MINIMAL GEODESICS ON GROUPS OF VOLUME-PRESERVING MAPS AND GENERALIZED SOLUTIONS OF THE EULER EQUATIONS

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Abstract.

The three-dimensional motion of an incompressible inviscid fluid is classically described by the Euler equations, but can also be seen, following Arnold [1], as a geodesic on a group of volume-preserving maps. Local existence and uniqueness of minimal geodesics have been established by Ebin and Marsden [16]. In the large, for a large class of data, the existence of minimal geodesics may fail, as shown by Shnirelman [26]. For such data, we show that the limits of approximate solutions are solutions of a suitable extension of the Euler equations or, equivalently, as sharp measure-valued solutions to the Euler equations in the sense of DiPerna and Majda [14].

1. Problems and results.

1.1. Lagrangian description of incompressible fluids.

Let D be the unit cube $[0,1]^d$ in $\mathbb{R}^d$ (or the flat torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$), let $T > 0$ be a finite fixed time and set $Q = [0,T] \times D$. The velocity field of an incompressible fluid moving inside D is mathematically defined as a time dependent, square integrable, divergence-free and parallel to the boundary ∂D vector field u. Let us denote by V the vector space of all such vector fieds having a sufficient smoothness, say continuous in $(t,x) \in Q$ and Lipschitz

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continuous in x, uniformly in t. For each $u \in V$ and each $a \in D$, we can define, according to the Cauchy-Lipschitz theorem, a unique trajectory $t \in [0,T] \to g_u(t,a) \in D$ by

$$g_u(0,a) = a, \quad \partial_t g_u(t,a) = u(t, g_u(t,a)).$$

At each time t, the map $g_u(t) = g_u(t,.)$ belongs to $G(D)$, the group (with respect to the composition rule) of all homeomorphisms h of D that are Lebesgue measure-preserving in the sense

$$\int_D f(h(x))dx = \int_D f(x)dx \tag{1}$$

for all continuous functions f on D. This set is included in $S(D)$, the semi-group of all Borel maps h of D that satisfy (1). We denote by $G_V(D)$ the set $\{g_u(T), \ \ u \in V, \ \ T > 0\}$ and call target each of its elements h.

1.2. Minimal geodesics and the Euler equations.

We discuss the following minimization problem : reach a given target h at a given time $T > 0$ with a field $u \in V$ of minimal energy

$$K(u) = \frac{1}{2}\int_Q |u(t,x)|^2 dtdx \tag{2}$$

(where we denote by $|.|$ the Euclidean norm in $\mathbb{R}^d$). In other words, find a minimizer for

$$I(h) = \inf\{K(u), \quad u \in V, \quad g_u(T) = h\}. \tag{3}$$

This amounts to look for a minimal geodesics connecting the identity map to h on the infinite dimensional pseudo-Lie group $G_V(D)$ equipped with the pseudo-Riemannian structure inherited from the Hilbert space $L^2(D)^d$. This follows from the identity

$$K(u) = \int_0^T \int_D \frac{1}{2}|\partial_t g_u(t,a)|^2 dtda.$$

We refer to [1], [16], [26], [2] for more details on this geometrical setting. The (formal) Euler-Lagrange equation of this minimization problem is nothing but the Euler equation of incompressible fluids (for which we refer to [11], [12], [19],[20], [22], etc...), namely

$$\nabla.u = 0, \quad \partial_t u + \nabla.(u \otimes u) + \nabla p = 0 \tag{4}$$

(where we denote by $a.b$ the inner product of two vectors a, b in $\mathbb{R}^d$, by $a\otimes b$ their tensor product and by $\nabla = (\partial_1, ..., \partial_d)$ the spatial partial derivatives), for some scalar field $p = p(t, x)$, which physically is the pressure field. In particular, one can show that, if (u, p) is a smooth solution to the Euler equation, then, for $T > 0$ sufficiently small and $h = g_u(T)$, $I(h)$ is achieved by u and only by u. (This is true as soon as $\Lambda T^2 < \pi^2$, where Λ is the supremum on Q of the largest eigenvalue of the hessian matrix of p. See [7], for instance, and Theorem 2..4 in the present paper.) So, the research of minimal geodesics is an alternative to the more classical initial value problem to study the Euler equations. (Of course, a better alternative would be to look for all critical points of $u \to K(u)$ on V, when $g_u(T)$ and $T > 0$ are prescribed, not only the minima.)

1.3. Non existence of minimal geodesics.

The classical paper of Ebin and Marsden [16] shows that, if h belongs to a sufficiently small neighborhood of the identity map for a suitable Sobolev norm, then there is a unique minimal geodesic connecting h and the identity map. However, in the large, a striking result of Shnirelman [26] shows that, for $D = [0, 1]^3$ and for a large class of data, there is no minimal geodesic. The argument is easy to explain. It is natural to denote by V_3 the vector space V and V_2 the subspace of all planar vector fields, of the form

$$u(t, x) = (u_1(t, x_1, x_2), u_2(t, x_1, x_2), 0).$$

We consider data $h = g_u(T)$ where $u \in V_2$. They necessarily are of the form $h(x_1, x_2, x_3) = (H(x_1, x_2), x_3)$. We respectively denote by $I_3(h) = I(h)$ and $I_2(h)$ the infimum of K on V_3 and V_2. Of course $I_2(h) \geq I_3(h)$. Assume that $I_2(h) > I_3(h)$ (which is possible [26],[28]). Let us consider a field $u \in V_3$ such that $K(u) < I_2(h)$. The third component u_3 cannot be identically equal to 0. Indeed, if $u_3 = 0$, then, for every fixed x_3, $u(., ., ., x_3)$ belongs to V_2, and therefore

$$K(u) = \int_0^1 K(u(., ., ., x_3))dx_3 \geq I_2(h).$$

We can rescale u by setting $\eta(x_3) = \min(2x_3, 2 - 2x_3)$,

$$\tilde{u}_i(t, x) = u_i(t, x_1, x_2, \eta(x_3))$$

for $i = 1, 2$, and

$$\tilde{u}_3(t, x) = \eta'(x_3)^{-1} u_3(t, x_1, x_2, \eta(x_3)).$$

This new field $\tilde{u}$ still belongs to V_3 and reaches h at time T, but with an energy strictly smaller than $K(u)$. So, $I(h)$ can never be achieved, and, since the rescaling process can be iterated as often as we wish, we see that all minimizing sequences must generate small scales in the vertical direction, which gives way to homogenization and weak convergence techniques to describe their limits.

1.4. Approximate solutions.

Let us introduce the following definition of approximate solutions :

Definition 1..1 *For $\epsilon > 0$, we say that a field $u_\epsilon \in V$ is an ϵ solution if*

$$K(u_\epsilon) + \frac{1}{2\epsilon} ||g_{u_\epsilon(T)} - h||^2_{L^2(D)} \leq I_\epsilon(h) + \epsilon,$$

where

$$I_\epsilon(h) = \inf\{K(u) + \frac{1}{2\epsilon} ||g_u(T) - h||^2_{L^2(D)}, \quad u \in V\}. \tag{5}$$

We also define

$$\underline{I}(h) = \lim_{\epsilon \to 0} I_\epsilon(h) = \sup_\epsilon I_\epsilon(h). \tag{6}$$

From the definitions, we get

$$0 \leq I_\epsilon(h) \leq \underline{I}(h) \leq I(h) \leq +\infty.$$

Of course, ϵ solutions differ from minimizing sequences since they don't exactly match the final conditions. Observe that each minimizing sequence is also a sequence of ϵ_n solutions for an appropriate sequence $\epsilon_n \to 0$, if and only if $\underline{I}(h) = I(h)$. According to Shnirelman's papers [26],[28], this holds true for all $h \in G_V([0, 1]^d)$ when $d \geq 3$, but may fail if $d = 2$. (This is a direct consequence of the group property of $G_V(D)$ and of the key result of Shnirelman showing that $h \to I(h)$ is continous on $G_V(D)$ with respect to the L^2 norm if and only if $d \geq 3$.) Notice that the concept of ϵ solution is less sensitive to the choice of the vector space V than the concept of minimizing sequences, and makes sense when the target h only belongs to the L^2 closure of $G_V(D)$, which is known to be the whole semi-group $S(D)$. (See a detailed proof in [23].)

1.5. Measure-valued solutions to the Euler equations.

The existence of global weak solutions to the initial value problem, for the Euler equations (4) in dimension 3, is unknown. (Notice however that Lions [19] has recently introduced a new concept of solutions, the so-called dissipative solutions, which are globally defined for any initial velocity field of bounded energy, are weak limits of the Leray solutions to the Navier-Stokes equations when the viscosity tends to 0 and coincide with the unique local smooth solution to the Euler equations when the initial condition is smooth.) About ten years ago, DiPerna and Majda introduced an extended notion of solutions for the Euler equations, the so called measure-valued solutions, [14], based on a concept of generalized functions, introduced by Young [30] for the calculus of variations and usually called Young's measures, that Tartar had shown to be a relevant tool in the field of non linear evolution PDEs [29]. DiPerna and Majda's measure-valued solutions can be described in the following way (up to the important energy concentration phenomenon, widely discussed in [14] and here neglected). They are nonnegative measures $\mu(t,x,\xi)$ of time, space and velocity variables $(t,x,\xi) \in \mathbb{R}_+ \times D \times \mathbb{R}^3$ satisfying the moment equations

$$\int \mu(t,x,d\xi) = 1, \tag{7}$$

$$\nabla_x \cdot \int \xi \mu(t,x,d\xi) = 0, \tag{8}$$

$$\partial_t \int \xi \mu(t,x,d\xi) + \nabla_x \cdot \int \xi \otimes \xi \mu(t,x,d\xi) + \nabla_x p(t,x) = 0, \tag{9}$$

for a distribution $p(t,x)$, that can be interpreted as the pressure field. The usual weak solutions to the Euler equations correspond to the special case when μ is a Dirac measure in ξ. Just by using the natural bound on the kinetic energy (without any control on the vorticity field), DiPerna and Majda obtained global measure-valued solutions from the global weak Leray solutions to the 3 dimensional Navier-Stokes equations, by letting go to zero the viscosity parameter. Unfortunately, (7),(8),(9) are not a complete set of equations to characterize these measure-valued solutions, since they involve only their first moments in ξ. Let us mention a related concept of relaxed solutions, due to Duchon and Robert [15], which is more precise but does not lead to any global existence result.
Our goal is to perform a similar analysis, this time in a variational context, for the research of minimal geodesics when the existence of a (classical)

solution fails. A first attempt was made in [4] and, later, in [5], with a purely Lagrangian concept of Young's measures, the so-called generalized flows. (A generalized incompressible flow is defined as a Borel probability measure ν on the product space $\Omega = D^{[0,T]}$ of all paths $t \in [0,T] \to \omega(t) \in D$, such that each projection ν_t, for $0 \leq t \leq T$ is the Lebesgue measure on D and the kinetic energy, namely

$$\int_\Omega \int_0^T \frac{1}{2}|\omega'(t)|^2 dt d\nu(\omega)$$

is finite. We recently discovered that this concept had been already used by Shelukhin, in a slightly different framework [25].) In that setting, we proved the existence of generalized solutions [4] and both existence and uniqueness of a pressure gradient, linked to them through a suitable Poisson equation [5], but we failed in obtaining for them a *complete* set of equations beyond the classical Euler equations (see also [7]). Such an achievement is made possible in the present paper to describe the limit of ϵ solutions as $\epsilon \to 0$, thanks to a different (but related) concept of Young's measures, with both Lagrangian and Eulerian features, and, thanks to an appropriate regularity result on the pressure gradient. So, in some sense, we obtain *sharp* measure-valued solutions to the Euler equations.

1.6. The main result.

It is convenient to make a clear distinction between the particle label and their initial location in D. The particle label is denoted by a and belongs to a metric compact space A with a Borel probability measure da. For the proofs of subsection 3.8., the most convenient choice is $A = \mathbb{T}$, where da is the Lebesgue measure. It is then possible to choose i as a Borel isomorphism transporting the Lebesgue measure from $\mathbb{T}$ to D. (Namely, i is one-to-one up to a negligible set, and, for all continuous function f on D,

$$\int_D f(x)dx = \int_{\mathbb{T}} f(i(a))da.$$

This is always possible, see [24], for instance.) We denote by Q' the compact metric space $Q \times A$, we consider the space of all Borel measures on Q', namely the dual space of $C(Q')$ (the space of all continuous function f on Q'), with its (sequential) weak-* topology. According to the circumstances, we denote by

$$< \nu, f >=< \nu(t,x,a), f(t,x,a) >= \int_{Q'} f(t,x,a) d\nu(t,x,a) = \int_{Q'} f d\nu,$$

the pairing of a measure ν and a continuous function f. We are now ready to state our main result :

Theorem 1..2 *Assume that $d = 3$, $D = [0,1]^3$, $Q = [0,T] \times D$ and the target $h \in S(D)$ satisfies*

$$h(x_1, x_2, x_3) = (H(x_1, x_2), x_3), \quad x \in [0,1]^3, \tag{10}$$

for some $H \in S([0,1]^2)$. Then
i) $\underline{I}(h) \leq 3T^{-1}$ and $K(u_\epsilon) \to \underline{I}(h)$ for every ϵ solution u_ϵ as $\epsilon \to 0$;
ii) There exists a unique distribution $\nabla p(t,x)$ in the interior of Q, depending only on h, such that, for every ϵ solution

$$\partial_t u_\epsilon + (u_\epsilon . \nabla) u_\epsilon + \nabla p \to 0, \tag{11}$$

in the sense of distributions in the interior of Q;
iii) ∇p is a locally bounded measure in the interior of Q;
iv) The respectively nonnegative and vector-valued measures on Q'

$$c_\epsilon(t,x,a) = \delta(x - g_{u_\epsilon}(t, i(a))), \quad m_\epsilon(t,x,a) = u_\epsilon(t,x)\delta(x - g_{u_\epsilon}(t, i(a))), \tag{12}$$

have weak- cluster points (c, m). For each of them, the measure m is absolutely continuous with respect to c, with a vector-valued density $v \in L^2(Q', dc)^d$, so that $m = cv$, and the following equations are satisfied*

$$\int_A c(t,x,da) = 1, \tag{13}$$

$$\partial_t c + \nabla_x . m = 0, \tag{14}$$

$$\partial_t (cv) + \nabla_x . (cv \otimes v) + \underline{c} \nabla_x p = 0, \tag{15}$$

in the sense of distributions in the interior of Q', where $\underline{c}$ is a natural extension of c, for which the product $\underline{c}(t,x,a)\nabla_x p(t,x)$ is well defined. Moreover

$$c(0,x,a) = \delta(x - i(a)); \quad c(T,x,a) = \delta(x - h(i(a))) \tag{16}$$

and the kinetic energy

$$\int_{D \times A} \frac{1}{2} |v(t,x,a)|^2 c(t, dx, da) \tag{17}$$

is time independent, with value $\underline{I}(h)/T$.

