FORMAL GROUPS

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1. Formal groups over fields

In this section we follow the presentation given in [Dem72].

Let k be a fixed field. All objects and morphisms of this section will be defined over k. We will consider formal schemes of a very particular type, as in the following

Definition 1.1. A formal scheme $\mathfrak{X}$ is an *affine* formal scheme $\mathfrak{X} = \mathrm{Spf}(A)$ where A is a *profinite* k-algebra. By this we mean that A is equipped with a decreasing sequence of ideals $\{\mathfrak{a}_i\}_{i \in \mathbb{N}}$ such that

- $A/\mathfrak{a}_i$ is a *finite* k-algebra (endowed with the discrete topology);
- the natural morphism $A \to \varprojlim_i A/\mathfrak{a}_i$ is an isomorphism.

If A is a finite k-algebra, we will always take $\mathfrak{a}_i = 0$ for i big enough. In this way, we can consider any finite k-scheme as a formal scheme.

If we want to stress the functorial point of view, we will consider a formal scheme $\mathfrak{X} = \mathrm{Spf}(A)$ as a functor

$$\text{finite } k\text{-algebras} \to \mathbf{set}$$
$$R \mapsto \varinjlim_i \mathrm{Hom}_k(A/\mathfrak{a}_i, R)$$

Note that Yoneda's lemma holds in this case, i.e. the functor $\mathrm{Spf}(A)$ defines A uniquely.

Let M be any k-vector space. We write M^* for its k-linear dual $\mathrm{Hom}_k(M, k)$. If M is finite dimensional over k, then $(M \otimes_k M)^* = M^* \otimes_k M^*$ (but this is false in general). From this observation we easily obtain the following

Lemma 1.2. *Let A be a finite k-algebra. Then A^* has a natural structure of finite k-coalgebra.*

On the other hand, if $C = (C, \Delta, \varepsilon)$ is any k-coalgebra (not necessarily finite), then C^* has an algebra structure, defined by,

$$\langle xy, u \rangle = \langle x \otimes y, \Delta u \rangle,$$

where $x, y \in C^*$ and $u \in C$.

Let C be a k-coalgebra. We define a functor $\mathrm{Spec}^*(C)\colon$ finite k-algebra $\to \mathbf{set}$ by the formula

$$\mathrm{Spec}^*(C)(R) = \{u \in C \otimes_k R \text{ such that } \Delta(u) = u \otimes u \text{ and } \varepsilon(u) = 1\}.$$

Proposition 1.3. *The above construction gives a functor*

$$\mathrm{Spec}^*\colon k\text{-}\textit{coalgebras} \to \textit{formal schemes.}$$

This functor is an equivalence of categories.

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Proof. Let A and R be finite k-algebras. Then one can easily check the following identity

$$\mathrm{Spf}(A)(R) = \{u \in A^* \otimes_k R \text{ such that } \Delta(u) = u \otimes u \text{ and } \varepsilon(u) = 1\}.$$

In particular we have $\mathrm{Spf}(A) = \mathrm{Spec}^*(A^*)$. This gives the proposition for finite objects. Since any k-coalgebra is can be written as the union of finite dimensional subcoalgebras, we see that $\mathrm{Spec}^*(C)$ is a formal scheme for any C.

If $\mathfrak{X} = \mathrm{Spf}(A)$ is a formal scheme, with $A = \varprojlim_i A_i$ then let $C := \varinjlim_i A_i^*$, that is a k-coalgebra. We have

$$\mathrm{Spec}^*(C)(R) = \varinjlim_i \mathrm{Spec}^*(A_i^*)(R) = \varinjlim_i \mathrm{Spf}(A_i)(R) = \mathrm{Spf}(A)(R).$$

□

Definition 1.4. A *formal group* is a group object in the category of formal schemes.

Example 1.5. Let X be any group scheme of finite type. The morphism $\mathrm{Spec}(k) \to X$ gives a closed point $x \in X$, and we denote with $\mathcal{O}_{X,x}$ the corresponding local ring, with maximal ideal $\mathfrak{m}$. We set $\mathfrak{a}_i := \mathfrak{m}^i$. In this way we clearly obtain a formal scheme $\mathfrak{X}$, the *completion* of X along its zero section.

