CONVERGENCE OF THE VLASOV-POISSON SYSTEM TO THE INCOMPRESSIBLE EULER EQUATIONS

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Résumé

On étudie la convergence du système de Vlasov-Poisson vers les équations d'Euler des fluides incompressibles dans deux régimes asymptotiques : la limite quasi-neutre et la limite gyrocinétique.

Abstract

The convergence of the Vlasov-Poisson system to the incompressible Euler equations is investigated in two asymptotic regimes: the quasi-neutral limit and the gyrokinetic limit.

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We consider the displacement of an electronic cloud generated by the local difference of charge with a uniform neutralizing background of non-moving ions. The equations are given by the Vlasov-Poisson system, with a coupling constant $\epsilon = (\frac{\tau}{2\pi})^2$ where τ is the (constant) oscillation period of the electrons. In the so-called quasi-neutral regime, namely as $\epsilon \to 0$, the current is expected to converge to a solution of the incompressible Euler equations, at least in the case of a vanishing initial temperature. This result is proved by adapting an argument used by P.-L. Lions [Li] to prove the convergence of the Leray solutions of the 3d Navier-Stokes equation to the so-called *dissipative* solutions of the Euler equations. For this purpose, the total energy of the system is modulated by a test-function. An alternative proof is given, based on the concept of measure-valued (mv) solutions introduced by DiPerna and Majda [DM] and already used by Brenier and Grenier [BG], [Gr2] for the asymptotic analysis of the Vlasov-Poisson system in the quasi-neutral regime. Through this analysis, a link is established between Lions' dissipative solutions and Diperna-Majda's mv solutions of the Euler equations. A second interesting asymptotic regime, still leading to the Euler equations, known as the gyrokinetic limit of the Vlasov-Poisson system, is obtained when the electrons are forced by a strong constant external magnetic field and has been investigated by Grenier [Gr3], Golse and Saint-Raymond [GSR]. As for the quasi-neutral limit, we justify the gyrokinetic limit by using the concepts of dissipative solutions and modulated total energy.

1 Formal analysis

1.1 The Vlasov-Poisson system

After suitable normalizations, the Vlasov-Poisson system reads (see [BR] for example) :

$$\partial_t f + \xi.\nabla_x f - \nabla_x \Phi.\nabla_\xi f = 0, \tag{1}$$

$$\int_{\mathbb{R}^d} f(d\xi) = 1 - \epsilon\Delta\Phi \tag{2}$$

where $(x,\xi) \in \mathbb{R}^{2d}$ is the position/velocity variable, with $d = 1, 2$ or 3, $f(t,x,\xi) \geq 0$ the electronic density, $\Phi(t,x) \in \mathbb{R}$ the electric potential and $\epsilon > 0$ the coupling constant between the Vlasov equation (1) and the Poisson equation (2). To complete this system, initial conditions

$$f(0,x,\xi) = f_0(x,\xi) \geq 0 \tag{3}$$

and $\mathbb{Z}^d$ periodicity in x are prescribed. Up to a change of sign, we call charge and current the two first moments

$$\rho(t,x) = \int f(t,x,d\xi), \quad J(t,x) = \int \xi f(d\xi). \tag{4}$$

Electrons are called *cold* electrons when the temperature, proportional to

$$\int |\xi - \frac{J}{\rho}|^2 f(t,x,d\xi), \tag{5}$$

vanishes.

The conservation of total energy reads

$$\int \frac{1}{2}|\xi|^2 f(t,dx,d\xi) + \int \frac{\epsilon}{2}|\nabla\Phi(t,x)|^2 dx \tag{6}$$

(where integrals in x are performed on the unit cube $[0,1]^d$), and the conservation laws for charge and current are :

$$\partial_t \int f(d\xi) + \nabla_x . \int \xi f(d\xi) = 0 \tag{7}$$

(or, equivalently because of (2),

$$\nabla_x . \int \xi f(d\xi) = \epsilon \partial_t \Delta\Phi), \tag{8}$$

$$\partial_t \int \xi f(d\xi) + \nabla_x : \int \xi \otimes \xi f(d\xi) + \nabla\Phi \tag{9}$$
$$= \epsilon\nabla : (\nabla\Phi \otimes \nabla\Phi) - \frac{\epsilon}{2}\nabla(|\nabla\Phi|^2).$$