Remark 1.

The measure c may naturally be considered as an element of the dual space of $L^1(dxdt, C(A))$ (the vector space of all Lebesgue integrable function of $(t,x) \in Q$ with values in the space $C(A)$ of all continuous function on A) and

$$c(t,x,a) = \lim_{\delta \to 0} \int_{-1/2}^{+1/2} d\theta \int c(t, x - 2\theta\delta e - \delta y, a)\gamma(y)dy, \tag{18}$$

holds true for the weak-* topology, when $\delta \to 0$, for any fixed $e \in \mathbb{R}^d$ and any nonnegative smooth radial compactly supported mollifier γ on $\mathbb{R}^d$. It will be shown that $\underline{c}$ can be defined in a similar way by

$$\underline{c}(t,x,a) = \lim_{\delta \to 0} \int_{-1/2}^{+1/2} d\theta \int c(t, x - 2\theta\delta e - \delta y, a)\gamma(y)dy, \tag{19}$$

for the weak-* topology of the dual space of $L^1(|\nabla p|, C(A))$ (the vector space of all $|\nabla p|$ integrable function of $(t,x) \in Q$ with values in the space $C(A)$ of all continuous function on A). This shows that $\underline{c}(t,x,a)$ coincide with $c(t,x,a)$, as a probability measure (in a), for almost every (t,x) with respect to the regular part of $|\nabla p|$ and is its natural extension to the singular set of $|\nabla p|$. Then, the product $\underline{c}\nabla p$ makes sense. Notice that a similar multiplication problem has been successfully handled by Majda and Zheng [21] for the Vlasov-Poisson system with singular initial conditions (electron sheets).

Remark 2.

The limit measures (c,m) are similar to Young's measures [30], [29], with a double character, both Eulerian and Lagrangian, through their dependence on both x (the Eulerian variable) and a (the Lagrangian variable). Equations (14), (15), (13) form a *complete* set of equations for (c,m). These equations are already known in the mathematical modelling of quasineutral plasmas [8],[17],[18] but the study of the corresponding initial value problem is widely open, even for short time and smooth data (see [18] for partial results).
We may also consider each particle label a as a phase label, $c(.,.,a)$ and $v(.,.,a)$ as the corresponding concentration and velocity fields. Then, we get a multiphase flow model, with a continuum of phases, labelled by $a \in A$. For a discrete label $a = 1, ..., N$, we would recover a classical crude model

of N phase flows [10]. This discrete model has been already studied from a variational point of view in [6]. As a matter of fact, some of the proofs of [6] can be straightforwardly extended to the continuous case and will be reproduced (with few changes), for the seek of completeness, in the present paper (subsections 3.2. and 3.3.).

Remark 3.

With each solution (c, m) of Theorem 1..2, we may associate a measure-valued solution μ, in the sense of DiPerna and Majda, by setting

$$\int_{Q\times\mathbb{R}^d} f(t,x,\xi)d\mu(t,x,\xi) = \int_{Q'} f(t,x,v(t,x,a))dc(t,x,a), \tag{20}$$

for every continuous function f on $Q \times \mathbb{R}^d$, with at most quadratic growth as $\xi \to \infty$ (since $v \in L^2(Q', dc)^d$). Equations (7),(8),(9) are automatically enforced, since they are nothing but (13),(14), (15), the two last one being averaged in a on A, with an obvious loss of information. So, the solutions (c, m) of Theorem 1..2 may be considered as *sharp* measure-valued solutions to the Euler equations.

Remark 4.

The uniqueness and the partial regularity of the pressure gradient are striking properties, certainly due the variational nature of our problem. For the initial value problem, there is no evidence at all that the pressure gradient should be a locally bounded measure in the large. The non uniqueness of the cluster points (c, m) is not surprising, since, already in the classical framework, $I(h)$ may be achieved by two different fields u. For instance, for $d = 2$, D the unit disk, $T = \pi$ and $h(x) = -x$, $I(h)$ is achieved by both $u(x_1, x_2) = (-x_2, x_1)$ and $-u$. (Notice that the pressure is the same for both, namely $\nabla p(x) = x$.) It would be interesting to see whether or not the uniqueness of the pressure gradient in the classical framework can be obtained by using only classical arguments (without refering to weak convergence tools).

Remark 5.

Some families of (stationary) solutions (u_ϵ, p_ϵ) to the Euler equations behave in the same way as the ϵ solutions discussed in Theorem 1..2. Let us consider, for instance,

$$u_\epsilon(x) = (-\cos(x_1)\sin(\epsilon^{-1}x_2), \epsilon\sin(x_1)\cos(\epsilon^{-1}x_2), 0),$$

$$p_\epsilon(x) = -\frac{1}{4}(\cos(2x_1) + \epsilon^2\cos(2\epsilon^{-1}x_2)).$$

As $\epsilon \to 0$, the pressure field strongly converges (with bounded second order derivatives) toward

$$p(x) = -\frac{1}{4}\cos(2x_1).$$

Meanwhile, the velocity field gets more and more oscillatory and the measures (c_ϵ, m_ϵ) defined by (12) converge to a solution (c, m) of equations (13), (14), (15). A second example is

$$u_\epsilon(x) = \exp(-\frac{x_1^2 + x_2^2}{2})\tilde{u}_\epsilon(x),$$

$$\tilde{u}_\epsilon(x) = (\cos(\epsilon^{-1}x_3), \sin(\epsilon^{-1}x_3), [-x_1\cos(\epsilon^{-1}x_3) + x_2\sin(\epsilon^{-1}x_3)]\epsilon),$$

for which the pressure is not even ϵ dependent

$$p(x) = -\frac{1}{2}\exp(-x_1^2 - x_2^2).$$

2. Main steps of the proof.

2.1. Reformulation of the minimal geodesic problem.

The first step is the complete reformulation of our problem in terms of the measures c and m associated with $u \in V$ through

$$c(t, x, a) = \delta(x - g_u(t, i(a))) \quad (21)$$

and

$$m(t, x, a) = u(t, x)c(t, x, a) = \partial_t g_u(t, i(a))\delta(x - g_u(t, i(a))). \quad (22)$$

We automatically get (14). The initial and final locations of the particles can be included in the weak form of this equation by

$$\int_{Q'}(\partial_t f(t,x,a)dc(t,x,a) + \nabla_x f(t,x,a).dm(t,x,a)) \tag{23}$$

$$= \int_A (f(T,h(i(a)),a) - f(0,i(a),a))da,$$

for every $f \in C(Q')$ such that both $\partial_t f$ and $\nabla_x f$ are continuous on Q'. The incompressibility condition can be expressed by

$$\int_{Q'} f(t,x)dc(t,x,a) = \int_Q f(t,x)dtdx, \tag{24}$$

for every continuous function f on Q, that is $\int_A c(t,x,da) = 1$. Now, an elementary calculation shows that we can (abusively) rewrite $K(u) = K(c,m)$, with

$$K(c,m) = \sup_{(F,\Phi)} \int_{Q'}(F(t,x,a)dc(t,x,a) + \Phi(t,x,a).dm(t,x,a)), \tag{25}$$

where the supremum is taken among all continuous functions F and Φ on Q', with values respectively in $\mathbb{R}$ and $\mathbb{R}^d$, such that

$$F(t,x,a) + \frac{1}{2}|\Phi(t,x,a)|^2 \leq 0, \tag{26}$$

pointwise.

So, the minimization problem becomes

$$I(h) = \inf_{(c,m)} \sup_{(\phi,p)} L(c,m,\phi,p), \tag{27}$$

where the Lagrangian L is given by

$$L(c,m,\phi,p) = K(c,m) - \int_{Q'}[\partial_t\phi(t,x,a) + p(t,x)]dc(t,x,a) \tag{28}$$

$$- \int_{Q'} \nabla_x\phi(t,x,a).dm(t,x,a)$$

$$+ \int_A (\phi(T,h(i(a)),a) - \phi(0,i(a),a))da + \int_Q p(t,x)dtdx,$$

and measures c, m are supposed to be of form (21),(22).

2.2. Introduction of a relaxed problem.

If we relax conditions (21),(22), we get a convex minimization problem set on the space of measures (c, m) (namely the dual space of $C(Q') \times C(Q')^d$) : minimize $K(c, m)$ (defined by (25)) subject to (23), (24). We denote by $I^*(h) \in [0, +\infty]$ the infimum. Of course, $I^*(h) \leq I(h)$. Then, we set :

Definition 2..1 *An admissible solution is a pair of measures (c, m) satisfying $K(c, m) < +\infty$, (23) and (24). An optimal solution is an admissible solution (c, m) such that $K(c, m) = I^*(h)$.*

The finiteness of $K(c, m) < +\infty$ is enough to enforce that c is a nonnegative measure positive on Q', m is absolutely continuous with respect to c, and its (vector-valued) density $v = v(t, x, a)$ is square integrable Q' with respect to c :

$$m(t, x, a) = v(t, x, a)c(t, x, a), \tag{29}$$

$$K(c, m) = \int_{Q'} \frac{1}{2}|v(t, x, a)|^2 dc(t, x, a). \tag{30}$$

(This will be shown in subsection 3.2..) In subsection 3.1., we prove :

Proposition 2..2 *Let $D = [0, 1]^d$ and $h \in S([0, 1]^d)$. Then, there is always an admissible solution (c, m) and*

$$I^*(h) \leq K(c, m) \leq dT^{-1} \tag{31}$$

The study of the relaxed problem, in subsections 3.2., 3.3., 3.4., 3.5., leads to :

Theorem 2..3 *Let $D = [0, 1]^d$ and $h \in S(D)$. Then there is at least one optimal solution (c, m) to the relaxed problem. We have $m = cv$ with $v \in L^2(Q', dc)^d$, equations (14), (13) and initial and final conditions (16) are satisfied, the kinetic energy (17) is time independent and bounded by (31). There exists a unique measure $\nabla p(t, x)$ in the interior of Q, depending only on h, for which (15) holds true for all optimal solutions.*

Then, in subsection 3.6., we prove that the relaxed problem is consistent with the local smooth solutions of the Euler equations.

Theorem 2..4 *Let (u, p) be a smooth solution to the Euler equations (4), $T > 0$ such that $\Lambda T^2 < \pi^2$, where Λ is the supremum on Q of the largest eigenvalue of the hessian matrix of p, and set $h = g_u(T)$. Then, the pair (c, m) associated with u through (21),(22) is the unique solution of the relaxed problem.*

Next, in subsection 3.7., we establish the following relationship between ϵ solutions (as defined in Definition 1..1), and optimal solutions to the relaxed problem, through the associated measures (c_ϵ, m_ϵ) (defined by (12)) :

Proposition 2..5 *Assume $\underline{I}(h) < +\infty$. Let u_ϵ be an ϵ solution and (c, m) be any cluster point (for the weak-* topology of measures on Q') of (c_ϵ, m_ϵ). Then*

$$I^*(h) \leq K(c, m) \leq \liminf K(u_\epsilon) \leq \limsup K(u_\epsilon) \leq \underline{I}(h) = \lim I_\epsilon(h). \quad (32)$$

In addition, if $I^*(h) = \underline{I}(h)$, then (c, m) is optimal,

$$I^*(h) = K(c, m) = \lim K(u_\epsilon) = \underline{I}(h) = \lim I_\epsilon(h) \quad (33)$$

and (11) holds true.

Finally, in subsection 3.8., we show :

Theorem 2..6 *Assume that $D = [0, 1]^3$ and $h \in S([0, 1]^3)$ satisfies (10) for some $H \in S([0, 1]^2)$, then $\underline{I}(h) = I^*(h)$.*

The proof is closely related to the one used by Shnirelman in [28] to show that classical smooth incompressible flows are dense among generalized flows (in the sense of [5]), in the case $D = [0, 1]^3$. We believe that Shnirelman's construction can be adapted for *any* data $h \in S([0, 1]^3)$, but for the sake of simplicity and completeness, we only discuss in this paper the special case (10), for which we provide all details. This concludes the overview of the main steps of the proof of Theorem 1..2.

2.3. Main steps of the study of the relaxed problem.

In subsection 3.2., elementary arguments from classical convex analysis show (as in [6]) :

Proposition 2..7 *The infimum $I^*(h)$ is always achieved and for every $\epsilon > 0$, there exist some continuous functions $\phi_\epsilon(t,x,a)$ on Q' and $p_\epsilon(t,x)$ on Q, with $\partial_t\phi_\epsilon$, $\nabla_x\phi_\epsilon$ continuous on Q' and $\int_D p_\epsilon(t,x)dx = 0$, such that, for every optimal solution (c,m),*

$$\partial_t\phi_\epsilon + \frac{1}{2}|\nabla_x\phi_\epsilon|^2 + p_\epsilon \leq 0 \tag{34}$$

and

$$\int_{Q'}(|\partial_t\phi_\epsilon + \frac{1}{2}|\nabla_x\phi_\epsilon|^2 + p_\epsilon| + |v - \nabla_x\phi_\epsilon|^2)dc \leq \epsilon^2. \tag{35}$$

Then, we get in subsection 3.3. an approximate regularity result :

Proposition 2..8 *Let $0 < \tau < T/2$ and $Q'_\tau = [\tau/2, T - \tau/2] \times D \times A$. Let $x \in D \to w(x) \in \mathbb{R}^d$ be a smooth divergence-free vector field, parallel to ∂D, and $s \in \mathbb{R} \to e^{sw}(x) \in D$ be the integral curve of w passing through x at $s = 0$. Then,*

$$\int_{Q'_\tau} |\nabla_x\phi_\epsilon(t,x,a) - v(t,x,a)|^2 dc(t,x,a) \leq C\epsilon^2, \tag{36}$$

$$\int_{Q'_\tau} |\nabla_x\phi_\epsilon(t,x,a)|^2 dc(t,x,a) \leq C, \tag{37}$$

$$\int_{Q'_\tau} |\nabla_x\phi_\epsilon(t+\eta, e^{\delta w}(x), a) - \nabla_x\phi_\epsilon(t,x,a)|^2 dc(t,x,a) \leq (\epsilon^2 + \eta^2 + \delta^2)C, \tag{38}$$

for all optimal solution (c,m) and all η, δ and $\epsilon > 0$ small enough, where C depends only on D, T, τ and w.

