Therefore, by taking an interval $[-M,M]$ containing $\mathrm{supp}(\nu_{\overline{X}})\cup\mathrm{supp}(\nu_{\overline{Y}})$, we obtain

$$\begin{aligned}\widehat{\mu}(\overline{X})&=\int_{-M}^{M}t\,\mathrm{d}\nu_{\overline{X}}(t)=-\int_{-M}^{M}t\,\mathrm{d}\,\mathrm{rk}(X_t)=\Big[-t\,\mathrm{rk}(X_t)\Big]_{-M}^{M}+\int_{-M}^{M}\mathrm{rk}(X_t)\mathrm{d}t\\&\leq M\,\mathrm{rk}(X_M)+\int_{-M}^{M}\mathrm{rk}(Y_t)\mathrm{d}t=M\,\mathrm{rk}(Y_M)+\int_{-M}^{M}\mathrm{rk}(Y_t)\mathrm{d}t=\widehat{\mu}(\overline{Y}).\end{aligned}$$

□

Let $\mathcal{D}$ be an Abelian category and rk be a rank function on $\mathcal{D}$. It is interesting to calculate explicitly the Harder-Narasimhan filtration of an object Y in $\mathcal{D}$, equipped with an $\mathbb{R}$-filtration $\mathcal{F}=(Y_\lambda)_{\lambda\in\mathbb{R}}$ which is separated, exhaustive, left continuous and of finite length. Let $U=\{\lambda_1>\cdots>\lambda_n\}$ be the minimal jumping set of the filtration $\mathcal{F}$, then

$$\begin{tikzcd}0\arrow[r] & Y_{\lambda_1}\arrow[r] & Y_{\lambda_2}\arrow[r] & \cdots\arrow[r] & Y_{\lambda_n}=Y\end{tikzcd}$$

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[9] Namely a concave function having value 0 at the origin and which is piecewise linear.

is the Harder-Narasimhan sequence of $\overline{Y}=(Y,[\mathcal{F}])$. Therefore, the Harder-Narasimhan filtration of $\overline{Y}$ is just the filtration $\mathcal{F}$ itself. So we have

$$P'_{\overline{Y}}(t)=\sum_{i=1}^{n}\lambda_i\,1\!\!1_{[t_{i-1},t_i[}$$

where $t_0=0$, and for any $1\leq i\leq n$, $t_i=\operatorname{rk}(Y_{\lambda_i})/\operatorname{rk}(Y)$. Furthermore,

$$\nu_{\overline{Y}}=\sum_{i=1}^{n}(t_i-t_{i-1})\delta_{\lambda_i}.$$

Let $F:\mathcal{C}\rightarrow\mathcal{D}$ be an exact functor from $\mathcal{C}$ to an Abelian category $\mathcal{D}$. The functor F induces a functor $\widehat{F}:\mathcal{C}_A\rightarrow\mathcal{D}$ which sends an arithmetic object $\overline{X}$ to $F(X)$, it also induces a homomorphism of groups $K_0(F):K_0(\mathcal{C},\mathcal{E},A)\rightarrow K_0(\mathcal{D})$. Since F is exact, it sends monomorphisms to monomorphisms, therefore it induces a functor $\widetilde{F}:\mathbf{Fil}^{\mathbb{R},\mathrm{self}}(\mathcal{C})\rightarrow\mathbf{Fil}^{\mathbb{R},\mathrm{self}}(\mathcal{D})$. If $\overline{X}$ is an arithmetic object of $(\mathcal{C},\mathcal{E},A)$, then $\widetilde{F}(\mathrm{HN}(\overline{X}))$ is an $\mathbb{R}$-filtration of $F(X)$. The following proposition shows that we can recover the Harder-Narasimhan polygon and the Harder-Narasimhan measure of $\overline{X}$ from the filtration $\widetilde{F}(\mathrm{HN}(\overline{X}))$.