By computing the divergence of the last equations and using the Poisson equation,

$$-(\epsilon\partial_{tt} + 1)\Delta\Phi - \nabla_x^2 : \int \xi \otimes \xi f(d\xi) = \tag{10}$$
$$= -\epsilon\nabla^2 : (\nabla\Phi \otimes \nabla\Phi) + \frac{\epsilon}{2}\Delta(|\nabla\Phi|^2)$$

is obtained for the electric potential Φ.

The mathematical analysis of the Vlasov-Poisson system is now well known, in particular after the recent contributions of Batt and Rein [BR], Lions and Perthame [LP], Pfaffelmoser [Pf], etc... Global existence and uniqueness of smooth solutions have been proved for smooth initial data $f_0(x,\xi)$, sufficiently decaying at infinity in ξ. Then, all the formal computations we have performed are fully justified.

1.2 The quasi-neutral regime

The asymptotic analysis $\epsilon \to 0$ is difficult and only partial results have been obtained, in particular by Grenier in [Gr1], [Gr2], [Gr3] (see also [Br]). The oscillatory behaviour of the linear part of equation (10) is one of the main difficulties.

Let us start by a purely formal analysis of the limit $\epsilon \to 0$. The Poisson equation (2) becomes

$$\int f(d\xi) = 1 \tag{11}$$

and we get from equations (8), (9)

$$\nabla_x . \int \xi f(d\xi) = 0 \tag{12}$$

$$\partial_t \int \xi f(d\xi) + \nabla_x : \int \xi \otimes \xi f(d\xi) + \nabla\Phi = 0. \tag{13}$$

For the potential, we find

$$-\Delta\Phi = \nabla_x^2 : \int \xi \otimes \xi f(d\xi). \tag{14}$$

For perfecty cold electrons, the probablity measure (in ξ) $f(t,x,\xi)$ is a delta function, which exactly means

$$f(t,x,\xi) = \delta(\xi - J(t,x)), \tag{15}$$

since J is the current and the charge ρ is identically equal to 1. In this particular case, we obtain

$$\nabla . J = 0 \tag{16}$$

$$\partial_t J + \nabla : (J \otimes J) + \nabla\Phi = 0, \tag{17}$$

which is nothing but the classical Euler equations for an incompressible fluid (with velocity J and pressure Φ), for which we refer to [AK], [Ch], [Li], [MP]...

The case of cold electrons is precisely the one for which we get a rigorous asymptotic result in the present paper.

2 The convergence result

Theorem 2.1 *Let $T > 0$ and $J_0(x)$ be a given divergence-free, $\mathbb{Z}^d$ periodic in x, square integrable vector field. Assume the initial data $f_0^\epsilon(x,\xi) \geq 0$ to be smooth, $\mathbb{Z}^d$ periodic in x, nicely decaying as $\xi \to \infty$, with total mass 1. In addition, we assume*

$$\int f_0^\epsilon(x,\xi)d\xi = 1 + o(\epsilon^{1/2}), \quad \epsilon \to 0, \tag{18}$$

in the strong sense of the space $H^{-1}(\mathbb{R}^d/\mathbb{Z}^d)$ and

$$\int |\xi - v_0(x)|^2 f_0^\epsilon(x,\xi)dxd\xi \to \int |J_0 - v_0(x)|^2 dx, \tag{19}$$

for all square integrable, divergence-free, $\mathbb{Z}^d$ periodic, vector field v_0.

Then, up to the extraction of a sequence $\epsilon_n \to 0$, the divergence-free component of the current J^ϵ converges in $C^0([0,T], D'(\mathbb{R}^d/\mathbb{Z}^d))$ to a dissipative solution $J \in C^0([0,T], L^2(\mathbb{R}^d/\mathbb{Z}^d) - w)$ of the Euler equations, in the sense of Lions [Li], with initial condition J_0. In particular, if J_0 is smooth and $d = 2$ (or $d = 3$ and T small), the entire family (without extraction of any subsequence) converges to the unique smooth solution of the Euler equations with J_0 as initial condition.