Since $c(t,x,a)$ is a measure, possibly highly concentrated (like a delta measure in x, as in the case of classical solutions), it is unclear how to deduce from Proposition 2..8 a bound such as

$$\int_{Q'_\tau} (|\partial_t v|^2 + |\nabla_x v|^2) dc \leq C, \tag{39}$$

by letting first $\epsilon \to$ (to get v instead of $\nabla_x \phi_\epsilon$), then $\delta, \eta \to 0$. Such a bound would be meaningful, if c could be bounded away from zero, which is exactly the contrary of the classical case and cannot be expected, anyway, because of the initial and final conditions. However a bound on $\int |\nabla p|$ is expectable. The formal (and, of course, not rigorous) argument is as follows. Starting from (35), we get

$$(\partial_t \phi + \frac{1}{2}|\nabla_x \phi|^2 + p)c = 0.$$

Differentiating in x, we formally get (15) or

$$(\partial_t v + (v.\nabla_x)v + \nabla_x p)c = 0.$$

Then, integrating in $a \in A$,

$$\int_A (\partial_t v + (v.\nabla_x)v)c(t,x,da) = -\nabla_x p,$$

and, by Schwarz inequality,

$$(\int |\nabla_x p|)^2 \leq \int |\partial_t v|^2 dc \int dc + \int |\nabla_x v|^2 dc \int |v|^2 dc.$$

All these calculations are incorrect. However, the formal idea can be made rigorous, by working only on the ϕ_ϵ and using finite differences instead of derivatives, and lead (in subsection 3.4.) to

Theorem 2..9 *The family (∇p_ϵ) converges in the sense of distributions toward a unique limit ∇p, depending only on h, which is a locally bounded measure in the interior of Q and is uniquely defined by*

$$\nabla p(t,x) = -\partial_t \int v(t,x,a)c(t,x,da) - \nabla_x . \int (v \otimes v)(t,x,a)c(t,x,da), \quad (40)$$

for ALL optimal solution $(c, m = cv)$.

Finally, in subsection 3.5., (15) is established.

3. Detailed proofs.

3.1. Existence of admissible solutions.

In this subsection we prove Proposition 2..2, with an explicit construction closely related to the one introduced in [4] for generalized flows on the torus $\mathbb{T}^d$ (and used in [23] and [28]). We only perform the construction in the cases $D = \mathbb{T}^d$ and $D = [0,1]^d$, the second one being an extension of the first one. (Extensions to general domains can be performed as in [23].)

Admissible solutions on the torus.

Let $h \in S(D)$ where $D = \mathbb{T}^d$. We introduce, for $(x, y, z) \in D^3$, a curve $t \in [0, T] \to \omega(t, x, y, z) \in D$, made, for $0 \leq t \leq T/2$, of a shortest path (with constant speed) going from x to y on the torus $\mathbb{T}^d$, and, for $T/2 \leq t \leq T$, of a shortest path (with constant speed) going from y to z. Such a curve is uniquely defined for Lebesgue almost every $(x, y, z) \in D^3$. Then, we set, for every continuous function f on $Q' = [0, T] \times D \times A$,

$$< c, f >= \int_{Q'} f(t, \omega(t, i(a), y, h(i(a))), a) dt dy da$$

$$< m, f >= \int_{Q'} \partial_t \omega(t, i(a), y, h(i(a))) f(t, \omega(t, i(a), y, h(i(a))), a) dt dy da.$$

(Intuitively, this amounts to define a generalized flow for which each particle a is first uniformly scattered on the torus at time $T/2$ and then focused to the target $h(i(a))$ at time T.) This makes (c, m) an admissible solution. Let us check, for instance, (13), by considering a continuous function f that does not depend on a and showing that $< c, f >= \int_Q f$. We split $< c, f >= I_1 + I_2$ according to $t \leq T/2$ or not. For $t \geq T/2$, we have, by definition, $\omega(t, i(a), y, h(i(a))) = \omega(t, y, y, h(i(a)))$. Since we work on the torus $\mathbb{T}^d$, $\omega(t, y, y, h(i(a))) = y + \omega(t, 0, 0, h(i(a)) - y)$. Thus

$$I_2 = \int_{[T/2,T] \times D \times D} f(t, y + \omega(t, 0, 0, x - y)) dx dy dt$$

(since $a \in A \to x = h(i(a)) \in D$ is Lebesgue measure preserving)

$$= \int_{[T/2,T] \times D \times D} f(t, y + \omega(t, 0, 0, x)) dx dy dt = \int_{[T/2,T] \times D \times D} f(t, y) dx dy dt$$

$$= \int_{[T/2,T] \times D} f(t, y) dy dt$$

(by using twice the translation invariance of the Lebesgue measure on the torus) and, doing the same for I_1, we conclude that $< c, f >= \int_Q f$. We also get, for $K(c, m)$ the following estimate that does not depend on the choice of $h \in S(D)$

$$2K(c, m) = \int_{[0,T] \times D \times A} |\partial_t \omega(t, i(a), y, h(i(a)))|^2 da dy dt$$

$$= T/2 \int_{D \times A} ((d_D(i(a), y)/(T/2))^2 + (d_D(y, h(i(a)))/(T/2))^2) da dy$$

$$= \frac{4}{T} \int_{D \times D} d_D(x, y)^2 dx dy \leq d/T$$

(where $d_D(.,.)$ denotes the geodesic distance on the torus).

Admissible solutions on the unit cube.

Let us now lift the unit cube to the torus by shrinking it by a factor 2 and reflecting it 2^d times through each face of its boundary. To do that, we introduce the Lipschitz continuous mapping

$$\Theta(x) = 2(\min(x_1, 1-x_1), ..., \min(x_d, 1-x_d)), \tag{41}$$

from $[0,1]^d$ onto $[0,1]^d$, and its 2^d reciprocal maps, each of them being denoted by Θ_k^{-1}, with $k \in \{0,1\}^d$, and mapping back $[0,1]^d$, one-to-one, to the cube $2^{-1}(k+[0,1]^d)$. We also define the (almost everywhere) one-to-one Lebesgue measure-preserving map

$$\theta_k^{-1}(a) = i^{-1}(\Theta_k^{-1}(i(a))) \tag{42}$$

from $\mathbb{T}$ into itself. Given $h \in S([0,1]^d)$, we associate $\tilde{h} \in S(\mathbb{T}^d)$ by setting

$$\tilde{h}(x) = \Theta_k^{-1}(h(\Theta(x)),$$

when $x \in \frac{1}{2}(k+[0,1]^d)$, $k \in \{0,1\}$. We consider the admissible pair $(\tilde{c}, \tilde{m})$ associated with $\tilde{h}$ and constructed exactly as in the previous subsection. Then we set

$$c(t,x,a) = \frac{1}{2^d} \sum_{k \in \{0,1\}^d} c(t, \Theta_k^{-1}(x), \theta_k^{-1}(a)),$$

$$v(t,x,a) = \frac{1}{2^d} \sum_{k \in \{0,1\}^d} [((v(t, \Theta_k^{-1}(x), \theta_k^{-1}(a))).\nabla_x)\Theta](\Theta_k^{-1}(x)).$$

Explicit calculations show that (c,m) is an admissible solution for h and that (as in [28])

$$K(c,m) \leq \frac{2d}{3T} \leq dT^{-1}.$$

Notice that it is important to use a continuous transform Θ (so that the particle trajectories are properly reflected at the boundary of the unit cube and not broken) in order to keep (23), but there is no need to use a one-to-one transform, which makes the construction very easy.

3.2. Duality and optimal solutions.

In this subsection, we prove Proposition 2..7 by using the same elementary arguments of convex analysis as in [6]. Let us fix an admissible solution

$(\overline{c}, \overline{m})$ (such a solution exists, as just shown in subsection 3.1., introduce the Banach space $E = C(Q') \times C(Q')^d$, and the following convex functions on E with values in $]-\infty, +\infty]$,

$$\alpha(F, \Phi) = 0 \quad if \quad F + \frac{1}{2}|\Phi|^2 \leq 0, \tag{43}$$

and $+\infty$ otherwise, for $(F, \Phi) \in E$,

$$\beta(F, \Phi) =< \overline{c}, F > + < \overline{m}, \Phi > \tag{44}$$

if there exist $p \in C(Q)$ and $\phi \in C(Q')$ with $\partial_t \phi \in C(Q')$ and $\nabla_x \phi \in C(Q')^d$ such that

$$F(t, x, a) + \partial_t \phi(t, x, a) + p(t, x) = 0, \quad \Phi(t, x, a) + \nabla_x \phi(t, x, a) = 0 \tag{45}$$

for all $(t, x, a) \in Q'$, and $\beta(F, \Phi) = +\infty$ otherwise. The Legendre-Fenchel-Moreau transforms (see [9], ch. I, for instance) of α and β are respectively given by

$$\alpha^*(c, m) = \sup\{< c, F > + < m, \Phi >; \ \ F + \frac{1}{2}|\Phi|^2 \leq 0\} \tag{46}$$

and

$$\beta^*(c, m) = \sup < c - \overline{c}, F > + < m - \overline{m}, \Phi > \tag{47}$$

where $(F, \Phi) \in E$ satisfies (45). In (46), we recognize definition (25) of $K(c, m)$. So K precisely is the Legendre transform of α. We see that $K(c, m)$ is finite if and only if c is a nonnegative measure and m is absolutely continuous with respect to c with a vector-valued density $v \in L^2(Q, dc)^d$ and, then,

$$K(c, m) = \frac{1}{2} \int_{Q'} |v|^2 dc.$$

Observe that $\beta^*(c, m) = 0$ if (c, m) satisfies

$$< \overline{c} - c, \partial_t \phi + p > + < \overline{m} - m, \nabla_x \phi >= 0, \tag{48}$$

for all suitable (ϕ, p), and $\beta^*(c, m) = +\infty$ otherwise. Thus, since (48) exactly means that (c, m) satisfies admissibility conditions (13) and (23), the infimum of $\alpha^*(c, m) + \beta^*(c, m)$ is nothing but the infimum of $K(c, m)$ among all admissible pairs (c, m), namely $I^*(h)$.

Functions α and β are convex with values in $]-\infty,+\infty]$. Moreover, there is at least one point $(F,B)\in E$, namely

$$F=-1,\ \ \Phi=0,$$

for which α is continuous (for the sup norm) and β is finite. Thus, by the Fenchel-Rockafellar duality Theorem (an avatar of the Hahn-Banach Theorem, see [9], ch. I, for instance), we have the duality relation

$$\inf\{\alpha^*(c,m)+\beta^*(c,m);\ \ (c,m)\in E'\} \tag{49}$$

$$=\sup\{-\alpha(-F,-\Phi)-\beta(F,\Phi);\ \ (F,\Phi)\in E\}$$

and the infimum is achieved. More concretely, we get

$$I^*(h)=\sup <\overline{c},\partial_t\phi+p>+<\overline{m},\nabla_x\phi> \tag{50}$$

with

$$\partial_t\phi+\frac{1}{2}|\nabla_x\phi|^2+p\leq 0.$$

Let us take any of the optimal solutions as a reference solution $(\overline{c},\overline{m})$. Then, the duality relation exactly means that, for all $\epsilon>0$, there exist p_ϵ and ϕ_ϵ, such that

$$\partial_t\phi_\epsilon+\frac{1}{2}|\nabla_x\phi_\epsilon|^2+p_\epsilon\leq 0$$

and

$$\frac{1}{2}<\overline{c},|v|^2>\leq<\overline{c},\partial_t\phi_\epsilon+p_\epsilon>+<\overline{m},\nabla_x\phi_\epsilon>+\epsilon^2.$$

Thus,

$$\frac{1}{2}<\overline{c},|v-\nabla_x\phi_\epsilon|^2>+<\overline{c},|\partial_t\phi_\epsilon+p_\epsilon+\frac{1}{2}|\nabla_x\phi_\epsilon|^2|>\leq\epsilon^2.$$

So, we have obtained the desired relationship (35) between the p_ϵ and the ϕ_ϵ and any *any* of the optimal solutions $(\overline{c},\overline{m})$ and the proof of Proposition 2..7 is now complete, noticing that $\int_D p_\epsilon(t,x)dx=0$ can be enforced by adding to the $\phi_\epsilon(t,x,a)$ a suitable function of t.

3.3. An approximate regularity result.

In this subsection again, we use some arguments of [6] to prove Proposition 2..8. Let $(c, m = cv)$ be an optimal solution. Let δ and η be two small parameters and $0 < \tau < T/2$. Let ζ be a smooth compactly supported function on $]0, T[$. Let $x \to w(x)$ be a smooth vector field on D, parallel to ∂D, and let $e^{sw}(x)$ be the integral curve passing through x at $s = 0$. We set

$$c^\eta(t,x,a) = c(t+\eta\zeta(t),x,a), \quad v^\eta(t,x,a) = v(t+\eta\zeta(t),x,a)(1+\eta\zeta'(t)),$$

and, for all $f \in C(Q')$,

$$\int_{Q'} f(t,x,a)dc^{\eta,\delta}(t,x,a) = \int_{Q'} f(t,e^{\delta\zeta(t)w}(x),a)dc^\eta(t,x,a), \tag{51}$$

$$\int_{Q'} f(t,x,a)dm^{\eta,\delta}(t,x,a) \tag{52}$$

$$= \int_{Q'} f(t,e^{\delta\zeta(t)w}(x),a)(\partial_t + v^\eta(t,x,a).\nabla_x)e^{\delta\zeta(t)w}(x)dc^\eta(t,x,a).$$

Notice that the new pair of measures $(c^{\eta,\delta}, m^{\eta,\delta} = c^{\eta,\delta}v^{\eta,\delta})$ is well defined, has finite energy and satisfies (23). (Use the chain rule and compute

$$\int_{Q'} \frac{d}{dt}[f(t,e^{\delta\zeta(t)w}(x),a)]dc^\eta(t,x,a).)$$

However, (13) is not satisfied unless w is divergence-free. Let us show :

Proposition 3..1 *Let (c,m) be an optimal solution. Then, for any η, δ,*

$$\int p_\epsilon dc^{\eta,\delta} - \int p_\epsilon dc + \frac{1}{2}\int |(\partial_t + v^\eta.\nabla_x)e^{\delta\zeta w} - \nabla_x\phi_\epsilon \circ e^{\delta\zeta w}|^2 dc^\eta \tag{53}$$

$$\leq \epsilon^2 + \frac{1}{2}\int |(\partial_t + v^\eta.\nabla_x)e^{\delta\zeta w}|^2 dc^\eta - \frac{1}{2}\int |v|^2 dc.$$

Proof.