Proposition 5.7 *Suppose given a rank function* rk *on* $K_0(\mathcal{D})$ *(which defines a Harder-Narasimhan category structure on* $\mathcal{D}$*) such that the functor* F *preserves rank functions (i.e.* $\operatorname{rk}(F(X))=\operatorname{rk}(X)$ *for any* $X\in\operatorname{obj}\mathcal{C}$*). Then for any arithmetic object* $\overline{X}$ *in* $\mathcal{C}_A$*, the normalized Harder-Narasimhan polygon of the filtration* $\overline{F(X)}=(F(X),[\widetilde{F}(\mathrm{HN}(\overline{X}))])$ *coincides with that of* $\overline{X}$*, and the Harder-Narasimhan measure of* $\overline{F(X)}$ *coincides with that of* $\overline{X}$*.*

Proof. Since the Harder-Narasimhan filtration of $\overline{F(X)}$ coincides with $\widetilde{F}(\mathrm{HN}(X))$, the function $t\longmapsto-\operatorname{rk}(\mathrm{HN}(\overline{X})(t))$ identifies with $t\longmapsto-\operatorname{rk}(\widetilde{F}(\mathrm{HN}(\overline{X}))(t))$. Therefore $\nu_{\overline{F(X)}}=\nu_{\overline{X}}$ and hence $P_{\overline{F(X)}}=P_{\overline{X}}$. □

5.4 Abelian category case

Let $(\mathcal{C},\mathcal{E},A)$ be an arithmetic exact category, $\widehat{\deg}$ be a degree function on $(\mathcal{C},\mathcal{E},A)$ and rk be a rank function on $(\mathcal{C},\mathcal{E})$. If $(\mathcal{C},\mathcal{E})$ is an Abelian category, then the axioms for $(\mathcal{C},\mathcal{E},A,\widehat{\deg},\operatorname{rk})$ to be a Harder-Narasimhan category can be considerably simplified. We shall show this fact in Proposition 5.8.

Proposition 5.8 *Supose that* $(\mathcal{C},\mathcal{E})$ *is an Abelian category. Then* $(\mathcal{C},\mathcal{E},A,\widehat{\deg},\operatorname{rk})$ *is a Harder-Narasimhan category if the following conditions are satisfied:*

1) for any non-zero arithmetic object $\overline{X}$, there exists a non-zero arithmetic subobject $\overline{Z}$ of $\overline{X}$ such that

$$\widehat{\mu}(\overline{Z}) = \sup\{\widehat{\mu}(\overline{Y}) \mid \overline{Y} \textit{ is a non-zero arithmetic subobject of } \overline{X}\}; \qquad (7)$$

2) for any non-zero object X in $\mathcal{C}$ and for any two arithmetic structures h_X and h'_X on X, if $\mathrm{Id}_X : (X, h_X) \to (X, h'_X)$ is compatible with arithmetic structures, then $\widehat{\mu}(X, h_X) \le \widehat{\mu}(X, h'_X)$.

Note that the condition 1) is verified once $\{\widehat{\mu}(\overline{Y}) \mid \overline{Y}$ is a non-zero arithmetic subobject of $\overline{X}\}$ is a finite set, or equivalently $\{\widehat{\deg}(\overline{Y}) \mid \overline{Y}$ is a non-zero arithmetic subobject of $\overline{X}\}$ is a finite set for any non-zero arithmetic object $\overline{X}$.

The following technical lemma, which is dual to Lemma 4.6, is useful for the proof of Proposition 5.8.

Lemma 5.9 *Let $(\mathcal{C}, \mathcal{E}, A)$ be an arithmetic exact category. Suppose that any monomorphism in $\mathcal{C}$ has a cokernel. Let (X, h_X) and (Y, h_Y) be two arithmetic objects and $f : X \to Y$ be a morphism in $\mathcal{C}$. Suppose that (Y, h_Y) is an arithmetic subobject of an arithmetic object (Z, h_Z) and $u : Y \to Z$ is the inclusion morphism. Then the morphism f is compatible with arithmetic structures if and only if it is the case for uf.*

Proof of Proposition 5.8. Suppose that $\overline{X}_{\mathrm{des}}$ is a non-zero arithmetic subobject of $\overline{X}$ verifying (7), whose rank r is maximal. Suppose that $\overline{Z}$ is another non-zero arithmetic subobject of $\overline{X}$ verifying (7). Consider the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & Z \cap X_{\mathrm{des}} \arrow[r] & Z \oplus X_{\mathrm{des}} \arrow[r] & Z + X_{\mathrm{des}} \arrow[r] & 0, \end{tikzcd}$$