Remark 1

Following Lions [Li], we say that J is a dissipative solution with initial condition J_0 if, for all smooth vector divergence-free vector fields v on $[0,T]\times \mathbb{R}^d/\mathbb{Z}^d$, almost every $t \in [0,T]$,

$$\int |J(t,x) - v(t,x)|^2 dx \leq \int |J_0(x) - v(0,x)|^2 dx \exp(\int_0^t 2||d(v(\theta))||) d\theta \tag{20}$$

$$+2\int_0^t \exp(\int_s^t 2||d(v(\theta))||d\theta)(\int A(v)(s,x).(v-J)(s,x))dsdx,$$

where $d(v)$ is the symmetric part of $Dv = ((Dv)_{ij}) = (\partial_j v_i)$

$$d_{ij}(v) = \frac{1}{2}(\partial_{x_i} v_j + \partial_{x_j} v_i), \tag{21}$$

$||d(v(t))||$ is the supremum in x of the spectral radius of $d(v)(t,x)$, and $A(v)$ is the acceleration operator

$$A(v) = \partial_t v + (v.\nabla)v. \tag{22}$$

Notice a slight change of definition with respect to [Li], since here we use the spectral radius of the entire matrix $d(v)$, not only its negative part.

Remark 2

The quasi-neutrality assumption (18) exactly means, because of (2),

$$\epsilon \int |\nabla\Phi^\epsilon(0,x)|^2 dx \to 0. \tag{23}$$

Assumption (19) means that the electrons are cold and the initial current converges to J_0. Indeed, we have (take $v_0 = J_0$ and $v_0 = 0$)

$$\int |\xi - J_0(x)|^2 f_0^\epsilon(x,\xi) dx d\xi \to 0, \tag{24}$$

$$\int |\xi|^2 f_0^\epsilon(x,\xi) dx d\xi \to \int |J_0(x)|^2 dx \tag{25}$$

3 Proofs

The proof is a simple adaptation of the way that Lions follows in [Li] to show the convergence of Leray solutions of the Navier-Stokes equations to the so-called dissipative solutions of the Euler equations. To do that, the total energy of the system is modulated by a test-function.

3.1 Control of the modulated total energy

Let us compute the time derivative of the total energy of the Vlasov-Poisson system, modulated by a test function $(t,x) \to v(t,x)$, $\mathbb{Z}^d$ periodic, divergence-free in x,

$$H_v^\epsilon(t) = \int \frac{1}{2}|\xi - v(t,x)|^2 f^\epsilon(t,x,\xi) dx d\xi + \int \frac{\epsilon}{2}|\nabla\Phi^\epsilon(t,x)|^2 dx. \tag{26}$$

Let us temporarily drop the index ϵ. Because of the total energy conservation, we have, for the charge ρ and the current J,

$$\frac{d}{dt}H_v(t) = \frac{d}{dt}\int \frac{1}{2}|v(t,x)|^2 \rho(t,x) dx - \int \partial_t(J(t,x).v(t,x)) dx. \tag{27}$$

Elementary calculations lead to

$$\frac{d}{dt}H_v(t) = -\int d(v)(t,x) : (\xi - v(t,x)) \otimes (\xi - v(t,x)) f(t,x,\xi) dx d\xi \quad (28)$$

$$+\epsilon \int d(v)(t,x) : \nabla\Phi(t,x) \otimes \nabla\Phi(t,x) dx$$

$$+ \int A(v)(t,x).(\rho(t,x)v(t,x) - J(t,x))dx$$

where $d(v)$ is the symmetrized gradient of v defined by (21) and $A(v)$ is the acceleration operator (22). Thus, we get, after rising index ϵ,

$$\frac{d}{dt}H_v^\epsilon(t) \leq 2||d(v(t))||H_v^\epsilon(t) + \int A(v)(\rho^\epsilon v - J^\epsilon)dx, \quad (29)$$

where H_v^ϵ is defined by (26) and $||d(v(t))||$ is the supremum in x of the spectral radius of $d(v)(t,x)$. We deduce, after integrating (29) in t,

$$H_v^\epsilon(t) \leq H_v^\epsilon(0) \exp(\int_0^t 2||d(v(\theta))||)d\theta \quad (30)$$

$$+ \int_0^t \exp(\int_s^t 2||d(v(\theta))||d\theta)(\int A(v)(s,x).(\rho^\epsilon v - J^\epsilon)(s,x))dsdx.$$