As a particular case of Proposition 1.3, we obtain the following

Lemma 1.6. *We have that* Spec* *gives an equivalence of categories between the category of Hopf k-algebra and the category of* commutative *formal schemes.*

Theorem 1.7 (Cartier duality)**.** *Let $G = \mathrm{Spec}(A)$ be any affine group scheme and let $G^\vee$ the functor that is Cartier dual to G. Then the restriction of $G^\vee$ to the category of finite k-algebras is representable by the formal scheme* $\mathrm{Spec}^*(A)$.

Proof. If R is a finite k-algebra, we have

$$G^\vee(R) = \mathrm{Hom}(G_R, \mathbb{G}_{\mathrm{m},R}) = \mathrm{Spec}^*(A)(R).$$

□

Definition 1.8. We say that a formal scheme $G = \mathrm{Spf}(A)$ is *connected* if A is a local ring. In this case, we say that G is of finite type if A is Noetherian and we define the dimension of G as the dimension of A.

Note that if X is a group scheme of finite type, then $\mathfrak{X}$ is a connected formal group.

Example 1.9. If X is any scheme, we can restrict the functor defined by X to the category of finite k-algebra and we obtain a formal scheme $\widehat{X}$. Explicitly, this formal scheme is given as follows: it is the disjoint union of $\mathrm{Spf}(\widehat{\mathcal{O}}_{X,x})$, where x runs through the points of X such that $[k(x) : k] < \infty$ and $\widehat{\mathcal{O}}_{X,x}$ is the completion of $\mathcal{O}_{X,x}$ with respect to powers of the maximal ideals. Note that if X is of finite type, then the condition $[k(x) : k] < \infty$ is equivalent to x being a closed point. Clearly $\widehat{X}$ is in general not connected. If X is a group scheme, then the same is true for $\widehat{X}$, and by definition we have that $\mathfrak{X}$ is the connected component of $\widehat{X}$.

Definition 1.10. Let $G = \mathrm{Spf}(A)$ be a formal group. We say that G is *smooth* if there is an isomorphism $A \cong k[[X_1, \dots, X_n]]$.

Since $k[[X_1,\ldots,X_n]]$ is a local ring, any smooth formal group is connected (of dimension n). Moreover, the comultiplication of any such formal group is given by a set of power series, as follows. Let $\underline{X}=(X_1,\ldots,X_n)$ and $\underline{Y}=(Y_1,\ldots,Y_n)$. Then $A\widehat{\otimes}A\cong k[[\underline{X},\underline{Y}]]$. The morphism $A\to A\widehat{\otimes}A$ is given by the images of the X_i's, hence by n power series in $2n$ variables $\underline{F}=(F_1,\ldots,F_n)$. It is easy to verify that the following conditions must be satisfied:

- $F_i(\underline{X},0)=F_i(0,\underline{X})=X_i$;
- $F_i(\underline{F}(\underline{X},\underline{Y}),\underline{Z})=F_i(\underline{X},\underline{F}(\underline{Y},\underline{Z}))$;
- there are n power series $\underline{H}=(H_1,\ldots,H_n)$ such that $F_i(\underline{X},\underline{H}(\underline{X}))=0$.

Furthermore, G is commutative if and only if $\underline{F}(\underline{X},\underline{Y})=\underline{F}(\underline{Y},\underline{X})$. Vice versa, to give a smooth formal group is the same as to specify $\underline{F}$ that satisfies the above conditions (the third condition can be omitted).

The smooth formal groups are the most important for us. They are also called formal group laws, and can be defined, via power series, over any base.

Example 1.11. Taking $n=1$ and $F(X,Y)=X+Y$ one gets the *formal additive group*, denoted $\widehat{\mathbb{G}}_{\mathrm{a}}$.

Example 1.12. Taking $n=1$ and $F(X,Y)=X+Y+XY$ one gets the *formal multiplicative group*, denoted $\widehat{\mathbb{G}}_{\mathrm{m}}$.