where $Z \cap X_{\mathrm{des}}$ is the fiber product $Z \times_X X_{\mathrm{des}}$ and $Z + X_{\mathrm{des}}$ is the canonical image of $Z \oplus X_{\mathrm{des}}$ in X. Therefore,

$$\widehat{\deg}(\overline{Z \cap X_{\mathrm{des}}}) + \widehat{\deg}(\overline{Z + X_{\mathrm{des}}}) = \widehat{\deg}(\overline{Z}) + \widehat{\deg}(\overline{X}_{\mathrm{des}}) = \alpha(\mathrm{rk}(Z) + \mathrm{rk}(X_{\mathrm{des}})),$$

so

$$\begin{aligned} \widehat{\deg}(\overline{Z + X_{\mathrm{des}}}) &= \alpha(\mathrm{rk}(Z) + \mathrm{rk}(X_{\mathrm{des}})) - \widehat{\deg}(\overline{Z \cap X_{\mathrm{des}}}) \\ &\ge \alpha(\mathrm{rk}(Z) + \mathrm{rk}(X_{\mathrm{des}}) - \mathrm{rk}(\overline{Z \cap X_{\mathrm{des}}})) = \alpha\,\mathrm{rk}(\overline{Z + X_{\mathrm{des}}}), \end{aligned}$$

which means that $\widehat{\mu}(\overline{Z + X_{\mathrm{des}}}) = \alpha$, and hence $\mathrm{rk}(Z + X_{\mathrm{des}}) = \mathrm{rk}(X_{\mathrm{des}})$ since $\mathrm{rk}(X_{\mathrm{des}})$ is maximal. As rk is a rank function, we obtain $Z = X_{\mathrm{des}}$. Therefore, the axiom (**HN**1) is fulfilled.

We now verify the axiom (**HN**2). Let $\overline{X} = (X, h_X)$ and $\overline{Y} = (Y, h_Y)$ be two semistable arithmetic objects. Suppose that there exists a non-zero morphism

$f : X \to Y$ which is compatible with arithmetic objects. Let Z be the image of f in Y, $u : Z \to Y$ be the canonical inclusion and $\pi : X \to Z$ be the canonical projection. The fact that f is compatible with arithmetic structures implies that the identity morphism $\mathrm{Id}_Z : (Z, \pi_* h_X) \to (Z, u^* h_Y)$ is compatible with arithmetic structures (after Lemmas 4.6 and 5.9). Therefore, the semistability of $\overline{X}$ and of $\overline{Y}$, combining the condition 2), implies that $\widehat{\mu}(\overline{X}) \le \widehat{\mu}(Z, \pi_* h_X) \le \widehat{\mu}(Z, u^* h_Y) \le \widehat{\mu}(\overline{Y})$.

□

Corollary 5.10 *Let $(\mathcal{C}, \mathcal{E})$ be an Abelian category equipped with a rank function* rk*, $n \ge 2$ be an integer, $(A_i)_{1 \le i \le n}$ be a family of arithmetic structures on $(\mathcal{C}, \mathcal{E})$ and $A = A_1 \times \cdots \times A_n$. Suppose given for any $1 \le i \le n$ a degree function $\widehat{\deg}_i$ on $(\mathcal{C}, \mathcal{E}, A_i)$ such that*

1) $\{\widehat{\deg}_i(\overline{Y}) \mid \overline{Y}$ is a non-zero arithmetic subobject of $\overline{X}\}$ is a finite set for any non-zero arithmetic objec $\overline{X}$;

2) $(\mathcal{C}, \mathcal{E}, A_i, \widehat{\deg}_i, \mathrm{rk})$ is a Harder-Narasimhan category.

Let $\alpha = (a_i)_{1 \le i \le n}$ be an arbitrary element in $\mathbb{R}^n_{\ge 0}$. If we denote by $\widehat{\deg}_\alpha = \sum_{i=1}^n a_i \widehat{\deg}_i$, then $(\mathcal{C}, \mathcal{E}, A, \widehat{\deg}_\alpha, \mathrm{rk})$ is a Harder-Narasimhan category.

6 Examples of Harder-Narasimhan categories

In this section, we shall give some example of Harder-Narasimhan categories.