Using definitions (51), (52), we get the following identities,

$$\int \partial_t\phi_\epsilon(dc^{\eta,\delta} - dc) = -\int \nabla_x\phi_\epsilon.(dm^{\eta,\delta} - dm)$$

$$= -\int ((\partial_t + v^\eta.\nabla_x)e^{\delta\zeta w}).\nabla_x\phi_\epsilon \circ e^{\delta\zeta w} dc^\eta + \int v.\nabla_x\phi_\epsilon dc$$

$$= \frac{1}{2}\int |(\partial_t + v^\eta.\nabla_x)e^{\delta\zeta w} - \nabla_x\phi_\epsilon \circ e^{\delta\zeta w}|^2 dc^\eta - \frac{1}{2}\int |(\partial_t + v^\eta.\nabla_x)e^{\delta\zeta w}|^2 dc^\eta$$

$$-\frac{1}{2}\int |\nabla_x\phi_\epsilon \circ e^{\delta\zeta w}|^2 dc^\eta + \int \ v.\nabla_x\phi_\epsilon dc.$$

We also have

$$\int |\nabla_x\phi_\epsilon|^2(dc^{\eta,\delta} - dc) = \int |\nabla_x\phi_\epsilon \circ e^{\delta\zeta w}|^2 dc^\eta - \int |\nabla_x\phi_\epsilon|^2 dc.$$

Thus,

$$\int (\partial_t\phi_\epsilon + \frac{1}{2}|\nabla_x\phi_\epsilon|^2)(dc^{\eta,\delta} - dc)$$

$$= \frac{1}{2}\int |(\partial_t + v^\eta.\nabla_x)e^{\delta\zeta w} - \nabla_x\phi_\epsilon \circ e^{\delta\zeta w}|^2 dc^\eta - \frac{1}{2}\int |(\partial_t + v^\eta.\nabla_x)e^{\delta\zeta w}|^2 dc^\eta$$

$$-\frac{1}{2}\int |\nabla_x\phi_\epsilon - v|^2 dc + \frac{1}{2}\int |v|^2 dc.$$

Since (c, m) is an optimal solution, we deduce from Proposition 2..7

$$\epsilon^2 \geq \int (p_\epsilon + \partial_t\phi_\epsilon + \frac{1}{2}|\nabla_x\phi_\epsilon|^2)(dc^{\eta,\delta} - dc)$$

$$+\frac{1}{2}\int |\nabla_x\phi_\epsilon - v|^2 dc,$$

which completes the proof of (53).
Now, we use Proposition 3..1 with different choices of parameters η, δ. First, in (53), we set $\delta = 0$ (so that $e^{\delta\zeta(t)w}(x) = x$). The first term of the left-hand side vanishes (since $\int_A c(t, x, da) = 1$ and $\int_D p_\epsilon(t, x)dx = 0$). Thus, we get

$$\frac{1}{2}\int |v^\eta - \nabla_x\phi_\epsilon|^2 dc^\eta \tag{54}$$

$$\leq \epsilon^2 + \frac{1}{2}\int |v^\eta|^2 dc^\eta - \frac{1}{2}\int |v|^2 dc.$$

We can bound from below the left-hand side by zero, let $\epsilon \to 0$ and get

$$0 \leq \int |v^\eta|^2 dc^\eta - \int |v|^2 dc \tag{55}$$

for any choice of ζ and η small enough. This implies that the kinetic energy defined as a measure on $[0, T]$ by (17) is time independent (therefore equal to $I^*(h)T^{-1}$) and the right-hand side of (55) is bounded by $C\eta^2$, where C depends only on D, T and ζ. Thus, we have obtained

$$\int |v^\eta - \nabla_x\phi_\epsilon|^2 dc^\eta \leq (\epsilon^2+\eta^2)C. \tag{56}$$

Next, we assume w to be divergence-free, which makes e^{sw} a Lebesgue measure-preserving map for all $s \in \mathbb{R}$ and cancels the first term of the left-hand side in (53). So we can rewrite

$$\int |(\partial_t + v.\nabla_x)e^{\delta\zeta w} - \nabla_x\phi_\epsilon \circ e^{\delta\zeta w}|^2 dc \tag{57}$$

$$\leq 2\epsilon^2 + \int |(\partial_t + v.\nabla_x)e^{\delta\zeta w}|^2 dc - \int |v|^2 dc.$$

Let us bound from below the left-hand side by zero and set $\epsilon = \eta = 0$ in the right-hand side. We get

$$0 \leq \int (|(\partial_t + v.\nabla_x)e^{\delta\zeta(t)w}(x)|^2 - |v|^2)dc.$$

Since $e^{\delta\zeta(t)w}(x) = x + \delta\zeta(t)w(x) + 0(\delta^2)$, this implies that

$$\int v.(\partial_t + v.\nabla_x)(\zeta w)dc = 0 \tag{58}$$

for all smooth ζ compactly supported in $]0,T[$ and all smooth divergence-free parallel to ∂D vector field w (which means that

$$\partial_t \int_A v(t,x,a)c(t,x,da) + \nabla_x \cdot \int_A v(t,x,a)\otimes v(t,x,a)c(t,x,da)$$

is a gradient in the sense of distribution). We can now rewrite (57)

$$\int |(\partial_t + v.\nabla_x)(e^{\delta\zeta w}(x) - x) + v(t,x,a) - \nabla_x\phi_\epsilon(x, e^{\delta\zeta w}(x), a)|^2 dc \tag{59}$$

$$\leq 2\epsilon^2 + \int |(\partial_t + v.\nabla_x)(e^{\delta\zeta w}(x) - x) + v|^2 dc - \int |v|^2 dc$$

and, after rearranging the squares,

$$\frac{1}{2}\int |v - \nabla_x\phi_\epsilon \circ e^{\delta\zeta w}|^2 dc$$

$$\leq \epsilon^2 - \int [v - \nabla_x\phi_\epsilon \circ e^{\delta\zeta w}].[(\partial_t + v.\nabla_x)(e^{\delta\zeta w}(x) - x)]dc$$

$$+\int v.(\partial_t + v.\nabla_x)(e^{\delta\zeta w}(x) - x)dc.$$

Since $e^{\delta\zeta(t)}(x) - x = \delta\zeta(t)w(x) + 0(\delta^2)$ we deduce from (58) that

$$\int |v - \nabla_x\phi_\epsilon \circ e^{\delta\zeta w}|^2 dc \leq C(\epsilon^2 + \delta^2), \tag{60}$$

where C depends only on D, T, ζ and w.
Finally, to get (38), assume that ζ takes its values in $[0,1]$ and satisfies $\zeta(t) = 1$ if $\tau \leq t \leq T - \tau$ and $\zeta(t) = 0$ if $t \leq \tau/2$ or $t \geq T - \tau/2$. For $(t,x,a) \in Q'_\tau$ and η small enough, we have $e^{\delta\zeta(t)w}(x) = e^{\delta w}(x)$,

$$c^\eta(t,x,a) = c(t+\eta,x,a), \quad v^\eta(t,x,a) = v(t+\eta,x,a).$$

So, by bounding from below the left-hand sides of (56) and (60) by the corresponding integrals performed on Q'_τ, we get (38) and complete the proof of Proposition 2..8. In addition, we have established the conservation of the kinetic energy (17) and the averaged momentum equation (58).

3.4. Existence, uniqueness and partial regularity of the pressure field.

In this subsection, we prove Theorem 2..9 in three steps : first, we show that the family (p_ϵ) is compact in the sense of distributions in the interior of Q, next, we establish a discrete version of (15), using divided differences instead of derivatives, then we deduce the convergence toward a unique limit of (∇p_ϵ) and finally we prove that the limit ∇p is a locally bounded measure in the interior of Q.

Compactness in the sense of distributions.

From (53) with $\eta = 0$, we get, for all smooth vector field w parallel to ∂D, all smooth function ζ compactly supported in $]0,T[$ and all $|\delta| \leq 1$, that

$$\int_Q p_\epsilon(t,y)(J(e^{-\delta\zeta(t)w}(y)) - 1)dydt \leq \epsilon^2 + C, \tag{61}$$

where C depends on ζ, w, and J denotes the Jacobian determinant. Moreover, we know that $\int_D p_\epsilon(t,x)dx = 0$ holds true for all $t \in [0,T]$. According to Moser's lemma (see [13] for a recent reference), by varying δ, ζ and w,

we can generate a sufficiently large set of smooth functions compactly supported in the interior of Q of form $(t,x) \to (J(e^{-\delta\zeta(t)w}(y) - 1)$ so that the boundedness of the family (p_ϵ) in the sense of distributions is enforced.
More precisely : Let $\zeta(t)$ and $\sigma(x)$ be two compactly supported smooth functions. We further assume

$$\int \sigma(x)dx = 0, \quad \sup|\sigma| < 1.$$

$1) Set \rho(s,x) = 1 + s\sigma(x), \quad s \in [0,1];$

$$2) Solve$$

-$\Delta\phi = \sigma$

$$with homogeneous Neumann boundary conditions; 3) Set$$

v(s,x)=$\nabla\phi(x)\overline{\rho(s,x),}$

$$so that$$

$\partial_s\rho + \nabla \cdot (\rho v) = 0.$

$$4) Solve$$

$\partial_s g(s,x) = v(s, g(s,x)), \quad g(0,x) = x.$

$$Then, we have$$

$\det(\mathrm{D}_x g(s,x)) = \rho(s,x).$

$$Thus$$

$\det(\mathrm{D}_x g(\zeta(t),x)) = 1 + \zeta(t)\sigma(x).$

$$So, we have obtained,$$

$\sup_\epsilon \int p_\epsilon(t,x)\zeta(t)\sigma(x)dxdt < +\infty,$

$$which proves that$$

p_ϵ is bounded (up to an irrelevant added function of t only) in the sense of distributions.

A discrete optimality equation.

Let $(c, m = cv)$ be a fixed *arbitrarily* chosen optimal solution. We call test function on Q' (resp. on Q) any function f continuous on Q' (resp. on Q), vanishing near $t = 0$ and $t = T$, with continuous derivatives in t and x. Notice that $f(t, x, a)$ does not necessarily vanish when x approaches ∂D. Let f be such a test function with values in $[0, 1]$ and w be a smooth divergence-free vector field on D, parallel to ∂D. We will denote by C_f a generic constant depending on T, D, w and f only through the sup norm of $\nabla_x f$, the $L^2(Q', dc)$ norm of $\partial_t f$ and the support of f in t. (It is important that C_f involves $\partial_t f$ only through its $L^2(Q', dc)$ norm.) Let us introduce

$$I = \int \frac{1}{\delta}(-\lambda_\epsilon(t, e^{\delta w}(x), a) + \lambda_\epsilon(t, x, a)) f(t, x, a) dc(t, x, a)$$

where $\lambda_\epsilon \geq 0$ is defined by

$$\lambda_\epsilon = -(\partial_t \phi_\epsilon + \frac{1}{2}|\nabla_x \phi_\epsilon|^2 + p_\epsilon). \tag{62}$$

From (35) we deduce that

$$I \leq \frac{1}{\delta} \int \lambda_\epsilon f dc \leq \frac{\epsilon^2}{\delta}. \tag{63}$$

We can split I into $I_1 + I_2 + I_3$ with

$$I_1 = \int \frac{1}{\delta}(p_\epsilon(t, e^{\delta w}(x)) - p_\epsilon(t, x)) f dc,$$

$$I_2 = \int \frac{1}{2\delta}[|\nabla_x \phi_\epsilon|^2(t, e^{\delta w}(x), a) - |\nabla_x \phi_\epsilon|^2(t, x, a)] f dc =$$

$$\int \frac{1}{2\delta}[\nabla_x \phi_\epsilon(t, e^{\delta w}(x), a) - \nabla_x \phi_\epsilon(t, x, a)].[\nabla_x \phi_\epsilon(t, e^{\delta w}(x), a) + \nabla_x \phi_\epsilon(t, x, a)] f dc$$

$$\geq \int \frac{1}{\delta} \int [\nabla_x \phi_\epsilon(t, e^{\delta w}(x), a) - \nabla_x \phi_\epsilon(t, x, a)].v(t, x, a) f dc - \frac{1}{\delta}(\delta^2 + \epsilon^2) C_f$$

(by Schwarz inequality and the resultats of Proposition 2..8) and

$$I_3 = \int \frac{1}{\delta}[\partial_t \phi_\epsilon(t, e^{\delta w}(x), a) - \partial_t \phi_\epsilon(t, x, a)] f dc$$

$$= \int \int_0^1 d\sigma [\partial_t \nabla_x \phi_\epsilon(t, e^{\sigma\delta w}(x), a).w(e^{\sigma\delta w}(x))] f dc = I_4 + I_5 + I_6,$$

where, by using (14) and by integrating by part in t,

$$I_4 = -\int\int_0^1 d\sigma \partial_t f \nabla_x \phi_\epsilon(t, e^{\sigma\delta w}(x), a).w(e^{\sigma\delta w}(x))dc$$

$$\geq -\int \partial_t f v.w dc - (\delta+\epsilon)C_f$$

(from the results of Proposition 2..8 and Schwarz inequality, which involves the $L^2(Q', dc)$ norm of $\partial_t f$, taken into account through C_f)

$$I_5 = -\int (v(t,x,a).\nabla_x f)(\nabla_x \phi_\epsilon(t, e^{\sigma\delta w}(x), a).w(e^{\sigma\delta w}(x))dc d\sigma.$$

$$\geq -\int (v(t,x,a).\nabla_x f)(v(t,x,a).w(x))dc - (\delta+\epsilon)C_f$$

(here C_f involves the sup norm of $\nabla_x f$) and,

$$I_6 = -\int ((v(t,x,a).\nabla_x)[\nabla_x \phi_\epsilon(t, e^{\sigma\delta w}(x), a).w(e^{\sigma\delta w}(x))]f dc d\sigma.$$

