DES EDP AU CALCUL SCIENTIFI UE, PARIS, 2-6 JUILLET 2007

Congrès en l'honneur de Luc Tartar

ON THE FIELD-MATTER INTERACTION IN ELECTRODYNAMICS: A WEAK CONVERGENCE APPROACH

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A VARIATIONAL SETTING FOR NON-LINEAR E-MAGNETISM

We look for electromagnetic fields $(\mathbf{E}(t,x),\mathbf{B}(t,x))$, where $x \in \mathbb{R}^3$, subject to the differential constraints

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$

that satisfy the following stationary action principle

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int \{L(\mathbf{E}(t,x) + \varepsilon\, \eta(t,x), \mathbf{B}(t,x) + \varepsilon\, \beta(t,x)) - L(\mathbf{E}(t,x),\mathbf{B}(t,x))\} dx\, dt = 0,$$

for all compactly supported perturbation (η,β) compatible with the differential constraints. Here L defines the model and is a given real function of $(\mathbf{E},\mathbf{B}) \in \mathbb{R}^6$, stricly convex in E, and depending on $(\mathbf{E},\mathbf{B})$ only through $\mathbf{E}^2 - \mathbf{B}^2$ and $\mathbf{E}\cdot\mathbf{B}$.
The simplest model, given by

$$L(\mathbf{E},\mathbf{B}) = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2),$$

leads to the classical linear (homogeneous and normalized) Maxwell's equations.

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THE HAMILTONIAN FORMULATION

Introducing the partial Legendre transform:

$$\mathbf{h}(\mathbf{D},\mathbf{B}) = \sup_{\mathbf{E}\in\mathbf{R}^3} \mathbf{E}\cdot\mathbf{D} - \mathbf{L}(\mathbf{E},\mathbf{B}), \quad \forall\, \mathbf{D}\in\mathbf{R}^3, \quad \forall\, \mathbf{B}\in\mathbf{R}^3$$

We get the 'Hamiltonian form'

$$\partial_t \mathbf{B} + \nabla\times(\mathbf{h}'_{\mathbf{D}}(\mathbf{D},\mathbf{B})) = 0, \quad \nabla\cdot\mathbf{B} = 0,$$

$$\partial_t \mathbf{D} - \nabla\times(\mathbf{h}'_{\mathbf{B}}(\mathbf{D},\mathbf{B})) = 0, \quad \nabla\cdot\mathbf{D} = 0,$$

with the additional 'energy-momentum' conservation laws (provided by Noether's theorem)

$$\partial_t(\mathbf{h}(\mathbf{D},\mathbf{B})) + \nabla\cdot(\mathbf{D}\times\mathbf{B}) = 0,$$

$$\partial_t(\mathbf{D}\times\mathbf{B}) + \nabla\cdot(\mathbf{\Pi}(\mathbf{D},\mathbf{B})) = 0,$$

where the flux Π can be computed explicitly.

cf. C. Dafermos, Hyperbolic conservation laws in continuum physics, Springer 2005,
D. Serre, Hyperbolicity of the nonlinear models of Maxwell's equations, ARMA (2004).

MAXWELL AND BORN-INFELD'S MODELS

The simplest model, given by

$$\mathbf{L(E,B)} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2), \quad \mathbf{h(D,B)} = \frac{1}{2}(\mathbf{D}^2 + \mathbf{B}^2),$$

corresponds to the classical Maxwell's equations. A non-linear correction, suggested in 1934 by Max Born and Leopold Infeld, is obtained with

$$\mathbf{L(E,B)} = -\sqrt{1 - \mathbf{E}^2 + \mathbf{B}^2 - (\mathbf{E}\cdot\mathbf{B})^2}, \quad \mathbf{h(D,B)} = \sqrt{1 + \mathbf{D}^2 + \mathbf{B}^2 + (\mathbf{D}\times\mathbf{B})^2}$$

and leads to the $\mathbf{BORN - INFELD}$ system. The Maxwell system is recovered as the low field limit of the BI system, as $\mathbf{B,D} << 1$.