6.1 Filtrations in an extension of Abelian categories

Let $\mathcal{C}$ and $\mathcal{C}'$ be two Abelian categories and $F : \mathcal{C} \to \mathcal{C}'$ be an exact functor which sends a non-zero object of $\mathcal{C}$ to a non-zero object of $\mathcal{C}'$. Let $\mathcal{E}$ (resp. $\mathcal{E}'$) be the class of all exact sequences in $\mathcal{C}$ (resp. $\mathcal{C}'$). Suppose given a rank function $\mathrm{rk}' : K_0(\mathcal{C}', \mathcal{E}') \to \mathbb{R}$. Let I be a non-empty subset of $\mathbb{R}$, equipped with the induced order. For any object X in $\mathcal{C}$, let $A(X)$ be the set of isomorphism classes of objects in $\mathbf{Fil}^{I,\mathrm{self}}_{F(X)}$. Suppose that $h = [\mathcal{F}]$ is an element in $A(X)$. For any monomorphism $u : X_0 \to X$, we define $u^*(h)$ to be the class $[F(u)^*\mathcal{F}] \in A(X_0)$. For any epimorphism $p : X \to Y$, we define $p_*(h)$ to be $[F(p)_*\mathcal{F}] \in A(Y)$. Similarly to the the case of filtrations in an Abelian category, $(\mathcal{C}, \mathcal{E}, A)$ is an arithmetic exact category. By definition we know that if $\overline{X}_i = (X_i, [\mathcal{F}_i])$ $(i = 1, 2)$ are two arithmetic objects, then a morphism $f : X_1 \to X_2$ in $\mathcal{C}$ is compatible with arithmetic structures if and only if $F(f)$ is compatible with filtrations $(\mathcal{F}_1, \mathcal{F}_2)$. For any arithmetic object $\overline{X}$ of $(\mathcal{C}, \mathcal{E}, A)$, we define the arithmetic degree of $\overline{X} = (X, [\mathcal{F}])$ to be the real number

$$\widehat{\deg}(\overline{X}) = \sum_{\lambda \in I} \lambda \Big(\operatorname{rk}'(\mathcal{F}(\lambda)) - \sup_{j > \lambda, j \in I} \operatorname{rk}'(\mathcal{F}(j)) \Big).$$

Since F is an exact functor, $\widehat{\deg}$ extends naturally to a homomorphism from $K_0(\mathcal{C}, \mathcal{E}, A)$ to $\mathbb{R}$. In the previous section we have shown that if we define, for any object X' in $\mathcal{C}'$, $A'(X')$ as the set of all isomorphism classes of objects in $\mathbf{Fil}_{X'}^{I,\mathrm{self}}$, then $(\mathcal{C}', \mathcal{E}', A')$ is an arithmetic category. Furthermore, if for any arithmetic object $\overline{X}' = (X', [\mathcal{F}'])$, we define

$$\widehat{\deg}'(\overline{X}') = \sum_{\lambda \in I} \lambda \Big(\operatorname{rk}'(\mathcal{F}'(\lambda)) - \sup_{j > \lambda, j \in I} \operatorname{rk}'(\mathcal{F}'(j)) \Big),$$

then $\widehat{\deg}'$ extends naturally to a homomorphism $K_0(\mathcal{C}', \mathcal{E}', A') \rightarrow \mathbb{R}$, and $(\mathcal{C}', \mathcal{E}', A', \widehat{\deg}', \operatorname{rk}')$ is a Harder-Narasimhan category. Notice that for any object $(X, [\mathcal{F}])$ in $\mathcal{C}_A$, we have

$$\widehat{\deg}(X, [\mathcal{F}]) = \widehat{\deg}'(F(X), [\mathcal{F}]).$$

Proposition 6.1 *Denote by* rk *the composition* $\operatorname{rk}' \circ K_0(F)$*. Then* $(\mathcal{C}, \mathcal{E}, A, \widehat{\deg}, \operatorname{rk})$ *is a Harder-Narasimhan category.*

Proof. Since F is an exact functor which sends non-zero objects to non-zero objects, the homomorphism rk is a rank function. Let $\overline{X} = (X, [\mathcal{F}])$ be a non-zero arithmetic object in $\mathcal{C}_A$. First we show that

$$S := \Big\{ \widehat{\deg}(\overline{Y}) \mid \overline{Y} \text{ is an arithmetic subobject of } \overline{X} \Big\}$$

is a finite set. Let $U = \{\lambda_1, \cdots, \lambda_n\}$ be a jumping set of $\mathcal{F}$. If $u : Y \rightarrow X$ is a monomorphism, then U is also a jumping set of $F(u)^*\mathcal{F}$, therefore,

$$\widehat{\deg}(Y, [F(u)^*\mathcal{F}]) \in \Big\{ \sum_{i=1}^n a_i \lambda_i \,\Big|\, \forall 1 \le i \le n,\ a_i \in \mathbb{N},\ 0 \le a_1 + \cdots + a_n \le \operatorname{rk}(X) \Big\}.$$