We get (after integrating in $\sigma \in [0,1]$)

$$I_6 = -\frac{1}{\delta}\int (v(t,x,a).\nabla_x)[\phi_\epsilon(t, e^{\delta w}(x), a)) - \phi_\epsilon(t,x,a))]f dc = I_7 + I_8,$$

where

$$I_7 = -\frac{1}{\delta}\int v(t,x,a).[\nabla_x \phi_\epsilon(t, e^{\delta w}(x), a)) - \nabla_x \phi_\epsilon(t,x,a))]f dc,$$

$$I_8 = -\frac{1}{\delta}\int v(t,x,a).((\nabla_x \phi_\epsilon(t, e^{\delta w}(x), a).\nabla_x)(e^{\delta w}(x) - x))f dc$$

$$\geq -\int v(t,x,a).((v(t,x,a).\nabla_x)w(x))f dc - \delta C_f.$$

Thus, since $I = I_1 + I_2 + I_3$ (where $I_3 = I_4 + I_5 + I_6$ and $I_6 = I_7 + I_8$) satisfies (63), we get

$$\int \frac{1}{\delta}(p_\epsilon(t, e^{\delta w}(x)) - p_\epsilon(t,x))f dc - \int v.(\partial_t + v.\nabla_x)(wf)dc \tag{64}$$

$$\leq \frac{1}{\delta}(\delta^2 + \epsilon^2)C_f.$$

Let $t \to \zeta(t)$ a cutoff function $[0,T]$, with values in $[0,1]$, vanishing near $t=0$, $t=T$, and with value 1 on the support in t of f. Since f takes its values in $[0,1]$, we can apply (64) to $(1-f)\zeta = \zeta - f$ instead of f. We get

$$\int \frac{1}{\delta}(p_\epsilon(t,e^{\delta w}(x)) - p_\epsilon(t,x))(\zeta - f)dc + \int v.(\partial_t + v.\nabla_x)(w\zeta - wf)dc$$

$$\leq \frac{1}{\delta}(\delta^2+\epsilon^2)C_f.$$

Both terms involving ζ vanish, the first one because $\int c(t,x,da) = 1$ and $e^{\delta w}$ is Lebesgue measure-preserving, the second one because of (58). So, we can put an absolute value on the left-hand side of (64), and, then, perform the symmetry $w \to -w$ to obtain a first discrete version of (15), namely

$$|\int \frac{1}{2\delta}(p_\epsilon(t,e^{\delta w}(x)) - p_\epsilon(t,e^{-\delta w}(x)))fdc - \int v.(\partial_t + v.\nabla_x)(fw)dc| \quad (65)$$

$$\leq \frac{1}{\delta}(\delta^2+\epsilon^2)C_f.$$

Existence and uniqueness of the pressure gradient.

If we use in (65) a test function $f(t,x,a) = f(t,x)$ that does not depend on a, we get, since $\int c(t,x,da) = 1$,

$$|\int \frac{1}{2\delta}(f(t,e^{-\delta w}(y)) - f(t,e^{\delta w}(y)))p_\epsilon(t,y)dtdy \quad (66)$$

$$-\int v.(\partial_t + v.\nabla_x)(wf)dc| \leq \frac{1}{\delta}(\delta^2+\epsilon^2)C_f,$$

where C_f, now, depends on f only through the support in t of f and on the norm

$$|||f||| = \sup_Q(|f| + |\nabla_x f|) + (\int_Q |\partial_t f|^2 dtdx)^{1/2}. \quad (67)$$

Since, in the sense of distributions, the family (p_ϵ) is bounded, there are cluster points. Let p be any one of the cluster points. Letting first $\epsilon \to 0$, and then $\delta \to 0$, in (66), shows that p satisfies (40). Since $(c, m = cv)$ can be *any* of the optimal solutions, this shows the uniqueness of ∇p and, therefore, the convergence of the entire family (∇p_ϵ) to ∇p in the sense of distributions. Notice also that (66) implies the convergence of

$$p_\epsilon(t,e^{\delta w}(x)) - p_\epsilon(t,e^{-\delta w}(x)) \to p(t,e^{\delta w}(x)) - p(t,e^{-\delta w}(x)), \quad (68)$$

as $\epsilon \to 0$, δ being fixed, for the weak-* topology of the dual space of the separable Banach space obtained by completion of the smooth functions compactly supported in $]0,T[\times D$ with respect to the norm defined by (67).

The pressure gradient is a measure.

To show Theorem 2..9, we have now to prove that ∇p is a locally bounded measure in the interior of Q. Let $t \to \zeta(t)$ a smooth cutoff function on $[0,T]$, with values in $[0,1]$, 0 near $t=0$ and $t=T$ and 1 away from a neighborhood of 0 and T. Let $\eta > 0$, $\delta > 0$ small enough so that $t \in \zeta(t-\theta\eta)$ vanishes near 0 and T, for every θ in $[-1,+1]$. It is difficult to estimate

$$\int_Q \zeta(t)|p_\epsilon(t, e^{\delta w}(x)) - p_\epsilon(t,x)|dxdt,$$

so we consider the following time regularization

$$I = \int_Q \zeta(t)|\int_0^1 (p_\epsilon(t+\eta\theta, e^{\delta w}(x)) - p_\epsilon(t+\eta\theta, x))d\theta|dxdt$$

$$= \int_{Q'} \zeta(t)|\int_0^1 (p_\epsilon(t+\eta\theta, e^{\delta w}(x)) - p_\epsilon(t+\eta\theta, x))d\theta|dc(t,x,a)$$

(using $\int_A c(t,x,da) = 1$). Let us consider $\lambda_\epsilon(t,x,a)$ defined by (62), which is nonnegative and of which the c integral is bounded by ϵ^2 (according to (35) in Proposition 2..7). We have $I \leq I_1 + I_2 + I_3$, where

$$I_1 = \int \zeta|\int_0^1 (\lambda_\epsilon(t+\eta\theta, e^{\delta w}(x), a) - \lambda_\epsilon(t+\eta\theta, x, a))d\theta|dc,$$

$$I_2 = \int \zeta|\int_0^1 (\partial_t\phi_\epsilon(t+\eta\theta, e^{\delta w}(x), a) - \partial_t\phi_\epsilon(t+\eta\theta, x, a))d\theta|dc,$$

$$I_3 = \int \zeta|\int_0^1 (\frac{1}{2}|\nabla_x\phi_\epsilon|^2(t+\eta\theta, e^{\delta w}(x), a) - \frac{1}{2}|\nabla_x\phi_\epsilon|^2(t+\eta\theta, x, a))d\theta|dc.$$

The third term is easy to handle, thanks to (38) and Schwarz inequality. It can be bounded by $(\epsilon+\delta+\eta)C$, where C, as all the coming up constants always denoted by C, depends only on D, T, ζ and w. The second term can be bounded, after using the mean value theorem and integrating in θ, by

$$I_2 \leq \delta \int \zeta \int_0^1 d\sigma|\frac{1}{\eta}[\nabla_x\phi_\epsilon(t+\eta, e^{\sigma\delta}(x), a) - \nabla_x\phi_\epsilon(t, e^{\sigma\delta}(x), a)].w(e^{\sigma\delta}(x))|dc$$

(here we see that the time regularization avoids the use of $\partial_t \nabla_x \phi_\epsilon$ on which we have no control)

$$\leq C\frac{\delta}{\eta}(\epsilon + \delta + \eta).$$

Let us now consider the most delicate term, I_1. We start with a rough estimate, using $\lambda_\epsilon \geq 0$ and $\int \lambda_\epsilon dc \leq \epsilon^2$, to obtain

$$I_1 \leq \int \zeta[\lambda_\epsilon(t+\eta\theta, e^{\delta w}(x), a) + \lambda_\epsilon(t+\eta\theta, x, a)]d\theta dc$$

$$\leq 2\epsilon^2 + \int \zeta[\lambda_\epsilon(t+\eta\theta, e^{\delta w}(x), a) + \lambda_\epsilon(t+\eta\theta, x, a) - 2\lambda_\epsilon(t, x, a)]d\theta dc$$

$$= 2\epsilon^2 - I_4 - I_5 - I_6$$

with

$$I_4 = \int \zeta[\partial_t\phi_\epsilon(t+\eta\theta, e^{\delta w}(x), a) + \partial_t\phi_\epsilon(t+\eta\theta, x, a) - 2\partial_t\phi_\epsilon(t, x, a)]d\theta dc,$$

$$I_5 = \frac{1}{2}\int \zeta[|\nabla_x\phi_\epsilon(t+\eta\theta, e^{\delta w}(x), a)|^2 + |\nabla_x\phi_\epsilon(t+\eta\theta, x, a)|^2$$
$$-2|\nabla_x\phi_\epsilon(t, x, a)|^2]d\theta dc,$$

$$I_6 = \int \zeta[p_\epsilon(t+\eta\theta, e^{\delta w}(x)) + p_\epsilon(t+\eta\theta, x) - 2p_\epsilon(t, x)]d\theta dc.$$

Notice that we go backward only apparently, since now there is no absolute value any longer in the integrals. In particular I_6 vanishes since $\int c(t, x, da) = 1$, e^{sw} is Lebesgue measure-preserving and $\int p_\epsilon(t, x)dx = 0$. The same bound is obtained for I_5 as for I_3 earlier. The treatment of I_4 needs more care. The idea is to get rid of the time derivatives by using (14). Since ζ vanishes near 0 and T, there will be no boundary terms coming out of the integration by parts, however $\partial_t\zeta$ has to be handled carefully because we wish to involve ϕ_ϵ only through its partial derivatives. We split $I_4 = I_7 + I_8$ with

$$I_7 = \int [\zeta(t)(\partial_t\phi_\epsilon(t+\eta\theta, e^{\delta w}(x), a) + \partial_t\phi_\epsilon(t+\eta\theta, x, a))$$
$$-2\zeta(t-\eta\theta)\partial_t\phi_\epsilon(t, x, a)]d\theta dc,$$

$$I_8 = \int 2(\zeta(t-\eta\theta) - \zeta(t))\partial_t\phi_\epsilon(t, x, a)d\theta dc.$$

To treat I_8, we have to go backward again

$$I_8 = \int 2(\zeta(t-\eta\theta) - \zeta(t))(-\lambda_\epsilon(t, x, a))d\theta dc$$

$$-\int(\zeta(t-\eta\theta)-\zeta(t))|\nabla_x\phi_\epsilon(t,x,a)|^2 d\theta dc - \int 2(\zeta(t-\eta\theta)-\zeta(t))p_\epsilon(t,x)d\theta dc.$$

The last term vanishes (since $\int c(t,x,da)=1$ and $\int p_\epsilon(t,x)dx=0$) and the two first ones are easily bounded by $C\eta$. Let us now consider $I_7=I_9+2I_{10}$, where

$$I_9=\int\zeta(t)[\partial_t\phi_\epsilon(t+\eta\theta,e^{\delta w}(x),a)-\partial_t\phi_\epsilon(t+\eta\theta,x,a)]d\theta dc,$$

$$I_{10}=\int[\zeta(t)\partial_t\phi_\epsilon(t+\eta\theta,x,a)-\zeta(t-\eta\theta)\partial_t\phi_\epsilon(t,x,a)]d\theta dc.$$

We have

$$I_9=\delta\int\zeta(t)\int_0^1 d\sigma\partial_t\nabla_x\phi_\epsilon(t+\eta\theta,e^{\sigma\delta w}(x),a).w(e^{\sigma\delta w}(x))d\theta dc$$

$$=\frac{\delta}{\eta}\int\zeta(t)\int_0^1 d\sigma[\nabla_x\phi_\epsilon(t+\eta,e^{\sigma\delta w}(x),a)-\nabla_x\phi_\epsilon(t,e^{\sigma\delta w}(x),a)].w(e^{\sigma\delta w}(x))dc,$$

which can be bounded as I_2 by $C\frac{\delta}{\eta}(\epsilon+\delta+\eta)$. We also have

$$I_{10}=\int\int_0^1 d\sigma\partial_t[\zeta(t-(1-\sigma)\eta\theta)\partial_t\phi_\epsilon(t+\eta\theta\sigma,x,a)]\eta\theta d\theta dc.$$

Here, we use (23) and get

$$-\int\int_0^1 d\sigma\zeta(t-(1-\sigma)\eta\theta)\nabla_x\partial_t\phi_\epsilon(t+\eta\theta\sigma,x,a).v(t,x,a)\eta\theta d\theta dc.$$

Thus $I_{10}=I_{11}+I_{12}$, where

$$I_{11}=-\int\int_0^1 d\sigma\partial_t[\zeta(t-(1-\sigma)\eta\theta)\nabla_x\phi_\epsilon(t+\eta\theta\sigma,x,a)].v(t,x,a)\eta\theta d\theta dc$$

$$=-\int[\zeta(t)\nabla_x\phi_\epsilon(t+\eta\theta,x,a)-\zeta(t-\eta\theta)\nabla_x\phi_\epsilon(t,x,a)].v(t,x,a)d\theta dc,$$

$$I_{12}=\int\int_0^1 d\sigma\zeta'(t-(1-\sigma)\eta\theta)\nabla_x\phi_\epsilon(t+\eta\theta\sigma,x,a).v(t,x,a)\eta\theta d\theta dc.$$

So, I_{10} is bounded by $(\epsilon+\delta+\eta)C$.
Finally, because $\zeta(t)=1$ away from $t=0$ and $t=T$, we have shown that, for $\tau>0$ small enough,

$$\int_{Q_\tau}|\int_0^1(p_\epsilon(t+\eta\theta,e^{\delta w}(x))-p_\epsilon(t+\eta\theta,x))d\theta|dxdt \tag{69}$$

$$\leq (1+\delta/\eta)(\eta+\delta+\epsilon)C,$$

where C depends only on D, T, τ and w. Letting $\epsilon \to 0$, with frozen $\eta = \delta$, in (69), shows that the limit p satisfies

$$\frac{1}{\delta}\int < p(t,x), f(t-\delta\theta, e^{-\delta w}(x)) - f(t-\delta\theta, x) > d\theta \leq C \sup |f|$$

for every smooth function $f(t,x) \geq 0$ with compact support in $0 < t < T$. Finally, we obtain, when $\delta \to 0$,

$$< \nabla p(t,x).w(x), f(t,x) > \leq C \sup |f|,$$

where C depends on D, T, w and the support of f, which shows that ∇p is a locally bounded measure in the interior of Q and completes the proof of Theorem 2..9.