In the electrostatic case, $\mathbf{B} = 0$, we get

$$\mathbf{L(E,0)} = -\sqrt{1 - \mathbf{E}^2}, \quad \nabla \times \mathbf{E} = 0.$$

Then, the electric field E is cutoff by 1, in appropriate physical units. (With Born's scaling BI fits Maxwell down to 10^{-15} meters.) This was Born's original motivation for a non-linear theory, in the spirit of special relativity where no speed is allowed to exceed the speed of light.

(cf. Born and Infeld, Proc. Roy. Soc. London, A 144 (1934), Born, Ann. Inst. H. Poincaré, 1937)

THE BORN-INFELD SYSTEM

The Born-Infeld system reads:

$$\partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times \mathbf{v} + \frac{\mathbf{D}}{\mathbf{h}}) = 0, \quad \nabla \cdot \mathbf{B} = 0,$$

$$\partial_t \mathbf{D} + \nabla \times (\mathbf{D} \times \mathbf{v} - \frac{\mathbf{B}}{\mathbf{h}}) = 0, \quad \nabla \cdot \mathbf{D} = 0,$$

where

$$\mathbf{h} = \sqrt{1 + \mathbf{D}^2 + \mathbf{B}^2 + (\mathbf{D} \times \mathbf{B})^2}, \quad \mathbf{v} = \frac{\mathbf{D} \times \mathbf{B}}{\mathbf{h}}.$$

This system is hyperbolic and linearly degenerate. Global smooth solutions have been proven to exist for small localized initial conditions by Chae and Huh, J. Math. Phys. 2003. The additional conservation law

$$\partial_t \mathbf{h} + \nabla \cdot (\mathbf{h}\mathbf{v}) = 0,$$

provides an 'entropy function' h which is a convex function of the unknown $\mathbf{D}, \mathbf{B}$ only in a neighborhood of $(0, 0)$.

cf. G. Boillat, in Boillat, Dafermos, Lax, Liu, CIME 1994-Springer lecture notes 1640.

THE AUGMENTED BORN-INFELD (ABI) SYSTEM

The 10×10 augmented Born-Infeld system (ABI) is made of the original BI system augmented by adding the 4 'energy-momentum' conservation laws:

$$\partial_t(\mathbf{hv}) + \nabla \cdot (\mathbf{hv} \otimes \mathbf{v} - \frac{\mathbf{B} \otimes \mathbf{B} + \mathbf{D} \otimes \mathbf{D}}{\mathbf{h}}) = \nabla(\frac{1}{\mathbf{h}}), \qquad \partial_t \mathbf{h} + \nabla \cdot (\mathbf{hv}) = \mathbf{0}$$

to the 6 original BI evolution equations

$$\partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times \mathbf{v} + \frac{\mathbf{D}}{\mathbf{h}}) = \mathbf{0}, \quad \nabla \cdot \mathbf{B} = \mathbf{0},$$

$$\partial_t \mathbf{D} + \nabla \times (\mathbf{D} \times \mathbf{v} - \frac{\mathbf{B}}{\mathbf{h}}) = \mathbf{0}, \quad \nabla \cdot \mathbf{D} = \mathbf{0},$$

while DISREGARDING the algebraic constraints

$$\mathbf{h} = \sqrt{\mathbf{1} + \mathbf{D^2} + \mathbf{B^2} + (\mathbf{D} \times \mathbf{B})^\mathbf{2}}, \quad \mathbf{v} = \frac{\mathbf{D} \times \mathbf{B}}{\mathbf{h}},$$

which define the 6 dimensional BI MANIFOLD. For smooth solutions, THE BI SYSTEM IS JUST EQUIVALENT TO THE AUGMENTED SYSTEM RESTRICTED TO THE BI MANIFOLD.