In particular, in the case $v = 0$, we recover the total energy bound

$$H_0^\epsilon(t) = \int \frac{1}{2}|\xi|^2 f^\epsilon(t,x,\xi)dxd\xi + \int \frac{\epsilon}{2}|\nabla\Phi^\epsilon(t,x)|^2 dx \leq H_0^\epsilon(0). \quad (31)$$

Remark

Here we use the spectral radius of the entire matrix $d(v)$ and not only its negative part (as in Lions' definition for dissipative solutions of the Euler equations). Indeed, in the right-hand side of (28), the first and the second terms involve $d(v)$ with opposite signs !

3.2 A priori bounds

The assumptions on the initial conditions and equations (18), (7), imply that

$$\int f^\epsilon(t,x,\xi)dxd\xi = \int \rho_0^\epsilon(x)dx = 1, \quad (32)$$

$$\int |\xi|^2 f_0^\epsilon(x,\xi)dxd\xi + \epsilon \int |\nabla\Phi^\epsilon(0,x)|^2 dx \to \int |J_0(x)|^2 dx. \tag{33}$$

From (31), we deduce that

$$\int |\xi|^2 f^\epsilon(t,x,\xi)dxd\xi + \epsilon \int |\nabla\Phi^\epsilon(t,x)|^2 dx \le C. \tag{34}$$

Thus J^ϵ is bounded in $L^\infty([0,T], L^1(\mathbb{R}^d/\mathbb{Z}^d))$ since

$$(\int |J^\epsilon(t,x)|dx)^2 \le \int |\xi|^2 f^\epsilon(t,x,\xi)d\xi dx \int f^\epsilon(t,x,\xi)d\xi dx \le C.$$

Up to the extraction of a sequence (ϵ_n), we can assume that J^ϵ has a vague limit J, in the sens of (Radon) measures on $[0,T]\times\mathbb{R}^d/\mathbb{Z}^d$. Similarly, from (32), (7) and (34), we get that $\rho^\epsilon(t,x)\ge 0$ converges to 1 in $C^0([0,T], D'(\mathbb{R}^d/\mathbb{Z}^d))$ and therefore in the vague sense of measures. Let us now consider the convex functional of (Radon) measures

$$K(\sigma,m) = \sup_b \int -\frac{1}{2}|b(t,x)|^2\sigma(dtdx) + b(t,x).m(dtdx),$$

where b spans the space of all continuous functions from $[0,T]\times\mathbb{R}^d/\mathbb{Z}^d$ to $\mathbb{R}^d$ and σ, m respectively denote nonnegative and vector-valued measures on $[0,T]\times\mathbb{R}^d/\mathbb{Z}^d$. When $\sigma(t,x)=1$ (the Lebesgue measure), we simply obtain

$$2K(\sigma,m) = \int |m(t,x)|^2 dtdx,$$

if m is a square integrable function and $+\infty$ otherwise. Functional K is lsc with respect to the vague convergence of measures. Since, for each nonnegative function $z\in C^0([0,T])$,

$$2K(z\rho^\epsilon, zJ^\epsilon) = \int \frac{|J^\epsilon(t,x)|^2}{\rho^\epsilon(t,x)} z(t)dtdx$$

$$\le \int |\xi|^2 f^\epsilon(t,x,\xi)z(t)dtdxd\xi \le C\int_0^T z(t)dt,$$

we deduce that

$$2K(z,zJ) \le C\int_0^T z(t)dt,$$

which exactly means that J belongs to $L^\infty([0,T], L^2(\mathbb{R}^d/\mathbb{Z}^d))$. From (8), we get that J is divergence-free in x and, from (9), that $\partial_t J$ is bounded in