3.5. Optimality equations.

In this subsection, we deduce (15) from (65) by letting $\epsilon \leq \delta \to 0$, which makes some problem since ∇p_ϵ converges to the measure ∇p only in the sense of distributions. In (65), let us assume that the support of f is compact in the interior of Q'. Then, given a fixed vector e in the unit ball of $\mathbb{R}^d$, we can choose w in such a way that, for $|\delta|$ small enough, $e^{\delta w}(x) = x + \delta e$ and $w(x) = e$ hold true for all (t,x) such that $f(t,x,a) \neq 0$. Then (65) becomes

$$|\int \frac{1}{2\delta}(p_\epsilon(t,x+\delta e) - p_\epsilon(t,x-\delta e)) f dc - \int v.(\partial_t + v.\nabla_x)(fe) dc| \qquad (70)$$

$$\leq \frac{1}{\delta}(\delta^2+\epsilon^2)C_f.$$

where C_f depends on D, T, the norm of f defined by (67) and the support of f.

Let γ a radial nonnegative mollifier on $\mathbb{R}^d$, supported in the ball centered at the origine and of radius $1/2$, and e a fixed vector in the same ball. For each y in this ball, we perform the change of variable $e \to e + y$ in (70), multiply by $\gamma(y)$ and integrate with respect to y. Then, we obtain

$$|\int \frac{1}{2\delta}(p_\epsilon(t,x+\delta(e+y)) - p_\epsilon(t,x-\delta(e+y))) f dc\, \gamma(y) dy$$

$$+ \int v.e(\partial_t f + v.\nabla_x f) dc| \leq \delta C_f.$$

Since γ is radial, we can replace the second y by $-y$ and rewrite

$$|\int \frac{1}{2\delta}(p_\epsilon(t,x+\delta e) - p_\epsilon(t,x-\delta e) f_{c,\delta,\gamma}(t,x)dtdx \tag{71}$$

$$+\int v.e(\partial_t f + v.\nabla_x f)dc| \leq \delta C_f,$$

where $f_{c,\delta,\gamma}$ is defined by

$$f_{c,\delta,\gamma}(t,x) = \int f(t,x-\delta y,a)c(t,x-\delta y,da)\gamma(y)dy. \tag{72}$$

This function belongs to the Banach space obtained by completion of the test functions with respect to the norm defined by (67). Indeed, it is compactly supported in time, smooth in x and sufficiently smooth in t, since, because of equation (14),

$$\partial_t f_{c,\delta,\gamma}(t,x) =$$

$$\int(\partial_t + v(t,x-\delta y,a).\nabla_x)f(t,x-\delta y,a)c(t,x-\delta y,da)\gamma(y)dy$$

$$-\nabla_x.\int v(t,x-\delta y,a)f(t,x-\delta y,a)c(t,x-\delta y,da)\gamma(y)dy,$$

which is Lebesgue square integrable on Q. Thus, for each fixed δ, we can take the limit of (71) when $\epsilon \to 0$, by using (68), and obtain

$$|<\nabla p(t,x).e, \int_{-1/2}^{+1/2} f_{c,\delta,\gamma}(t,x-2\theta\delta e)d\theta> + \int v.e(\partial_t f + v.\nabla_x f)dc| \tag{73}$$

$$\leq \delta C_f,$$

where $<.,.>$ denotes the duality bracket pairing measures and continuous function on Q and the mean value theorem has been used to rewrite the divided difference as an integral in θ. Let us introduce

$$c_{\delta,e,\gamma}(t,x,a) = \int_{-1/2}^{+1/2} d\theta \int c(t,x-2\theta\delta e-\delta y,a)\gamma(y)dy, \tag{74}$$

which is a mollification of c, continuous in t and smooth in x with values in the space of Borel probability measures in a. (The smoothness in x comes from the mollification by γ and the continuity in t from (14).) Definitions (72) and (74) show that

$$|\int_{-1/2}^{+1/2} f_{c,\delta,\gamma}(t,x-2\theta\delta e)d\theta - \int f(t,x,a)c_{\delta,e,\gamma}(t,x,da)| \leq C_f\delta,$$

so we can rewrite (73) as

$$|<\nabla p(t,x).e, \int f(t,x,a)c_{\delta,e,\gamma}(t,x,da)> + \int v.e(\partial_t f + v.\nabla_x f)dc| \tag{75}$$
$$\leq \delta C_f.$$

Since ∇p is a locally bounded measure in the interior of Q and $c_{\delta,e,\gamma}(t,x,a)$ is a probability measure in a, we get

$$|<\nabla p(t,x).e, \int f(t,x,a)c_{\delta,e,\gamma}(t,x,da)>| \tag{76}$$
$$\leq\ <|\nabla p(t,x).e|, \sup_{a\in A}|f(t,x,a)|>$$

Thus, the $c_{\delta,e,\gamma}$ can be seen as a bounded subset of $L^\infty(|\nabla p|, C(A))$, defined as the dual space of the (separable) Banach space $L^1(|\nabla p|, C(A))$, made of all functions $f(t,x,a)$ which are $|\nabla p|$ integrable in (t,x) with values in $C(A)$, the space of continuous functions in a. So this family is weak-* sequentially relatively compact. Because of (75), for each fixed e and γ, there is a unique possible cluster point, when $\delta \to 0$, denoted by $c_{e,\gamma}$ and satisfying

$$<\nabla p(t,x).e, \int f(t,x,a)c_{e,\gamma}(t,x,da)> = -\int v.e(\partial_t f + v.\nabla_x f)dc. \tag{77}$$

Since the right-hand side does not depend on γ and depends linearly on e, the limit la limite $c_{e,\gamma}$ cannot depend on γ and e. Thus, this limit can be denoted by $\underline{c}$ and satisfies

$$<\nabla p(t,x).e, \int f(t,x,a)\underline{c}(t,x,da)> = -\int v.e(\partial_t f + v.\nabla_x f)dc. \tag{78}$$

Thus, we deduce from definition (74) that (19) holds true for the weak-* topology of the dual space of $L^1(|\nabla p|, C(A))$ and we can use it as the definition of $\underline{c}$ as the extension of c with respect to the measure $dtdx + |\nabla p(t,x)|$. This completes the obtention of (15) and the proof of Theorem 2..3.

3.6. Consistency of the relaxed problem with local smooth solutions of the Euler equations.

In this subsection, we prove Theorem 2..4. Given a local smooth solution (u,p) of the Euler equations, it is enough to show that any admissible solution (c,m) satisfying $K(c,m) \leq 1/2 \int_Q |u|^2$ necessarily is the pair associated

with u through (21),(22).
Let us consider

$$f(t,x,a) = \partial_t g(t,i(a)).(x - g(t,i(a)))$$

which is a smooth function of $(t,x,a) \in Q'$ and satisfies $f(T,h(i(a)),a) = f(0,i(a),a) = 0$. Thus, we can use (23) and get

$$-\int v.\nabla_x f dc = \int \partial_t f dc.$$

So,

$$-\int v(t,x,a).\partial_t g(t,i(a)))dc(t,x,a)$$

$$= \int (\partial_t^2 g(t,i(a)).(x - g(t,i(a))) - |\partial_t g(t,i(a))|^2) dc(t,x,a).$$

Since (u,p) is a smooth solution to the Euler equations, we have

$$\partial_t^2 g(t,i(a)) = -(\nabla_x p)(t,g(t,i(a))).$$

Thus

$$\int \frac{1}{2}|v(t,x,a) - \partial_t g(t,i(a))|^2 dc(t,x,a)$$

$$= \int [\frac{1}{2}(|v(t,x,a)|^2 - |\partial_t g(t,i(a))|^2)$$

$$-(\nabla_x p)(t,g(t,i(a))).(x - g(t,i(a)))] dc(t,x,a).$$

But, by assumption,

$$K(c,m) \le \frac{1}{2}\int_Q |u|^2 dt dx$$

where the right-hand side is

$$\int \frac{1}{2}|\partial_t g(t,i(a))|^2 dt da = \int \frac{1}{2}|\partial_t g(t,i(a))|^2 dc(t,x,a)$$

(since $\int c(t,dx,a) = \int c(0,dx,a) = 1$, which follows from d'après (23)). So,

$$\int \frac{1}{2}|v(t,x,a) - \partial_t g(t,i(a))|^2 dc(t,x,a)$$

$$\le \int (-\nabla_x p)(t,g(t,i(a))).(x - g(t,i(a))) dc(t,x,a).$$

By definition of Λ as the largest eigenvalue of the second space derivatives of p, and by the mean value theorem, we have

$$p(t,x) - p(t,g(t,i(a))) - (\nabla_x p)(t,g(t,i(a))).(x - g(t,i(a)))$$
$$\leq \frac{1}{2}\Lambda|x - g(t,i(a))|^2.$$

Since

$$\int (p(t,x) - p(t,g(t,i(a))))dc(t,x,a) = 0$$

(indeed g is volume preserving and $\int c(t,dx,a) = 1$, $\int c(t,x,da) = 1$), we have finally obtained

$$\int (|v(t,x,a) - \partial_t g(t,i(a))|^2 - \Lambda|x - g(t,i(a))|^2)dc(t,x,a) \leq 0. \tag{79}$$

Since $\Lambda T^2 < \pi^2$, the proof of theorem 2..4 is completed, once we notice that

$$\int |x - g(t,i(a))|^2 dc(t,x,a) = 0$$

follows from the following generalization of the classical one dimensional Poincaré inequality :

Proposition 3..2 *Let $t \in [0,T] \to z(t) \in \mathbb{R}^d$ an absolutely continuous path suth that $\int_0^T |z'(t)|^2 dt < +\infty$. Let (c,m) be a pair of respectively nonnegative and vector-valued measures on $Q = [0,T] \times \mathbb{R}^d$, such that $m = cv$, with $v \in L^2(Q,dc)^d$, and*

$$\int (\partial_t \phi + v.\nabla_x \phi)dc = \phi(T,z(T)) - \phi(0,z(0)) \tag{80}$$

for all smooth functions ϕ on Q. Then

$$\int (|v(t,x) - z'(t)| - \Lambda|x - z(t)|^2)dc(t,x) \geq 0, \tag{81}$$

when $\Lambda T^2 < \pi^2$.

Proof of Proposition 3..2.

Set $z(t) = z(0)$ for $t \leq 0$ and fix $\tau > 0$. Introduce

$$\phi(t,x) = \inf \int_{-\tau}^{t} \frac{1}{2}[|\zeta'(\theta)|^2 - \Lambda|\zeta(\theta)|^2]d\theta, \tag{82}$$

where the infimum is performed on all paths $t \in [-\tau, T] \to \zeta(t) \in \mathbb{R}^d$ such that

$$\zeta(-\tau) = 0, \quad \zeta(t) = x - z(t).$$

By the classical one-dimensional Poincaré inequality,

$$\int_{-\tau}^{t} (|\zeta'(\theta)|^2 - \Lambda|\zeta(\theta)|^2) d\theta \geq 0,$$

for all paths ζ such that $\zeta(t) = 0$ for $t = -\tau$ and $t = T$, provided that $\Lambda(\tau + T)^2 \leq \pi^2$, which holds true if $\tau > 0$ is chosen small enough. We deduce

$$\phi(t, z(t)) = 0$$

(because the infimum is achieved by $\zeta = z$). By standard calculus, ϕ is a C^1 solution on Q of the Hamilton-Jacobi equation [3]

$$\partial_t \phi + \frac{1}{2}|\nabla_x \phi + z'|^2 + \frac{1}{2}[|z'|^2 + \Lambda|x - z|^2] = 0. \tag{83}$$

(We write, for small enough $\epsilon > 0$,

$$\phi(t, x) = \inf_{y,\zeta} [\phi(t - \epsilon, x - \epsilon y) + \int_{t-\epsilon}^{t} \frac{1}{2}[|\zeta'(\theta)|^2 - \Lambda|\zeta(\theta)|^2] d\theta,$$

where the infimum is taken over all $y \in \mathbb{R}^d$ and all paths ζ starting from $x - z(t) - \epsilon y$ at time $t - \epsilon$ and reaching $x - z(t)$ at time t. Then, we perform a Taylor expansion about $\epsilon = 0$.) By applying (80) to ϕ, which satisfies $\phi(T, z(T)) = \phi(0, z(0)) = 0$, and by using (83), we get

$$\int (v.\nabla\phi - \frac{1}{2}[|\nabla\phi + z'|^2 - |z'|^2 + \Lambda|x - z|^2]) dc = 0.$$

By this quantity exactly is

$$\frac{1}{2} \int (|v - z'|^2 - \Lambda|x - z|^2 - |v - \nabla\phi - z'|^2) dc,$$

which completes the proof of (81), Proposition 3..2 and Theorem 2..4.