cf. YB, Arch. Rat. Mech. Analysis 2004

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SOME PROPERTIES OF THE AUGMENTED BI SYSTEM

The 10×10 ABI (augmented Born-Infeld) system

$$\partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times \mathbf{v} + \frac{\mathbf{D}}{\mathbf{h}}) = \mathbf{0}, \quad \nabla \cdot \mathbf{B} = \mathbf{0},$$

$$\partial_t \mathbf{D} + \nabla \times (\mathbf{D} \times \mathbf{v} - \frac{\mathbf{B}}{\mathbf{h}}) = \mathbf{0}, \quad \nabla \cdot$$

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THE NON-CONSERVATIVE VERSION OF THE ABI SYSTEM

$$\partial_{\mathbf{t}}\mathbf{b} + (\mathbf{v}\cdot\nabla)\mathbf{b} = (\mathbf{b}\cdot\nabla)\mathbf{v} - \tau\nabla\times\mathbf{d}, \quad \partial_{\mathbf{t}}\mathbf{d} + (\mathbf{v}\cdot\nabla)\mathbf{d} = (\mathbf{d}\cdot\nabla)\mathbf{v} + \tau\nabla\times\mathbf{b},$$

$$\partial_{\mathbf{t}}\tau + (\mathbf{v}\cdot\nabla)\tau = \tau\nabla\cdot\mathbf{v}, \quad \partial_{\mathbf{t}}\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = (\mathbf{b}\cdot\nabla)\mathbf{b} + (\mathbf{d}\cdot\nabla)\mathbf{d} + \tau\nabla\tau,$$

where

$$\tau = \frac{\mathbf{1}}{\mathbf{h}}, \quad \mathbf{b} = \frac{\mathbf{B}}{\mathbf{h}}, \quad \mathbf{d} = \frac{\mathbf{D}}{\mathbf{h}}.$$

This system is quadratic, symmetric and well defined for ALL real values of τ (including $\tau < 0$, $\tau = 0$).

It is useful for a rigorous asymptotic analysis of the "high field regimes" $h \sim \infty$, which include Shallow-water MHD equations (without gravity), strings etc..., at least when the limit solutions are smooth.

In non-conservative variables, the Born-Infeld Manifold is defined by:

$$\tau > 0, \quad \tau^2 + \mathbf{v}^2 + \mathbf{b}^2 + \mathbf{d}^2 = 1, \quad \tau\mathbf{v} = \mathbf{d}\times\mathbf{b}.$$

cf. YB, Wen-an Yong, Derivation of particle, string and membrane motions from the Born-Infeld Electromagnetism, J. Math. Physics 2005

THE FIELD-MATTER INTERACTION AND THE WEAK BI MANIFOLD

The 10×10 ABI (augmented Born-Infeld) system is *linearly degenerate* and stable under weak convergence: weak limits of uniformly bounded sequences in L^∞ of smooth solutions depending on one space variable only are still solutions.
(This can be proven by using the 'div-curl' lemma, while the problem is open in higher dimensions.)
Thus, the CONVEX HULL of the BI – MANIFOLD can be conjectured to be the *natural* set for intial conditions to the ABI system, attainable by oscillations of the original BI system. (As a matter of fact, the differential constraints $\nabla \cdot D = \nabla \cdot B = 0$ must be taken into account.) This convex hull has full dimension and was computed by D. Serre:

$$\mathbf{h} \geq \sqrt{\mathbf{1} + \mathbf{D}^2 + \mathbf{B}^2 + (\mathbf{hv})^2 + \mathbf{2}|\mathbf{D} \times \mathbf{B} - \mathbf{hv}|}.$$

cf. D. Serre, A remark on Y. Brenier's approach to Born-Infeld electro-magnetic fields, Contemp. Math., 371, Amer. Math. Soc., 2005.