$L^\infty([0,T],D'(\mathbb{R}^d/\mathbb{Z}^d))$, since J is divergence-free (which allows us to ignore $\nabla\Phi^\epsilon$ in (9), although this term could be of size $O(\epsilon^{-1/2})$). It follows that the vague limit $J(t,x)$ of $J^\epsilon(t,x)$ is a divergence-free vector field belonging to $C^0([0,T],L^2(\mathbb{R}^d/\mathbb{Z}^d)-w)$. For the same reasons, the divergence-free (or solenoidal) part of J_ϵ converges toward J, not only in the vague sense of measures, but also in $C^0([0,T],D'(\mathbb{R}^d/\mathbb{Z}^d))$.

3.3 Convergence

We can rewrite (29) in weak form

$$-\int H_v^\epsilon(t)z'(t)dt - z(0)H_v^\epsilon(0) \leq \int 2||d(v(t))||H_v^\epsilon(t)z(t)dt \tag{35}$$

$$+\int A(v)(\rho^\epsilon v - J^\epsilon)(t,x)z(t)dtdx,$$

for all test function $z \geq 0$ in $D'([0,T[)$, where $H_v^\epsilon(t)$ is defined by (26). Let us introduce

$$h_v^\epsilon(t) = \int \frac{|J^\epsilon(t,x) - v(t,x)\rho^\epsilon(t,x)|^2}{2\rho^\epsilon(t,x)}dx \tag{36}$$

$$= \sup_b \int [-\frac{1}{2}|b(x)|^2\rho^\epsilon(t,x) + b(x).(J^\epsilon - v\rho^\epsilon)(t,x)]dx,$$

where b spans the space of all continuous functions from $\mathbb{R}^d/\mathbb{Z}^d$ to $\mathbb{R}^d$, which is, for each fixed t, a convex fuction of $J^\epsilon(t,.)$ and $\rho^\epsilon(t,.)$. (It is a just a modulated version of functional K, with a test function v.) By Cauchy-Schwarz inequality, we have

$$h_v^\epsilon(t) \leq \int \frac{1}{2}|\xi - v(t,x)|^2 f^\epsilon(t,x,\xi)dxd\xi \leq \int H_v^\epsilon(t).$$

The a priori bound previously obtained show that, for fixed v, $H_v^\epsilon(t)$ and $h_v^\epsilon(t)$ are bounded functions in $L^\infty([0,T])$ and, up to the extraction of a sequence (ϵ_n), respectively converge, in the weak-* sense, to some limits $H_v(t)$ and $h_v(t)$, with $H_v \geq h_v$. Since $\rho^\epsilon \to 1$ and $J^\epsilon \to J$ in the vague sense of measures, by convexity of the functional defined by (36), we get

$$\int |J(t,x) - v(t,x)|^2 dx \leq 2h_v(t). \tag{37}$$

The assumptions on the initial conditions mean

$$2H_v^\epsilon(0) = \int |\xi - v(0,x)|^2 f_0^\epsilon(x,\xi)dxd\xi + \epsilon\int |\nabla\Phi^\epsilon(0,x)|^2 dx \to 2H_{0,v} \tag{38}$$

where we set

$$H_{0,v} = \frac{1}{2}\int |J_0(x) - v(0,x)|^2 dx. \tag{39}$$

Then, we can pass to the limit in (35) to get

$$-\int H_v(t)z'(t)dt - z(0)H_{0,v} \leq \int 2||d(v(t))||H_v(t)z(t)dt \tag{40}$$

$$+\int A(v)(v-J)(t,x)z(t)dtdx.$$

By integrating in t, we get

$$H_v(t) \leq H_{0,v}\exp(\int_0^t 2||d(v(\theta))||)d\theta \tag{41}$$

$$+\int_0^t \exp(\int_s^t 2||d(v(\theta))||d\theta)(\int A(v)(s,x).(v-J)(s,x))dsdx.$$

Thus

$$h_v(t) \leq H_{0,v}\exp(\int_0^t 2||d(v(\theta))||)d\theta \tag{42}$$

$$+\int_0^t \exp(\int_s^t 2||d(v(\theta))||d\theta)(\int A(v)(s,x).(v-J)(s,x))dsdx$$

and, therefore, (20) holds true, which concludes the proof.