The latter is clearly a finite set. Therefore, the condition 1) of Proposition 5.8 is satisfied. If X is an object in $\mathcal{C}$ and if $\mathcal{F}$ and $\mathcal{G}$ are two filtrations of $F(X)$ such that $\mathrm{Id}_{F(X)} = F(\mathrm{Id}_X)$ is compatible to filtrations $(\mathcal{F}, \mathcal{G})$, then after Proposition 5.6, $\widehat{\deg}'(F(X), [\mathcal{F}]) \le \widehat{\deg}'(F(X), [\mathcal{G}])$ and therefore $\widehat{\mu}(X, [\mathcal{F}]) \le \widehat{\mu}(X, [\mathcal{G}])$. After Proposition 5.8, $(\mathcal{C}, \mathcal{E}, A, \widehat{\deg}, \operatorname{rk})$ is a Harder-Narasimhan category. □

Remark 6.2 By Corollary 5.10, we can easily generalize the formalism of Harder and Narasimhan to the case of bojects in $\mathcal{C}$ equipped with several filtrations of their images by F in $\mathcal{C}'$.

6.2 Filtered (φ, N)-modules

Let K be a field of characteristic 0, equipped with a discrete valuation v such that K is complete for the topology defined by v. Suppose that the residue field k of K is of characteristic $p > 0$. Let K_0 be the fraction field of Witt vector ring $W(k)$ and $\sigma : K_0 \to K_0$ be the absolute Frobenius endomorphism. We call (φ, N)*-module* (see [8], [21], and [6] for details) any finite dimensional vector space D over K_0, equipped with

1) a bijective σ-linear endomorphism $\varphi : D \to D$,
2) a K_0-linear endomorphism $N : D \to D$ such that $N\varphi = p\varphi N$.

Let $\mathcal{C}$ be the category of all (φ, N)-modules. It's an Abelian category. We denote by $\mathcal{E}$ the class of all short exact sequences in $\mathcal{C}$. There exists a natural rank function rk on the category $\mathcal{C}$ defined by the rank of vector space over K_0. Furthermore, we have an exact functor F from $\mathcal{C}$ to the category $\mathbf{Vec}_K$ of all finite dimensional vector spaces over K, which sends a (φ, N)-module D to $D \otimes_{K_0} K$. Consider the arithmetic structure A on $(\mathcal{C}, \mathcal{E})$ such that, for any (φ, N)-module D, $A(D)$ is the set of isomorphism classes of $\mathbb{Z}$-filtrations of $F(D) = D \otimes_{K_0} K$. Then $(\mathcal{C}, \mathcal{E}, A)$ becomes an arithmetic exact category. The objects in $\mathcal{C}_A$ are called *filtered* (φ, N)*-modules.*

To each (φ, N)-module D we associate an integer $\deg_\varphi(D) = -v(\det\varphi)$. If $\overline{D} = (D, [\mathcal{F}])$ is a filtered (φ, N)-module, we define

$$\deg_F(\overline{D}) := \sum_{i\in\mathbb{Z}} i\Big(\operatorname{rk}_K(\mathcal{F}(i)) - \operatorname{rk}_K(\mathcal{F}(i+1))\Big) \qquad \text{and} \qquad \widehat{\deg}(\overline{D}) = \deg_F(\overline{D}) + \deg_\varphi(D).$$

It is clear that $\widehat{\deg}$ is a degree function on $(\mathcal{C}, \mathcal{E}, A)$.