Example 1.13. Let A be an abelian variety. One can prove that $\widehat{A}$ is a smooth formal group, of dimension $n=\dim(A)$.

Remark 1.14. Note that the above examples make sense over any base.

If $G=\mathrm{Spf}(k[[\underline{X}]])$ and $G'=\mathrm{Spf}(k[[\underline{Y}]])$ are two smooth formal groups of dimension n and m, given by $\underline{F}$ and $\underline{G}$, then any morphism $G\to G'$ is given by m power series H_i of n variables (the images of the Y_i's), that satisfy

$$\underline{H}(\underline{F}(\underline{X},\underline{Y})=\underline{G}(\underline{H}(\underline{X}),\underline{H}(\underline{Y})).$$

Example 1.15. Let $\mathrm{char}(k)=0$ (one can more generally work over any $\mathbb{Q}$-algebra in this example). Then the power series given by $\log(1+X)$, i.e.

$$\sum_{i=1}^{\infty}\frac{(-1)^i X^i}{i}$$

defines an isomorphism $\widehat{\mathbb{G}}_{\mathrm{m}}\cong\widehat{\mathbb{G}}_{\mathrm{a}}$. The inverse is the power series of $\exp(X)-1$, i.e.

$$\sum_{i=1}^{\infty}\frac{X^i}{i!}.$$

Theorem 1.16 (Cartier)**.** *Let $G=\mathrm{Spf}(A)$ be a connected formal group of finite type. Then*

- *if $\mathrm{char}(k)=0$ then G is smooth;*
- *if $\mathrm{char}(k)=p\neq 0$, then the following are equivalent*
  1. *G is smooth;*
  2. *$A\otimes_k k^{p^{-1}}$ is reduced;*
  3. *the Frobenius morphism $\mathrm{F}_G\colon G\to G^{(p)}$ is an epimorphism.*

Definition 1.17. Let G be a formal group. We say that G is a *p-formal group* if it is of p-torsion, i.e. if the natural morphism $\varinjlim_i G[p^i]\to G$ is an isomorphism.

Example 1.18. Let us suppose that $\mathrm{char}(k) = p$ and let A be an abelian variety. One can prove that $\widehat{A} = \varinjlim_i \ker(\mathrm{F}^i_A)$ and that $\ker(\mathrm{F}^i_A)$ is finite of p-power order for each i. In particular, we have that $\widehat{A}$ is a p-formal group (it is even p-divisible or Barsotti-Tate).

2. Formal groups over general rings

In this section we present the theory of formal Lie groups over a general ring R. We will follow [Zin84], and we will simply say 'formal group' for a formal Lie group. Unfortunately, this is in contrast with the previous section, so in the case $R = k$ we do not obtain definitions equivalent to the ones above. In any case, for smooth formal groups of finite type, that is the most important case, the two theories are the same. The reason why we are forced to change the definition is that the theory of finite R-algebras is not so good in general as in the case $R = k$. We then have to work with *nilpotent* algebras, that of course cannot have unity, so theory can seem a little strange at a first sight. From now on, all formal groups will be commutative.

Definition 2.1. We write Nil be the category of commutative, nilpotent R-algebras and NilAug for the category of commutative, augmented, nilpotent R-algebras. These are couples (A, ε), where A is a commutative R-algebra with $1 \in A$, and $\varepsilon\colon A \to R$ is a morphism of R-algebras with nilpotent kernel.

Geometrically, objects of Nil are infinitesimal thickening of $\mathrm{Spec}(R)$.

Remark 2.2. The category Nil admits finite direct sums and fibred products. We have also that Nil admits infinite direct sums of set of nilpotent algebras whose nilpotency index is bounded.