4 An alternative proof

Let us sketch an alternative proof, which can be seen as a natural extension of the analysis made in [BG] (stationary case) and [Gr2] (general case) to study the defect measures of the Vlasov-Poisson system in the quasi-neutral regime.

After adaptating the proof (which requires an a priori L^∞ bound for f^ϵ, which is not acceptable in the framework of the present paper), we can show 1) the existence of $f(t,x,\xi)$, a nonnegative measure f in $(x,\xi) \in \mathbb{R}^d/\mathbb{Z}^d \times \mathbb{R}^d$, measurable in t, as the vague limit of f^ϵ, with enough tightness in ξ to allow the zero and first order moments in ξ (namely the charge and the current) to pass to the limit; 2) the existence of $\nu_K(t,x,\eta)$ and $\nu_E(t,x,\eta)$, two defect measures in $(x,\eta) \in \mathbb{R}^d/\mathbb{Z}^d \times S^{d-1}$, measurable in t, that correspond respectively to the defect of kinetic and potential energies; 3) the existence of two defect electric fields $E_+(t,x)$ and $E_-(t,x) \in L^\infty([0,T], L^2(\mathbb{R}^d/\mathbb{Z}^d))$, taking into account the temporal oscillations of the electric field generated by (10); 4) the convergence of the solenoidal part of J^ϵ toward $J = \int \xi f(d\xi)$

in $C^0([0,T],D'(\mathbb{R}^d/\mathbb{Z}^d))$. This is enough to enforce 1) the conservation in time of the total energy with defects

$$2H(t) = \int |\xi|^2 f(t,dx,d\xi) + \int (\nu_K + \nu_E)(t,dx,d\eta) \tag{43}$$

$$+ \int (|E_+(t,x)|^2 + |E_-(t,x)|^2)dx,$$

2) the following properties for the current $J(t,x) = \int \xi f(t,x,d\xi)$:

$$\nabla.J = 0, \tag{44}$$

$$\partial_t J + \nabla : Q = 0, \tag{45}$$

where

$$Q = \int \xi \otimes \xi f(d\xi) + \int \eta \otimes \eta (\nu_K - \nu_E)(d\eta) \tag{46}$$

$$-E_+ \otimes E_+ - E_- \otimes E_-.$$

(Note the change of sign between $\nu_K + \nu_E$ and $\nu_K - \nu_E$ when we switch from the energy conservation to the current conservation.) From these relations, we deduce that the weak-* L^∞ limit of the modulated total energy $H_v^\epsilon(t)$ is given by

$$2H_v(t) = \int |\xi - v(t,x)|^2 f(t,dx,d\xi) + \int (\nu_K + \nu_E)(t,dx,d\eta) \tag{47}$$

$$+ \int (|E_+(t,x)|^2 + |E_-(t,x)|^2)dx.$$

Thus, we directly get

$$\frac{d}{dt}H_v(t) = -\int d(v)(t,x) : (\xi - v(t,x)) \otimes (\xi - v(t,x)) f(t,dx,d\xi) \tag{48}$$

$$- \int d(v)(t,x) : \eta \otimes \eta (\nu_K - \nu_E)(t,dx,d\eta)$$

$$+ \int d(v)(t,x) : (E_+ \otimes E_+ + E_- \otimes E_-)(t,x)dx$$

$$+ \int A(v)(t,x).(v(t,x) - J(t,x))dx$$

$$\leq 2||d(v(t))||H_v(t) + \int A(v)(t,x).(v(t,x) - J(t,x))dx, \tag{49}$$

and we conclude as in the first proof.