PROPERTIES OF THE WEAK BI MANIFOLD

The weak BI manifold

$$\mathbf{h} \geq \sqrt{\mathbf{1} + \mathbf{D}^2 + \mathbf{B}^2 + (\mathbf{hv})^2 + 2|\mathbf{D} \times \mathbf{B} - \mathbf{hv}|}$$

can be also defined by

$$\tau \geq \mathbf{0}, \quad \tau^2 + \mathbf{v}^2 + \mathbf{b}^2 + \mathbf{d}^2 + 2|\mathbf{d} \times \mathbf{b} - \tau\mathbf{v}| \leq \mathbf{1}, \quad \tau = \frac{\mathbf{1}}{\mathbf{h}}, \quad \mathbf{b} = \frac{\mathbf{B}}{\mathbf{h}}, \quad \mathbf{d} = \frac{\mathbf{D}}{\mathbf{h}}.$$

On this manifold:

1) The electromagnetic field $(\mathbf{D}, \mathbf{B})$ and the 'density and velocity' fields $(\mathbf{h}, \mathbf{v})$ can be chosen *independently* of each other, as long as they satisfy the required *inequality*. There is no longer any algebraic dependence between them! So, through this weak convergence viewpoint, we get a *coupled* system between a 'fluid' and an 'electromagnetic field', just as in classical MHD.

PROPERTIES OF THE WEAK BI MANIFOLD, continued

2) 'Matter' may exist without electromagnetic field: $B = D = 0$, which leads to the Chaplygin gas (a possible model for 'dark energy' or 'vacuum energy')

$$\partial_t(hv) + \nabla\cdot(hv\otimes v) = \nabla(\frac{1}{h}), \qquad \partial_t h + \nabla\cdot(hv) = 0$$

(for which the pressure is negative=-1/density and the sound speed=1/density),

3) Velocities are 'subluminal': $|v| \leq 1$ and 'moderate' Galilean transforms are allowed

$$(t,x) \to (t, x + U\,t), \quad (h,v,D,B) \to (h, v - U, D, B)$$

(which is impossible on the original BI manifold). This is left from special relativity under weak completion ('subrelativistic' conditions.)

cf. YB, Non relativistic strings may be approximated by relativistic strings, Methods Appl. Anal. 12 (2005)

INTEGRABILITY OF THE ABI SYSTEM IN 1 SPACE DIMENSION

In one space dimension (say x_1), introducing

$$\mathbf{z} = \sqrt{\mathbf{b_1^2 + d_1^2} + \tau^2}\ , \quad \mathbf{u} = \left(\frac{\mathbf{b_1}}{\mathbf{z}}, \frac{\mathbf{d_1}}{\mathbf{z}}, \frac{\tau}{\mathbf{z}}\right)\ , \quad \mathbf{w} = (\mathbf{b_2 + ib_3}, \mathbf{d_2 + id_3}, \mathbf{v_2 + iv_3})\ ,$$

using a Lagrangian coordinate s, and defining $\mathbf{X}, \mathbf{U}, \mathbf{W}$ by:

$$\partial_{\mathbf{t}}\mathbf{X}(\mathbf{t},\mathbf{s}) = \mathbf{v_1}(\mathbf{t}, \mathbf{X}(\mathbf{t},\mathbf{s})), \quad \partial_{\mathbf{s}}\mathbf{X}(\mathbf{t},\mathbf{s}) = \mathbf{z}(\mathbf{t}, \mathbf{X}(\mathbf{t},\mathbf{s})),$$

$$\mathbf{U}(\mathbf{t},\mathbf{s}) = \mathbf{u}(\mathbf{t}, \mathbf{X}(\mathbf{t},\mathbf{s})), \quad \mathbf{W}(\mathbf{t},\mathbf{s}) = \mathbf{w}(\mathbf{t}, \mathbf{X}(\mathbf{t},\mathbf{s})),$$

the one-dimensional ABI system reduces to

$$\partial_{\mathbf{tt}}\mathbf{X} = \partial_{\mathbf{ss}}\mathbf{X}, \quad \partial_{\mathbf{t}}\mathbf{U} = \mathbf{0}, \quad \partial_{\mathbf{t}}\mathbf{W} = \mathbf{A}(\mathbf{U})\partial_{\mathbf{s}}\mathbf{W},$$

where

$$\mathbf{A}(\mathbf{U}) = \begin{pmatrix} 0 & -iU_3 & U_1 \\ iU_3 & 0 & U_2 \\ U_1 & U_2 & 0 \end{pmatrix}$$

The only propagation speeds of this system are $0, +1, -1$.