We have a pair of adjoint equivalences between Nil and NilAug:

$$\mathcal{N} \mapsto R \oplus \mathcal{N}$$
$$\ker(\varepsilon) \leftarrow\!\shortmid (A, \varepsilon)$$

The multiplication in $R \oplus \mathcal{N}$ is given by $(r_1, n_1)(r_2, n_2) = (r_1 r_2, r_1 n_2 + r_2 n_1 + n_1 n_2)$. In particular we have

$$\mathrm{Hom}_{\mathrm{Nil}}(\ker(\varepsilon), \mathcal{N}) = \mathrm{Hom}_{\mathrm{NilAug}}(A, R \oplus \mathcal{N}).$$

In this way, we will consider functors $\mathrm{Nil} \to \mathbf{set}$ as functors $\mathrm{NilAug} \to \mathbf{set}$, and we will write the functor $\mathcal{N} \mapsto F(\mathcal{N})$ also as $A \mapsto F(A)$, where $A := R \oplus \mathcal{N}$.

If $\mathcal{N} \in \mathrm{Nil}$, we write $\mathrm{Spf}(A)$ (recall that $A = R \oplus \mathcal{N} \in \mathrm{NilAug}$) for the functor

$$\mathrm{Nil} \to \mathbf{set}$$
$$\mathcal{M} \mapsto \mathrm{Hom}_{\mathrm{Nil}}(\mathcal{N}, \mathcal{M}).$$

Remark 2.3. Note that $\mathrm{Spec}(A)$ represents a functor $\mathrm{Sch} \to \mathbf{set}$ that, when restricted to (the opposed category of) NilAug, is $\mathrm{Spf}(A)$. In this way any affine finite group scheme over R gives a functor $\mathrm{Nil} \to \mathbf{set}$.

Example 2.4. If G is a smooth formal group of dimension n, given by $\underline{F}$, we then obtain a functor

$$\mathrm{Nil} \to \mathbf{Ab}$$
$$\mathcal{N} \mapsto \mathcal{N}^n$$

where the sum in $\mathcal{N}^n$ is given by $\underline{a}+\underline{b} := \underline{F}(\underline{a},\underline{b})$. Conversely, if $G\colon \mathrm{Nil} \to \mathbf{Ab}$ is a functor that maps $\mathcal{N}$ to $\mathcal{N}^n$, then G must be given by a set of power series $\underline{F}$.

Example 2.5. We have $\widehat{\mathbb{G}}_{\mathrm{a}}(\mathcal{N}) = \mathcal{N}$, with the natural sum. We also have $\widehat{\mathbb{G}}_{\mathrm{m}}(\mathcal{N}) = \mathcal{N}$, but the operation is defined by $F(a,b) = a+b+ab$.

Definition 2.6. Let Compl be the category of complete, augmented R-algebras. Its objects are commutative R-algebras A, with $1 \in A$, together with a morphism $\varepsilon\colon A \to R$ such that A is complete with respect to the topology defined by the augmentation ideal. Morphisms between complete augmented R-algebras are (continuous) morphisms that respect the augmentation.

If $A \in \mathrm{Compl}$, we write $\mathrm{Spf}(A)$ for the functor

$$\mathrm{Compl} \to \mathbf{set}$$
$$B \mapsto \mathrm{Hom}_{\mathrm{Compl}}(A,B).$$

Definition 2.7. Let $F\colon \mathrm{Compl} \to \mathbf{set}$ be a functor. We say that F is *continuous* if the natural map

$$F(A) \to \varprojlim_i F(A_i)$$

is an isomorphism, where $A = \varprojlim_i A_i$.

If $\mathcal{N} \in \mathrm{Nil}$, we can view $A = R \oplus \mathcal{N}$ as an object of Compl. In this way we get a functor

$$\mathrm{Nil} \to \mathrm{Compl}\,.$$

This gives an inclusion of the category of continuous functors $\mathrm{Compl} \to \mathbf{set}$ into the one of functors $\mathrm{Nil} \to \mathbf{set}$. This inclusion has an adjoint, called completion. Indeed, any functor $F\colon \mathrm{Nil} \to \mathbf{set}$ can be extended to a continuous functor

$$F\colon \mathrm{Compl} \to \mathbf{set}$$
$$A = \varprojlim_i A_i \mapsto \varprojlim_i F(A_i)$$

Clearly, if $A \in \mathrm{NilAug}$, then the completion of $\mathrm{Spf}(A)$ is just $\mathrm{Spf}(A)$, obtained considering A as on object of Compl.