Proposition 6.3 $(\mathcal{C}, \mathcal{E}, A, \widehat{\deg}, \operatorname{rk})$ *is a Harder-Narasimhan category.*

Proof. Let $\overline{X} = (X, [\mathcal{F}])$ be a non-zero filtered (φ, N)-module. We have shown in the previous example that $S_F = \{\deg_F(\overline{Y}) \mid \overline{Y}$ is an arithmetic subobject of $\overline{X}\}$ is a finite set. By the isoclinic decomposition we obtain that $S_\varphi = \{\deg_\varphi(Y) \mid Y$ is a subobject of $X\}$ is also a finite set. Therefore,

$$\widetilde{S} = \{\widehat{\mu}(\overline{Y}) \mid \overline{Y} \text{ is an arithmetic subobject of } \overline{X}\}$$

is a finite set, and hence the condition 1) of Proposition 5.8 is verified.

Suppose that X is a (φ, N)-module and $\mathcal{F}$ and $\mathcal{G}$ are two $\mathbb{Z}$-filtrations of X such that Id_X is compatible with filtrations $(\mathcal{F}, \mathcal{G})$. We have shown in the previous example that $\deg_F(X, \mathcal{F}) \leq \deg_F(X, \mathcal{G})$. Hence $\widehat{\deg}(X, \mathcal{F}) \leq \widehat{\deg}(X, \mathcal{G})$. Therefore, the condition 2) of Proposition 5.8 is verified, and hence

$(\mathcal{C},\mathcal{E},A,\widehat{\deg},\mathrm{rk})$ is a Harder-Narasimhan category. □

Note that semistable filtered (φ,N)-modules having slope 0 are nothing but admissible filtered (φ,N)-modules. In classical literature, such filtered (φ,N)-modules are said to be weakly admissible. In fact, Colmez and Fontaine [6] have proved that all weakly admissible (φ,N)-modules are admissible, which had been a conjecture of Fontaine.

6.3 Torsion free sheaves on a polarized projective variety

Let X be a geometrically normal projective variety of dimension $d\geq 1$ over a field K and L be an ample invertible $\mathcal{O}_X$-module. We denote by $\mathbf{TF}(X)$ the category of torsion free coherent sheaves on X. Notice that if $0\longrightarrow E'\longrightarrow E\longrightarrow E''\longrightarrow 0$ is an exact sequence of coherent $\mathcal{O}_X$-modules such that E' and E'' are torsion free, then also is E. Therefore, $\mathbf{TF}(X)$ is an exact sub-category of the Abelian category of all coherent $\mathcal{O}_X$-modules on X. Let $\mathcal{E}$ be the class of all exact sequences in $\mathbf{TF}(X)$ and let A be the trivial arithmetic structure on it. If E is a torsion free coherent $\mathcal{O}_K$-module, we denote by $\mathrm{rk}(E)$ its rank and by $\deg(E)$ the intersection number $c_1(L)^{d-1}c_1(E)$. The mapping deg (resp. rk) extends naturally to a homomorphism from $K_0(\mathbf{TF}(X))$ to $\mathbb{R}$ (resp. $\mathbb{Z}$). A classical result [15] (see also [19]) shows that $(\mathbf{TF}(X),\mathcal{E},A,\deg,\mathrm{rk})$ is in fact a Harder-Narasimhan category.

6.4 Hermitian vector bundles on the spectrum of an algebraic integer ring

Let K be a number field and $\mathcal{O}_K$ be its integer ring. We denote by $\mathbf{Pro}(\mathcal{O}_K)$ the category of all projective $\mathcal{O}_K$-modules of finite type. Let $\mathcal{E}$ be the family of short exact sequences of projective $\mathcal{O}_K$-modules of finite type. Then $(\mathbf{Pro}(\mathcal{O}_K),\mathcal{E})$ is an exact category.

We denote by Σ_f the set of all finite places of K which identifies with the set of closed points of $\operatorname{Spec}\mathcal{O}_K$. If $\mathfrak{p}$ is an element in Σ_f, we denote by $v_{\mathfrak{p}}:K^\times\rightarrow\mathbb{Z}$ the valuation associated to $\mathfrak{p}$ which sends a non-zero element $a\in\mathcal{O}_K$ to the length of the Artinian local ring $\mathcal{O}_{K,\mathfrak{p}}/a\mathcal{O}_{K,\mathfrak{p}}$. Let $\mathbb{F}_{\mathfrak{p}}:=\mathcal{O}_{K,\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{K,\mathfrak{p}}$ be the residue field and $N_{\mathfrak{p}}$ be its cardinal. We denote by $|\cdot|_{\mathfrak{p}}$ the absolute value on K such that $|x|_{\mathfrak{p}}=N_{\mathfrak{p}}^{-v_{\mathfrak{p}}(x)}$ for any $x\in K^\times$. Let Σ_∞ be the set of all embeddings of K in $\mathbb{C}$, whose cardinal is $[K:\mathbb{Q}]$. For any $\sigma\in\Sigma_\infty$, let $|\cdot|_\sigma:K\rightarrow\mathbb{R}_{\geq 0}$ be the Archimedian absolute value such that $|x|_\sigma=|\sigma(x)|$. The complex conjugation defines an involution $\sigma\mapsto\overline{\sigma}$ on Σ_∞. The product formula asserts that for any