SINGULARITIES AND VISCOSITY SOLUTIONS

The *linear* wave equation does not preserve the inversibility condition $\partial_s X(t,s) > 0$ in the large (large data or large times). This show that SINGULARITIES may develop in finite time for the ABI system.
This also corresponds to the CONCENTRATION of the Eulerian density field $h(t,x)$ as a singular measure.
Solutions can be extended beyond singularities by *adding* the unilateral constraint $\partial_s X(t,s) \geq 0$, which can be done easily in the framework of MAXIMAL MONOTONE OPERATORS IN L2. The resulting dissipative solutions no longer preserves energy. In Eulerian coordinates, this amount to add a vanishing viscosity to the momentum equation

$$\partial_{\mathbf{t}}(\mathbf{h}\mathbf{v_1}) + \frac{\partial}{\partial \mathbf{x_1}}(\mathbf{h}\mathbf{v_1^2}) + \cdots = \epsilon \frac{\partial^{\mathbf{2}}}{\partial {\mathbf{x_1}}^{\mathbf{2}}} \mathbf{v_1}, \quad \epsilon \to \mathbf{0},$$

Notice that this is a REALISTIC (Navier-Stokes style) viscosity.
A similar idea was used recently to provide a COMPLETELY HILBERTIAN formulation of MULTIDIMENSIONAL NON-LINEAR SCALAR CONSERVATION LAWS (not relying on L1 and BV spaces).

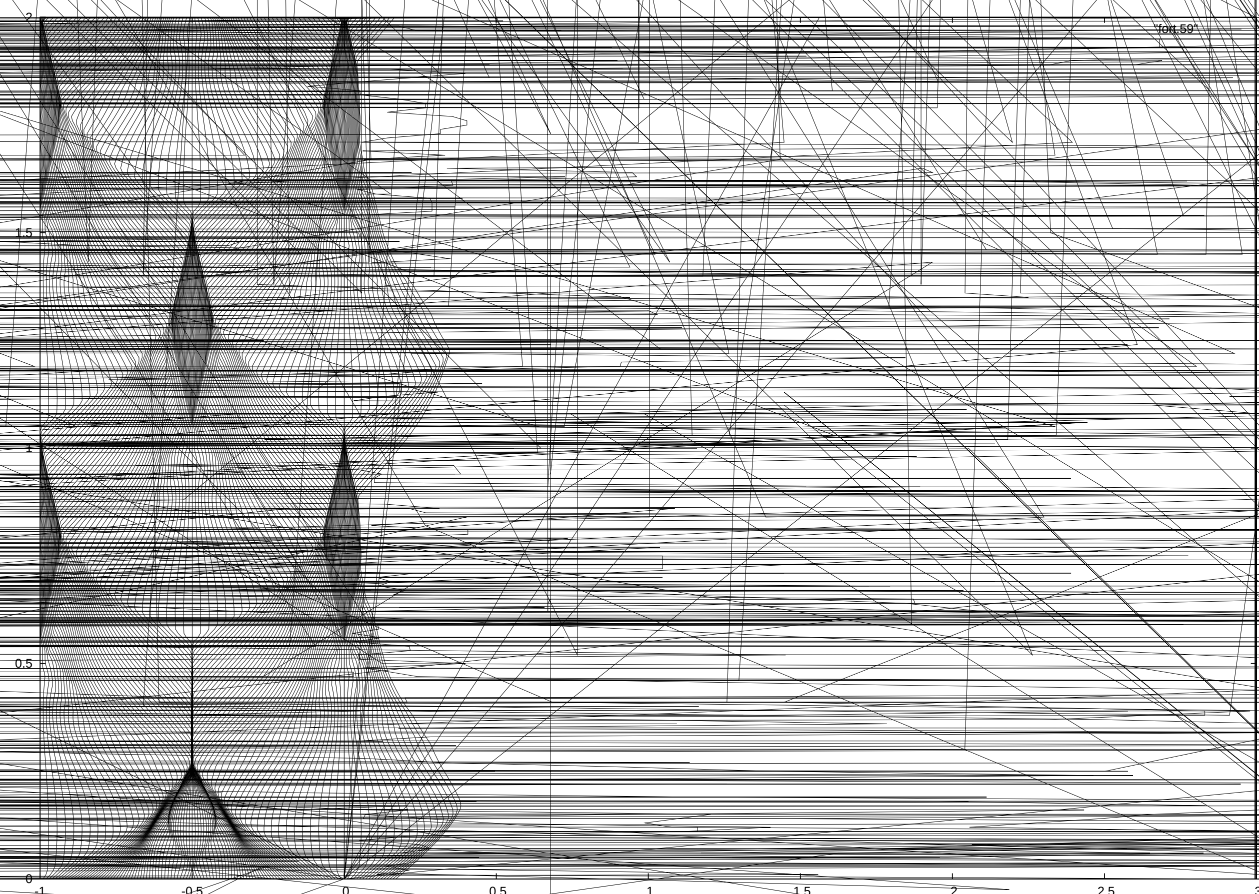
cf. YB Well-ordered vibrating strings, Methods Appl. Anal. 2004,