Definition 2.8. We say that a functor $\mathrm{Nil} \to \mathbf{set}$ is *prorepresentable* if it is of the form $\mathrm{Spf}(A)$ for some $A \in \mathrm{Compl}$.

The above discussion implies, by Yoneda, that $\mathrm{Spf}(A)$ (as a functor from Nil) defines A (as an object of Compl) uniquely.

Example 2.9. Let P be a *projective finitely generated* R-module. We consider the functor

$$h_P\colon \mathrm{Nil} \to \mathbf{set}$$
$$\mathcal{N} \mapsto \mathcal{N} \otimes_R P$$

This functor is prorepresentable by $\widehat{\mathrm{Sym}(P^*)}$, the completion of the symmetric algebra $\mathrm{Sym}(P^*)$ with respect to its natural augmentation ideal, where $P^* = \mathrm{Hom}_R(P,R)$ and $A = R \oplus \mathcal{N}$. Indeed, we have

$$\mathcal{N} \otimes_R P = \mathrm{Hom}_R(P^*, \mathcal{N}) = \mathrm{Hom}_{R\text{-alg}}(\mathrm{Sym}(P^*), A) = \mathrm{Hom}_{\mathrm{Compl}}(\widehat{\mathrm{Sym}(P^*)}, A).$$

Clearly, if h_P factors through $\mathbf{Ab}$ and P is free, we have that h_P is a formal group law.

Definition 2.10. A sequence of morphism in Nil

$$\mathcal{N}_1 \to \mathcal{N}_2 \to \mathcal{N}_3$$

is said to be exact (left exact) if it is exact as sequence of abelian group (left exact as sequence of abelian group and the image of $\mathcal{N}_2$ is an ideal of $\mathcal{N}_3$).

Definition 2.11. A functor $F\colon \mathrm{Nil} \to \mathbf{Ab}$ is *exact* (*left exact*) if it sends exact (left exact) sequences in Nil into exact (left exact) sequences in **Ab**.

Theorem 2.12. *A functor $F\colon \mathrm{Nil} \to \mathbf{Ab}$ is left exact if and only if it commutes with fibred products.*

Proof. See [Zin84], Theorem 2.16. □

Let Mod be category of R-modules. We have a natural functor

$$\mathrm{Mod} \to \mathrm{Nil}$$

given by setting $M^2 = 0$.

Example 2.13. Starting with the R-module R, the corresponding object of NilAug is clearly the ring of dual numbers $R[\varepsilon]$.

Definition 2.14. Let $F\colon \mathrm{Nil} \to \mathbf{set}$ be a functor. The *tangent space* of F is the functor $t_F\colon \mathrm{Mod} \to \mathbf{set}$ obtained composing F with $\mathrm{Mod} \to \mathrm{Nil}$.

Lemma 2.15. *Let $t\colon \mathrm{Mod} \to \mathbf{set}$ be a functor that commutes with finite products. Then t factors through $\mathrm{Mod} \to \mathbf{set}$.*

Proof. The R-module structure on a set M is given by certain commutative diagrams, that must be preserved by t. □

Note that if a functor t as above factors through $\mathbf{Ab} \to \mathbf{set}$, then the two additions that exist on $t(M)$ are the same.

Example 2.16. If $F\colon \mathrm{Nil} \to \mathbf{set}$ is representable by $\mathcal{N}$, then we have $t_F(R) = (\mathcal{N}/\mathcal{N}^2)^*$.

Let $t\colon \mathrm{Mod} \to \mathrm{Mod}$ be a functor that commutes with finite products. By the universal property of R as an R-module, for any $M \in \mathrm{Mod}$, we have a natural morphism

$$M \otimes_R t(R) \to t(M)$$
$$x \otimes y \mapsto t(c_x)(y)$$

where $c_x\colon R \to M$ is the unique R-linear morphism $R \to M$ that satisfies $c_x(1) = x$.