3.7. Limit behaviour of ϵ solutions as ϵ goes to 0.

In this subsection we prove Proposition 2..5. Let us consider an ϵ solution (as defined in Definition 1..1), and the associated measures (c_ϵ, m_ϵ) (defined by (12)). Let (c, m) be a cluster point of (c_ϵ, m_ϵ), when $\epsilon \to 0$, with respect to the weak-* topology. Since K is weak-* lower semicontinuous (as a supremum of continuous linear forms), we have

$$K(c, m) \leq \liminf K(u_\epsilon).$$

Since u_ϵ is an ϵ solution, the right-hand side is less than $\underline{I}(h) = \lim I_\epsilon(h)$, by definitions (1..1), (6). The weak-* convergence of (c_ϵ, m_ϵ) and the strong L^2 convergence of $g_{u_\epsilon}(T, .)$ toward h enforce (23). Thus (c, m) is admissible and $I^*(h) \leq K(c, m)$ follows, which completes the proof of the first part of Proposition 2..5. Let us now consider the case when $I^*(h) = \underline{I}(h)$. In subsection 3.8., we show that this case includes all data $h \in S([0, 1]^3)$ satisfying condition (10) and, in subsection 3.1., we have also obtained the energy bound (31). Then we immediately get (33). Let us now prove that

$$\partial_t u_\epsilon + (u_\epsilon.\nabla)u_\epsilon \to \tag{84}$$

$$\partial_t \int v(t, x, a)c(t, x, da) + \nabla. \int v(t, x, a) \otimes v(t, x, a)c(t, x, da),$$

holds true in the sense of distributions, which is enough to get (11), by using (40) and completes the proof of Proposition 2..5. In (84), only the quadratic term makes problem. Therefore, it is enough to show

Proposition 3..3 *Assume that (c_ϵ, m_ϵ) weak-* converges toward (c, m), with $K(c_\epsilon, m_\epsilon) \to K(c, m)$. Then*

$$\int v_\epsilon(t, x, a) \otimes v_\epsilon(t, x, a)c_\epsilon(t, x, da) \to \int v(t, x, a) \otimes v(t, x, a)c(t, x, da), \tag{85}$$

in the sense of measures in (t, x).

Up to a subsequence, we have

$$\int v_\epsilon(t, x, a) \otimes v_\epsilon(t, x, a)c_\epsilon(t, x, da) \to \mu(t, x),$$

for some measure μ with values in the convex cone of all nonnegative symmetric square matrices. Let us set

$$\overline{\mu}(t, x) = \int v(t, x, a) \otimes v(t, x, a)c(t, x, da),$$

which is a measure of the same nature. Since $K(c_\epsilon, m_\epsilon) \to K(c,m)$, we have

$$\int_Q trace(\mu)(dt,dx) = \int_Q trace(\overline{\mu})(dt,dx).$$

Thus, to conclude that $\mu = \overline{\mu}$, it is now enough to prove $\overline{\mu} \leq \mu$ in the sense of nonnegative symmetric matrices, that is

$$\int \frac{1}{2}\overline{\mu}(dt,dx)(\xi(t,x),\xi(t,x)) \leq \int \frac{1}{2}\mu(dt,dx)(\xi(t,x),\xi(t,x)), \tag{86}$$

for every continuous function $\xi(t,x)$ on Q with values in $\mathbb{R}^d$. We observe that the left-hand side and the right-hand can be rewritten respectively $K_\xi(c,m)$ and $\lim_{\epsilon\to 0} K_\xi(c_\epsilon, m_\epsilon)$, where we set

$$K_\xi(\tilde{c},\tilde{m}) = \int \frac{1}{2}|\tilde{v}(t,x,a).\xi(t,x)|^2 d\tilde{c}(t,x,a)$$

$$= \sup_{\phi\in C(Q')} \int [\phi(t,x,a)\xi(t,x).d\tilde{m}(t,x,a) - \frac{1}{2}\phi(t,x,a)^2 d\tilde{c}(t,x,a)],$$

for any pair of measures $(\tilde{c}, \tilde{m} = \tilde{c}\tilde{v})$ such that $K(\tilde{c},\tilde{m}) < +\infty$. Thus, (86) immediately follows from the obvious lower weak-* semi-continuity of K_ξ. So the proofs of Proposition 3..3 and Proposition 2..5 are now complete.

3.8. Construction of an ϵ solution from an optimal solution to the relaxed problem.

In this subsection, we prove Theorem 2..6, by constructing a family $u_\epsilon \in V$ such that $K(u_\epsilon)$ is asymptotically not larger than $I^*(h)$ and

$$||g_{u_\epsilon}(T) - h||_{L^2} \to 0,$$

as $\epsilon \to 0$. (Indeed, then, up to a relabelling of ϵ, we may assume

$$K(u_\epsilon) \leq I^*(h) + \epsilon/2, \quad ||g_{u_\epsilon}(T) - h||_{L^2} \leq \epsilon,$$

which implies

$$I_\epsilon(h) \leq K(u_\epsilon) + \frac{1}{2\epsilon}||g_{u_\epsilon}(T) - h||_{L^2}^2 \leq I^*(h) + \epsilon$$

and we know, by Proposition 2..5, that $I^*(h) \leq \underline{I}(h) = \lim I_\epsilon(h)$.)
Let us fix an optimal solution $(c, m = cv)$, so that $K(c,m) = I^*(h)$. We first observe

Proposition 3..4 *Assume that h satisfies (10) for a given* $H \in S([0,1]^2)$*. Then* $(c^*, m^* = c^* v^*)$ *is an optimal solution to the three dimensional relaxed problem with* $D = [0,1]^3$ *and data h if and only if*

$$c^*(t,x,a) = \delta(x_3 - i_3(a))c(t,x_1,x_2,a), \tag{87}$$

$$v^*(t,x,a) = (v_1(t,x_1,x_2,a), v_2(t,x_1,x_2,a), 0)$$

where $(c, m = cv)$ *is an optimal solution to the two dimensional relaxed problem with* $D = [0,1]^2$ *and data H. In addition,* $I^*(h) = I^*(H)$*.*

Proof of Proposition 3..4.

Let (c^*, m^*) an optimal solution to the three dimensional problem with data h. We first observe that necessarily $m_3^* = 0$. Otherwise, the rescaled solution

$$\tilde{c}(t,x,a) = c^*(t,x_1,x_2,\eta(x_3)),$$
$$\tilde{m}_1(t,x,a) = m_1^*(t,x_1,x_2,\eta(x_3)),$$
$$\tilde{m}_2(t,x,a) = m_2^*(t,x_1,x_2,\eta(x_3)),$$
$$\tilde{m}_3(t,x,a) = \eta'(x_3)^{-1} m_3^*(t,x_1,x_2,\eta(x_3)),$$

where

$$\eta(x_3) = \min(2x_3, 2 - 2x_3),$$

would be admissible with a strictly lower energy $K(\tilde{c},\tilde{m}) < K(c^*, m^*) = I^*(h)$, which is absurd. Now, since $m_3^* = 0$, we deduce from (14) that

$$\frac{d}{dt}\int |x_3 - i_3(a)|^2 c^*(t,dx,da) = -\int \nabla_x(|x_3 - i_3(a)|^2).m^*(t,dx,da) = 0.$$

This shows that $x_3 = i_3(a)$ holds true for c^* almost every (t,x,a), which implies that $\delta(x_3 - i_3(a))$ can be factorized in the expression of c^* and the x_3 dependence of v can be ignored. So (c^*, m^*) satisfies (87). Now, $(c, m = cv)$ is necessarily admissible for the two dimensional problem with data H and satisfies

$$I^*(H) \le \int_{[0,T]\times[0,1]^2\times\mathbb{T}} |v|^2 dc = \int_{[0,T]\times[0,1]^3\times\mathbb{T}} |v^*|^2 dc^* = I^*(h).$$

But, $I^*(H) \ge I^*(h)$, so (c, m) is an optimal solution to the two dimensional problem with data H and $K(c,m) = I^*(H) = I^*(h)$. The converse part is then straightforward and the proof of Proposition 3..4 is now complete.

Continuation of the proof of Theorem 2..6.

Next, we wish to mollify (c^*, m^*), or more precisely its two dimensional counterpart (c, m), according to (87). Let us introduce the following notation

$$\underline{i}(a) = (i_1(a), i_2(a)) \in [0,1]^2, \quad a \in \mathbb{T},$$

so that

$$c(0,x,a) = \delta(x - \underline{i}(a)), \quad c(T,x,a) = \delta(x - H(\underline{i}(a))),$$

where $x \in [0,1]^2$. We first perform a change of variable in t so that the particle do not move when t is close to 0 or T. Namely, we introduce a waiting time $0 < \epsilon < T/4$ and set

$$\eta(t) = T \max(0, \min(\frac{t - 2\epsilon}{T - 4\epsilon}, 1)),$$

for $\epsilon > 0$ small enough, and set

$$\tilde{c}(t,x,a) = c(\eta(t), x, a), \quad \tilde{m}(t,x,a) = \eta'(t) m(\eta(t), x, a),$$

for $t \in \mathbb{R}$, $x \in [0,1]^2$ and $a \in \mathbb{T}$. (Or, more precisely,

$$< \tilde{c}, f >= \int_0^{2\epsilon} \int_A f(t, i(a), a) da dt + \int_{T-2\epsilon}^{T} \int_A f(t, h(i(a)), a) da dt$$

$$+(1 - 4\epsilon T^{-1}) < c(t,x,a), f(2\epsilon + t(1 - 4\epsilon T^{-1}), x, a) >,$$

$$< \tilde{m}, f >=< m(t,x,a), f(2\epsilon + t(1 - 4\epsilon T^{-1}), x, a) >,$$

for all continuous functions f on $[0,t] \times \mathbb{R}^2 \times \mathbb{T}$.)
So, for $t \leq 2\epsilon$,

$$\tilde{c}(t,x,a) = \delta(x - \underline{i}(a)), \quad \tilde{m}(t,x,a) = 0,$$

and, for $t \geq T - 2\epsilon$,

$$\tilde{c}(t,x,a) = \delta(x - H(\underline{i}(a))), \quad \tilde{m}(t,x,a) = 0.$$

Since the kinetic energy of (c, m) is time independent, we have

$$K(\tilde{c}, \tilde{m}) = K(c,m) \frac{1}{T} \int_0^T \eta'(t)^2 dt = K(c,m) \frac{T}{T - 4\epsilon}$$

arbitrarily close to $K(c,m)$ as $\epsilon \to 0$ and, of course, $(\tilde c, \tilde m)$ is still admissible. To keep simple notations, let us now denote $(\tilde c, \tilde m)$ by (c,m).
Next, we mollify our new (c,m) in $t \in [0,T]$, $a \in \mathbb{T}$ and get a new $(\tilde c, \tilde m)$. We will perform the more delicate mollification in $x \in [0,1]^2$ later. We denote by γ_q $(q = 1,2)$ a nonnegative even mollifier on $\mathbb{R}^q$, compactly supported in $]-1/2,+1/2[^q$, and set for all positive ϵ

$$\gamma_{q,\epsilon}(y) = \epsilon^{-q}\gamma_1(\epsilon^{-1}(y)), \quad y \in \mathbb{R}^q.$$

For the mollification with respect to $a \in \mathbb{T}$, we use an everywhere positive periodic convolution kernel

$$\gamma_{\mathbb{T},\epsilon}(b) = \sum_{k\in\mathbb{Z}} \epsilon^{-1}\gamma_0(\epsilon^{-1}(b+k)), \quad b \in \mathbb{T},$$

where γ_0 is a fixed positive function in the Schwartz class of integral equal to 1. Then, we set for all $(t,x,a) \in [0,T] \times [0,1]^2 \times \mathbb{T}$

$$\tilde c(t,x,a) = \int_{\mathbb{R}} ds \gamma_{1,\epsilon}(t-s) \int_{\mathbb{T}} db \gamma_{\mathbb{T},\epsilon}(a-b) c(s,x,b),$$

$$\tilde m(t,x,a) = \int_{\mathbb{R}} ds \gamma_{1,\epsilon}(t-s) \int_{\mathbb{T}} db \gamma_{\mathbb{T},\epsilon}(a-b) m(s,x,b).$$

Since $\gamma_{\mathbb{T},\epsilon}$ is bounded away from 0 and $+\infty$ on $\mathbb{T}$ and $\int c(t,x,da) = 1$, we have

$$0 < \inf \gamma_{\mathbb{T},\epsilon} \le \tilde c \le \sup \gamma_{\mathbb{T},\epsilon} < +\infty$$

and, therefore, $\tilde c$ can now be considered as a positive bounded function. For t near 0, we have

$$\tilde c(t,x,a) = \int_{\mathbb{T}} \delta(x - \underline{i}(b))\gamma_{\mathbb{T},\epsilon}(b-a)db, \quad \tilde v(t,x,a) = 0$$

and, for t near T,

$$\tilde c(t,x,a) = \int_{\mathbb{T}} \delta(x - H(\underline{i}(b)))\gamma_{\mathbb{T},\epsilon}(a-b)db, \quad \tilde v(t,x,a) = 0.$$

So the time boundary conditions are no longer satisfied by $(\tilde c, \tilde m)$. However, since we have performed a convex linear combination of translations of (c,m), (13) and (14) are still satisfied and, by convexity of K, $K(\tilde c, \tilde m) \le K(c,m)$. Therefore, $K(\tilde c, \tilde m)$ is asymptotically not larger than $I^*(h)$ as $\epsilon \to 0$. To keep simple notations again, let us denote $(\tilde c, \tilde m)$ by (c,m) to perform now the mollification in $x \in [0,1]^2$. We rescale the transform Θ introduced in subsection 3.1. by setting

$$\Theta(x) = (\min(x_1, 2-x_1), ..., \min(x_d, 2-x_d)), \tag{88}$$

for $x \in [0,2]^d$ and we extend Θ as a $(2\mathbb{Z})^d$ periodic map from $\mathbb{R}^d$ onto $[0,1]^d$. We denote by I_k the square $k+[0,1]^2$ for each $k \in \mathbb{Z}^2$ and by Θ_k^{-1} the reciprocal map from $[0,1]^2$ into I_k. Notice that the Jacobian matrix of Θ_k^{-1} is s_k times the identity matrix with $s_k \in \{-1,+1\}$. Next, we define a mollification kernel adapted to $[0,1]^2$ by setting, for all $\epsilon > 0$, $x, y \in [0,1]^2$

$$\Gamma_{2,\epsilon}(x,y) = \sum_{k\in\mathbb{Z}^2} \epsilon^{-2}\gamma_2(\frac{x-\Theta_k^{-1}(y)}{\epsilon}).$$