Y.B. L2-formulation of multid scalar conservation laws, 2006,

http://arxiv.org/pdf/math.AP/0609761

1D Chaplygin gas

Trajectories of the gas particles (vertical time, horizontal space). Observe the concentration effect.



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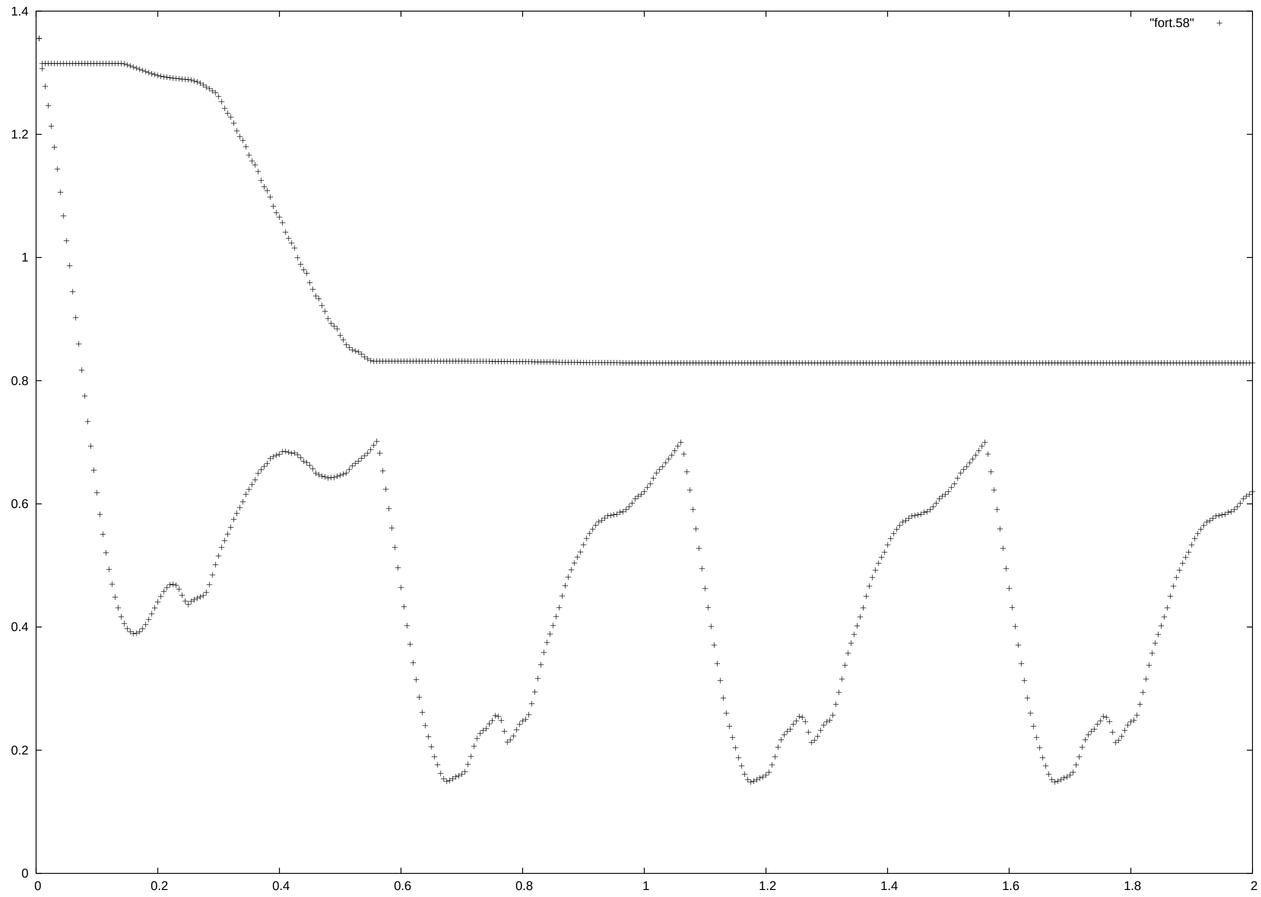
1D Chaplygin gas