$x \in K^\times$, $|x|_{\mathfrak{p}} = 1$ for almost all finite places $\mathfrak{p}$, and we have

$$\prod_{\mathfrak{p}\in\Sigma_f} |x|_{\mathfrak{p}} \prod_{\sigma\in\Sigma_\infty} |x|_\sigma = 1.$$

Notice that a Hermitian vector bundle over $\operatorname{Spec}\mathcal{O}_K$ is nothing other than a pair $\overline{E} = (E, (\|\cdot\|_\sigma)_{\sigma\in\Sigma_\infty})$, where E is a projective $\mathcal{O}_K$-module of finite type E, and for any $\sigma \in \Sigma_\infty$, $\|\cdot\|_\sigma$ is a Hermitian metric on $E \otimes_{\mathcal{O}_K,\sigma} \mathbb{C}$ such that $\|x \otimes z\|_\sigma = \|x \otimes \overline{z}\|_{\overline{\sigma}}$. The rank of the Hermitian vector bundle $\overline{E}$ is just defined to be that of E. The rank function on $\mathbf{Pro}(\mathcal{O}_K)$ extends naturally to a homomorphism from $K_0(\mathbf{Pro}(\mathcal{O}_K))$ to $\mathbb{Z}$. If $\overline{E}$ is a Hermitian vector bundle of rank r, the (normalized) Arakelov degree of $\overline{E}$ is by definition

$$\widehat{\deg}_n\overline{E} = \frac{1}{[K:\mathbb{Q}]}\Big(\log\#(E/\mathcal{O}_K s_1 + \cdots + \mathcal{O}_K s_r) - \frac{1}{2}\sum_{\sigma\in\Sigma_\infty}\log\det(\langle s_i, s_j\rangle_\sigma)\Big),$$

where $(s_1, \cdots, s_r) \in E^r$ is an arbitrary element in E^r which defines a basis of E_K over K. This definition doesn't depend on the choice of $(s_1, \cdots, s_r)$. For more details, see [1] and [4].

If for any projective $\mathcal{O}_K$-module of finite type E, we denote by $A(E)$ the set of all Hermitian structures on E, then $(\mathbf{Pro}(\mathcal{O}_K), \mathcal{E}, A)$ is an arithmetic exact category, as we have shown in the previous section. The category $\mathbf{Pro}(\mathcal{O}_K)_A$ is the category of all Hermitian vector bundles over $\operatorname{Spec}\mathcal{O}_K$ and all homomorphism of $\mathcal{O}_K$-modules having norm ≤ 1 at every $\sigma \in \Sigma_\infty$. Furthermore, if $0 \longrightarrow \overline{E}' \longrightarrow \overline{E} \longrightarrow \overline{E}'' \longrightarrow 0$ is a sequence in $\mathcal{E}_A$, then we have the equality $\widehat{\deg}_n(\overline{E}) = \widehat{\deg}_n(\overline{E}') + \widehat{\deg}_n(\overline{E}'')$. Therefore, $\widehat{\deg}_n$ extends to a homomorphism from $K_0(\mathbf{Pro}(\mathcal{O}_K), \mathcal{E}, A)$ to $\mathbb{R}$. The results of Stuhler [20] and Grayson [10] show that $(\mathbf{Pro}(\mathcal{O}_K), \mathcal{E}, A, \widehat{\deg}_n, \operatorname{rk})$ is a Harder-Narasimhan category.

A recent work of Moriwaki [16] generalizes the notion of semistability and Harder-Narasimhan flag to Hermitian torsion free coherent sheaves on normal arithmetic varieties. His appoach may also be adapted into the framework of Harder-Narasimhan categories.

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