We are now ready to define

$$\tilde{c}(t,x,a) = \int_{[0,1]^d} \Gamma_{2,\epsilon}(x,y)c(t,y,a)dy$$

and

$$\tilde{m}(t,x,a) = \int_{[0,1]^d} \sum_{k\in\mathbb{Z}^2} \epsilon^{-2} s_k \gamma_2(\frac{x-\Theta_k^{-1}(y)}{\epsilon})m(t,y,a)dy,$$

so that (14) is satisfied. Observe that

$$\int_{[0,1]^d} \Gamma_{2,\epsilon}(y,x)dy = \int_{[0,1]^d} \Gamma_{2,\epsilon}(x,y)dy = 1,$$

for all $x \in [0,1]^d$. It follows that (13) is satisfied (because $\int c(t,x,da) = 1$) and, by Schwartz inequality, $K(\tilde{c},\tilde{m}) \le K(c,m)$ (which implies that $K(\tilde{c},\tilde{m})$ is asymptotically not larger than $I^*(h)$ as $\epsilon \to 0$), and

$$0 < \inf \gamma_{\mathbb{T},\epsilon} \le \tilde{c} \le \sup \gamma_{\mathbb{T},\epsilon} < +\infty.$$

Thus, both $\tilde{c}$ and $\tilde{v} = \tilde{m}/\tilde{c}$ are well defined and smooth. In addition, for each $a \in \mathbb{T}$ and $t \in [0,T]$, the field $x \to \tilde{v}(t,x,a)$ is parallel to the boundary of $[0,1]^2$, because of the symmetries of $\Gamma_{2,\epsilon}$. Let us denote $(\tilde{c},\tilde{m})$ by (c,m), again, and start the construction of an ϵ solution, after noticing that the boundary values in time of c are given by :

$$\tilde{c}(0,x,a) = \int_{[0,1]^d} \Gamma_{2,\epsilon}(x,\underline{i}(b))\gamma_{\mathbb{T},\epsilon}(a-b)db, \tag{89}$$

$$\tilde{c}(T,x,a) = \int_{[0,1]^d} \Gamma_{2,\epsilon}(x, H(\underline{i}(b)))\gamma_{\mathbb{T},\epsilon}(a-b)db. \tag{90}$$

The idea is to use the particle label a as the vertical coordinate z of $[0,1]^3$ and to construct a smooth divergence free velocity field

$$(t,x,z) \in [0,T] \times [0,1]^2 \times [0,1] \to (u(t,x,z), w(t,x,z)) \in \mathbb{R}^2 \times \mathbb{R}$$

out of the smooth field

$$(t,x,a) \in [0,T] \times [0,1]^2 \times [0,1] \to (v(t,x,a), c(t,x,a)) \in \mathbb{R}^2 \times]0,+\infty[.$$

For that purpose, let us set, for $x \in [0,1]^2$ and $a \in [0,1]$,

$$Z(t,x,a) = \int_0^a c(t,x,b)db,$$

which is smooth on $[0,T] \times [0,1]^3$. Since $Z(t,x,0) = 0$, $Z(t,x,1) = 1$ and $\partial_a Z > 0$, for each fixed $t \in [0,T]$, $(x,a) \to (x, Z(t,a))$ is a diffeomorphism of $[0,1]^3$ with a Jacobian determinant equal to $\partial_a Z(t,x,a) = c(t,x,a)$. Let us now introduce

$$u_i(t,x,Z(t,x,a)) = v_i(t,x,a),$$

for $i = 1,2$ and

$$w(t,x,z) = -\int_0^z (\partial_1 u_1 + \partial_2 u_2)(t,x,z')dz'$$

which satisfies $w(t,x,1) = w(t,x,0) = 0$. Indeed, by integrating (14) in $a \in [0,1]$, we get

$$0 = \nabla_x \cdot \int_0^1 v(t,x,a)c(t,x,a)da$$

$$= \int_0^1 \nabla_x . v(t,x,a)c(t,x,a)da + \int_0^1 v(t,x,a).\nabla_x c(t,x,a)da$$

$$= \int_0^1 \nabla_x . u(t,x,Z(t,x,a))c(t,x,a)da$$

$$+ \int_0^1 \partial_z u(t,x,Z(t,x,a)).\nabla_x Z(t,x,a)c(t,x,a)da$$

$$+ \int_0^1 u(t,x,Z(t,x,a)).\nabla_x \partial_a Z(t,x,a)da$$

$$= \int_0^1 \nabla_x . u(t,x,Z(t,x,a))\partial_a Z(t,x,a)da$$

$$+\int_0^1 \partial_a[u(t,x,Z(t,x,a)).\nabla_x Z(t,x,a)]da$$
$$= \int_0^1 \nabla_x.u(t,x,z')dz'.$$

Next, let us define $\xi(t,x,z) \in [0,1]^2$, $\zeta(t,x,z) \in [0,1]$ and $X(t,x,a) \in [0,1]^2$ by

$$\xi(t,x,z) = x + \int_0^t u(s,\xi(s,x,z),\zeta(s,x,z))ds,$$
$$\zeta(t,x,z) = z + \int_0^t w(s,\xi(s,x,z),\zeta(s,x,z))ds,$$
$$X(t,x,a) = x + \int_0^t v(s,X(s,x,a),a)ds.$$

Then, let us prove :

$$\xi(t,x,Z(0,x,a)) = X(t,x,a), \quad \zeta(t,x,Z(0,x,a)) = Z(t,X(t,x,a),a) \quad (91)$$

with the help of :

$$\partial_t Z(t,x,a) + u(t,x,Z(t,x,a)).\nabla_x Z(t,x,a) = w(t,x,Z(t,x,a)). \quad (92)$$

To prove (91), we use the following short notations, where x and a are frozen,

$$\xi(t) = \xi(t,x,Z(0,x,a)), \quad \zeta(t) = \zeta(t,x,Z(0,x,a)),$$
$$X(t) = X(t,x,a), \quad Z(t) = Z(t,X(t,x,a),a)$$

and denote by $'$ time derivatives. We have

$$\xi' = u(t,\xi,\zeta), \quad \zeta' = w(t,\xi,\zeta),$$
$$X' = v(t,X,a) = u(t,X,Z),$$
$$Z' = (\partial_t Z)(t,X,a) + (\nabla_x Z)(t,X,a).u(t,X,Z) = w(t,X,Z)$$

(by (92)). So

$$(X,Z)' = (u,w)(t,X,Z).$$

Since

$$\xi(0) = x = X(0), \quad \zeta(0) = Z(0,x,a) = Z(0),$$

it follows that $\xi = X$, $\zeta = Z$, which completes the proof of (91). Let us now show (92). We have

$$\partial_t Z(t,x,a) = \int_0^a \partial_t c(t,x,b)db = -\int_0^a \nabla_x.(cv)(t,x,b)db$$

$$= -\int_0^a (v.\nabla_x)c(t,x,b)db - \int_0^a (c\nabla_x.v)(t,x,b)db = I_1 + I_2,$$

where

$$I_1 = -\int_0^a u(t,x,Z(t,x,b)).\nabla_x c(t,x,b)db$$

$$= -\int_0^a u(t,x,Z(t,x,b)).\nabla_x \partial_a Z(t,x,b)db$$

$$= -\nabla_x Z(t,x,a).u(t,x,Z(t,x,a)) + \nabla_x Z(t,x,a=0).u(t,x,Z(t,x,a=0))$$

$$+\int_0^a \partial_z u(t,x,Z(t,x,b)).\nabla_x Z(t,x,b)c(t,x,b)db,$$

where the boundary term at $a=0$ vanishes (since $Z(t,x,a=0)=0$).

$$I_2 = -\int_0^a \nabla_x.(u(t,x,Z(t,x,b)))c(t,x,b)db$$

$$= \int_0^a (\partial_z w)(t,x,Z(t,x,b))\partial_a Z(t,x,b)db.$$

$$-\int_0^a (\partial_z u)(t,x,Z(t,x,b)).(c\nabla_x Z)(t,x,b)db.$$

Thus

$$I_1 + I_2 = -\nabla_x Z(t,x,a).u(t,x,Z(t,x,a)) + w(t,x,Z(t,x,a)),$$

which completes the proof of (92).

Let us now show that the divergence-free vector field

$$(t,x,z) \in [0,T]\times[0,1]^2\times[0,1] \to (u(t,x,z),w(t,x,z)) \in \mathbb{R}^3,$$

almost reaches the target H in the sense that

$$\int_{[0,1]^2\times[0,1]} |\xi(T,x,z) - H(x)|^2 dxdz \to 0, \tag{93}$$

as $\epsilon \to 0$. We set

$$\delta^2 = \int |\xi(T,x,z) - H(x)|^2 dxdz$$

$$= \int |\xi(T,x,Z(0,x,a)) - H(x)|^2 c(0,x,a)dxda$$

$$= \int |X(T,x,a) - H(x)|^2 c(0,x,a)dxda$$

(according to (91)), where

$$c(0,x,a) = \int \Gamma_{2,\epsilon}(x,\underline{i}(b))\gamma_{\mathbb{T},\epsilon}(a-b)db$$

is highly concentrated along $x = \underline{i}(a)$ for most of the $x \in [0,1]^2$. We have $\delta \leq \delta' + \delta''$ where

$$\delta'^2 = \int |X(T,x,a) - H(\underline{i}(a))|^2 c(0,x,a)dxda,$$

$$\delta''^2 = \int |H(x) - H(\underline{i}(a))|^2 c(0,x,a)dxda.$$

We can rewrite

$$\delta''^2 = \int |H(x) - H(\underline{i}(a))|^2 \gamma_{\mathbb{T},\epsilon}(a-b)\Gamma_{2,\epsilon}(x,\underline{i}(b))dbdxda.$$

If H' denotes a mollification of H, then

$$\int |H'(x) - H'(\underline{i}(a))|^2 \gamma_{\mathbb{T},\epsilon}(a-b)\Gamma_{2,\epsilon}(x,\underline{i}(b))dbdxda$$

$$\leq C' \int |x - \underline{i}(a)|^2 \gamma_{\mathbb{T},\epsilon}(a-b)\Gamma_{2,\epsilon}(x,\underline{i}(b))dbdxda,$$

where C' denotes any constant depending only on H', γ_1 and γ_2,

$$\leq C' \int (|x - \underline{i}(b)|^2 + |\underline{i}(b) - \underline{i}(a)|^2)\gamma_{\mathbb{T},\epsilon}(a-b)\Gamma_{2,\epsilon}(x,\underline{i}(b))dbdxda$$

$$\leq C'(\epsilon^2 + \int |\underline{i}(b) - \underline{i}(a)|^2 \gamma_{\mathbb{T},\epsilon}(a-b)dbda),$$

which tends to 0 as $\epsilon \to 0$, H' being fixed. Since,

$$\int (|H'(x) - H(x)|^2 + |H'(\underline{i}(a)) - H(\underline{i}(a))|^2)\gamma_{\mathbb{T},\epsilon}(a-b)\Gamma_{2,\epsilon}(x,\underline{i}(b))dbdxda$$

$$= 2\int |H'(x) - H(x)|^2 dx,$$

we deduce that $\delta'' \to 0$ as $\epsilon \to 0$. Let us now look at δ'. Since

$$\partial_t X(t,x,a) = v(t,X(t,x,a),a)$$

and

$$\partial_t c + \nabla_x.(cv) = 0,$$

$c(T, x, a)$ is nothing but the image measure of $c(0, x, a)$ by the map $(x, a) \to (X(t, x, a), a)$. Thus

$$\delta' = \int |x - H(\underline{i}(a))|^2 c(T, x, a) dx da$$

$$= \int |x - H(\underline{i}(a))|^2 \gamma_{\mathbb{T},\epsilon}(a - b) \Gamma_{2,\epsilon}(x, H(\underline{i}(b))) db dx da,$$

which can be estimated in an even simpler way than δ'' and completes the proof of (93).

So far, we have found a smooth divergence free vector field $(t, x, z) \in [0, T] \times [0, 1]^2 \times [0, 1] \to (u(t, x, z), w(t, x, z))$, with integral curves $(t, x, z) \to (\xi(t, x, z), \zeta(x, y, z))$, such that

$$\xi(t, x, Z(0, x, a)) = X(t, x, a)$$

and

$$\int |\xi(T, x, z) - H(x)|^2 dx dz \to 0$$

as $\epsilon \to 0$. We may now rescale this field by setting

$$\tilde{u}(t, x, z) = u(t, x, \eta_N(z)), \quad \tilde{w}(t, x, z) = \eta'_N(z)^{-1} w(t, x, \eta_N(z)),$$

where $N > 0$ is a large integer, η_N is the $2N^{-1}$ periodic Lipschitz continuous function defined by

$$\eta_N(z) = \min(Nz, 2 - Nz), \quad 0 \le z \le 2N^{-1}.$$

The corresponding integral curves are denoted by $(\tilde{\xi}, \tilde{\zeta})$. Since

$$w(t, x, z = 0) = w(t, x, z = 1) = 0,$$

$$\sup_{t,x,z} |\tilde{\zeta}(t, x, z) - z| \le N^{-1}$$

holds true and, therefore,

$$\int (|\tilde{\xi}(T, x, z) - H(x)|^2 + |\tilde{\zeta}(T, x, z) - z|^2) dx dz \to 0,$$

as ϵ and N^{-1} tend to 0. This shows that the field $(\tilde{u}, \tilde{w})$ almost reaches the target $h(x, z) = (H(x), z)$ in the L^2 norm. Next,

$$\int (|\tilde{u}(t, x, z)|^2 + \tilde{w}^2(t, x, z)) dt dx dz = \int (|u(t, x, z)|^2 + N^{-2} w^2(t, x, z)) dt dx dz,$$

which tends to

$$\int |u(t,x,z)|^2 dtdxdz = \int |v(t,x,a)|^2 c(t,x,a) dtdxda$$

as $N^{-1} \to 0$ (ϵ being fixed) and, therefore, is asymptotically not larger than $2K(c,m) = 2I^*(h)$ as $\epsilon \to 0$, as already shown. This completes the proof of Theorem 2..6. Notice that $(\tilde{u}, \tilde{w})$ is an ϵ' solution for some $\epsilon' \to 0$ as $\epsilon \to 0$.

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