Dissipation of the total energy and evolution of the kinetic energy.



kinetic/total
energy vs time

PRESSURELESS SHALLOW-WATER MHD

If we set $d = \tau = 0$ in the non-conservative form of the ABI system, we get the pressureless version of the **Shallow water MHD system**

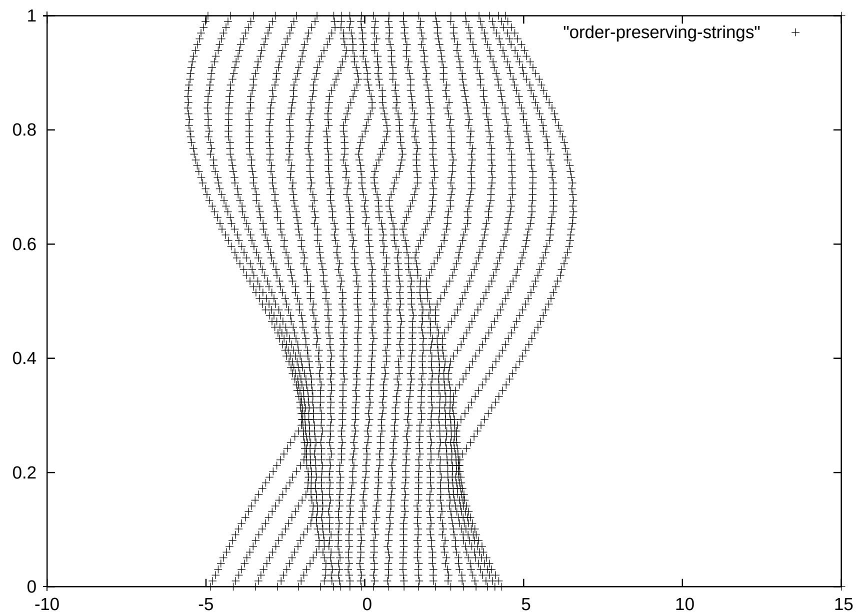
$$\boxed{\partial_{\mathbf{t}}\mathbf{b} + (\mathbf{v}\cdot\nabla)\mathbf{b} = (\mathbf{b}\cdot\nabla)\mathbf{v}, \qquad \partial_{\mathbf{t}}\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = (\mathbf{b}\cdot\nabla)\mathbf{b}.}$$

This system was also introduced for 'optimal transportation of currents' (a generalization of the optimal transportation of densities).

cf. YB A note on deformations of 2D fluid motions using 3D Born-Infeld equations. Monatsh. Math. 142 (2004), no. 1-2, 113–122

SW-MHD

Drawing of the magnetic lines.



String integrator

Purely Lagrangian