Proposition 2.17. *Let $t\colon \mathrm{Mod} \to \mathrm{Mod}$ be a right-exact functor that commutes with arbitrary direct sums. Then the map $M \otimes_R t(R) \to t(M)$ is an isomorphism. If t is moreover exact, then $t(R)$ is a flat R-module.*

Proof. The proposition is trivial for $M = R$. Since t commutes with arbitrary direct sums, the proposition is true also for R^I, for any set I. Now take any R-module M, and choose a resolution

$$R^I \to R^J \to M \to 0$$

We get a commutative diagram

$$\begin{array}{ccccccc} R^I \otimes_R t(R) & \longrightarrow & R^J \otimes_R t(R) & \longrightarrow & M \otimes_R t(R) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ t(R^I) & \longrightarrow & t(R^J) & \longrightarrow & t(M) & \longrightarrow & 0 \end{array}$$

with exact rows. Since the two left vertical maps are isomorphisms, we get the claim by the five lemma. Since we have proved that the two functors $t(\cdot)$ and $\cdot \otimes_R t(R)$ are the same, we see that is t is exact then $t(R)$ is flat. □

Definition 2.18. Let $F\colon \mathrm{Nil} \to \mathbf{set}$ be a functor. We say that F is *smooth* if it sends surjections to surjections.

Definition 2.19. Let $G\colon \mathrm{Nil} \to \mathbf{Ab}$ be a functor. We say that G is a *formal group* if it is left exact and commutes with arbitrary direct sums (provided that the direct sum exists in Nil).

Definition 2.20. A morphism $f\colon \mathcal{N} \to \mathcal{M}$ in Nil is a *small surjection* if it is a surjection and $\mathcal{N}\ker(f) = 0$.

Remark 2.21. Any surjection $f\colon \mathcal{N} \twoheadrightarrow \mathcal{M}$ can be decomposed is a sequence of small surjection

$$\mathcal{N}_0 = \mathcal{N} \twoheadrightarrow \mathcal{N}_1 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{N}_k = \mathcal{M}$$

Proposition 2.22. *Let $\alpha\colon F \to G$ be a morphism of left exact functors $\mathrm{Nil} \to \mathbf{Ab}$. We assume that α gives an isomorphism $t_F \cong t_G$ and F is smooth. Then α is an isomorphism.*

Proof. Let $\mathcal{N} \in \mathrm{Nil}$. We are going to prove that $\alpha_{\mathcal{N}}\colon F(\mathcal{N}) \to G(\mathcal{N})$ is an isomorphism. We decompose the surjection $\mathcal{N} \twoheadrightarrow 0$ as a sequence of small surjections

$$\mathcal{N}_0 = \mathcal{N} \twoheadrightarrow \mathcal{N}_1 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{N}_k = 0$$

We prove the claim by induction on k, the case $k = 0$ being trivial. We have a commutative diagram

$$\begin{array}{ccc} F(\mathcal{N}) & \overset{u}{\twoheadrightarrow} & F(\mathcal{N}_1) \\ \downarrow & & \downarrow{\scriptstyle \alpha_{\mathcal{N}_1}} \\ G(\mathcal{N}) & \overset{v}{\twoheadrightarrow} & G(\mathcal{N}_1) \end{array}$$

Here, u is surjective by smoothness of F and, by induction hypothesis, $\alpha_{\mathcal{N}_1}$ is an isomorphism. This implies that v is surjective. Let $\eta \in F(\mathcal{N}_1)$ and let ξ be its image in $G(\mathcal{N}_1)$. We write $F(\mathcal{N})_\eta$ for $u^{-1}(\eta)$ and $G(\mathcal{N})_\xi$ for $v^{-1}(\xi)$ (these sets are non empty since u and y are surjections). It is enough to prove that

$$F(\mathcal{N})_\eta \to G(\mathcal{N})_\xi$$

is an isomorphism. Let $\mathcal{K}$ be the kernel of $\mathcal{N} \to \mathcal{N}_1$. Since $\mathcal{N} \to \mathcal{N}_1$ is a small surjection, we have $F(\mathcal{K}) = t_F(\mathcal{K}) \cong t_G(\mathcal{K}) = G(\mathcal{K})$. The commutative diagram

$$\begin{array}{ccc} \mathcal{K} \times \mathcal{N} & \overset{+}{\longrightarrow} & \mathcal{N} \\ \downarrow{\scriptstyle \pi_2} & & \downarrow \\ \mathcal{N} & \longrightarrow & \mathcal{N}_1 \end{array}$$

is Cartesian (here $+$ is the natural addition, that is a morphism in Nil since $\mathcal{K}\mathcal{N} = 0$). Since F is left exact, by Theorem 2.12, the diagram

$$\begin{array}{ccc} F(\mathcal{K}) \times F(\mathcal{N}) & \xrightarrow{F(+)} & F(\mathcal{N}) \\ \downarrow{\scriptstyle \pi_2} & & \downarrow \\ F(\mathcal{N}) & \longrightarrow & F(\mathcal{N}_1) \end{array}$$

is again Cartesian. Moreover, the diagram gives an action of $F(\mathcal{K})$ on $F(\mathcal{N})$. The fact the the diagram is Cartesian implies that the action of $F(\mathcal{K})$ on $F(\mathcal{N})_\eta$ is simply transitive. The same argument gives that the action of $G(\mathcal{K})$ on $G(\mathcal{N})_\xi$ is simply transitive. The proposition follows since $F(\mathcal{K}) \cong G(\mathcal{K})$. □

Theorem 2.23. *Let $F\colon \mathrm{Nil} \to \mathbf{set}$ be a functor. We assume that*

- *F is left exact and smooth;*
- *t_F commutes with arbitrary direct sums;*
- *$t_F(R)$ is a projective R-module of finite rank.*

Then F is prorepresentable.

Proof. Since F is smooth, t_F is right exact, so, by Proposition 2.17, we have that

$$t_F(M) = M \otimes_R P = \mathrm{Hom}_R(P^*, M)$$

for all $M \in \mathrm{Mod}$, where $P := t_F(R)$. In particular, we have

$$t_F(P^*) = \mathrm{Hom}_R(P^*, P^*).$$

On the other hand we have, by definition, $t_F(P^*) = F(R \oplus P^*)$. Let $\xi_1 \in F(R \oplus P^*)$ be the element corresponding to the identity of P^*, that by Yoneda gives a natural transformation

$$\mathrm{Spf}(R \oplus P^*) \to F.$$

This natural transformation induces an isomorphism on tangent spaces since, for all $M \in \mathrm{Mod}$, we have

$$\mathrm{Spf}(R \oplus P^*)(M) = \mathrm{Hom}_R(P^*, M) = t_F(M).$$

We now consider $\widehat{\mathrm{Sym}(P^*)}$ (see Example 2.9). We have a natural surjective morphism $\widehat{\mathrm{Sym}(P^*)} \to R \oplus P^*$, that by [Zin84], Lemma 2.29, gives a surjection

$$F(\widehat{\mathrm{Sym}(P^*)}) \to F(R \oplus P^*).$$

Let $\xi \in F(\widehat{\mathrm{Sym}(P^*)})$ be an inverse image of ξ_1, that by Yoneda gives a natural transformation

$$\mathrm{Spf}(\widehat{\mathrm{Sym}(P^*)}) \to F.$$

Since this an isomorphism on tangent space, it is an isomorphism by Proposition 2.22. □

Corollary 2.24. *Let $G\colon \mathrm{Nil} \to \mathbf{set}$ be a smooth formal group. If $t_G(R)$ is a projective R-module of finite rank, then G is prorepresentable. If moreover $t_G(R)$ is free as R-module, then G is given by a formal group law.*

References

[Dem72] Michel Demazure, *Lectures on p-divisible groups*, Lecture Notes in Mathematics, Vol. 302, Springer-Verlag, Berlin, 1972.

[Zin84] Thomas Zink, *Cartiertheorie kommutativer formaler Gruppen*, Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], vol. 68, BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1984, With English, French and Russian summaries.

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