5 Comparison of dissipative and mv solutions to the Euler equations

Our analysis makes a link between Lions' concept of dissipative solutions [Li] and Diperna-Majda's concept of measure-valued solutions ("mv solutions) [DM], both introduced to describe the vanishing viscosity limit of the Navier-Stokes equations [Li]. If we get back to [DM], we obtain, as before, two limits f, $J = \int \xi f(d\xi)$, and a kinetic defect measure ν_K (the only relevent defect measure when approaching the Euler equations from the Navier-Stokes side and not from the Vlasov-Poisson side). We get for J (44) and (45) with,

$$Q = \int \xi \otimes \xi f(d\xi) + \int \eta \otimes \eta \nu_K(d\eta). \tag{50}$$

In addition, the total kinetic energy, *including defects*, namely :

$$= \int |\xi|^2 f(t, dx, d\xi) + \int \nu_K(t, dx, d\eta) \tag{51}$$

is decaying in time. Thus, after the same kind of manipulations we already used, we see that J is a dissipative solution of the Euler equations. Thus, the mv solutions are not as different from the dissipative solutions as they look. Anyway, the concept of dissipative solutions clarify the relationship between mv solutions and classical solutions, which was not discussed in [DM].

6 The gyrokinetic limit

There is a second asymptotic regime of the Vlasov-Poisson system leading to the Euler equations, the so-called gyrokinetic limit. We consider, as in [Gr3] (see also the included references) or in [GSR] (with a different scaling), the effect of a large external magnetic field. If this magnetic field is parallel to the third coordinate x_3, we get the following two-dimensional (in both x and ξ) Vlasov-Poisson system

$$\partial_t f^\epsilon + \xi.\nabla_x f^\epsilon + \frac{1}{\epsilon}(-\nabla\Phi^\epsilon +^\perp \xi).\nabla_\xi f^\epsilon = 0, \tag{52}$$

$$\rho^\epsilon = 1 - \Delta\Phi^\epsilon, \quad \int \rho^\epsilon(t,x)dx = 1, \tag{53}$$

where $x \in \mathbb{R}^2/\mathbb{Z}^2$, $\xi \in \mathbb{R}^2$, and $^\perp\xi = (-\xi_2, \xi_1)$ is the additional term due to the external magnetic field. We assume the total mass of ρ^ϵ to be equal to one at time 0 to enforce global neutrality. The total energy is still conserved and, here, defined by

$$\epsilon \int \frac{1}{2}|\xi|^2 f^\epsilon(t, dx, d\xi) + \int \frac{1}{2}|\nabla \Phi^\epsilon(t, x)|^2 dx \tag{54}$$

(notice that the magnetic field is not involved). In addition, we get

$$\partial_t \rho^\epsilon + \nabla . J^\epsilon = 0, \tag{55}$$

$$\partial_t J^\epsilon + \nabla_x : \int \xi \otimes \xi f^\epsilon(d\xi) \tag{56}$$

$$= \frac{1}{\epsilon}(-\rho^\epsilon \nabla \Phi^\epsilon +^\perp J^\epsilon).$$

By combining (56) and (55), we also get

$$\partial_t(\rho^\epsilon - \epsilon^\perp \nabla . J^\epsilon) +^\perp \nabla .(\rho^\epsilon \nabla \Phi^\epsilon) \tag{57}$$

$$= \epsilon^\perp \nabla .(\nabla_x : \int \xi \otimes \xi f^\epsilon(d\xi)).$$

Formally, as ϵ goes to zero, we expect for the limits ρ, J and Φ, the self-consistent system :

$$\partial_t \rho + \nabla . J = 0, \tag{58}$$

$$-\rho \nabla \Phi +^\perp J = 0, \quad \rho = 1 - \Delta \Phi, \tag{59}$$

which is nothing but the Euler equations written in the so-called vorticity formulation, with $\rho - 1$ standing for the vorticity and Φ for the stream-function. The limit $\epsilon \to 0$ has been successfully investigated in [Gr3] for monokinetic data and small time, as well as in [GSR] for a different scaling and global weak solutions of the Euler equation in Delort's sense (see [De]).

We can perform the same kind of analysis as for the quasi-neutral limit, and show :

Theorem 6.1 *Let $T > 0$ and $J_0(x) = -^\perp \nabla \Phi_0$ be a given divergence-free, $\mathbb{Z}^2$ periodic in x, square integrable vector field. Assume the initial data $f_0^\epsilon(x, \xi) \geq 0$ to be smooth, $\mathbb{Z}^2$ periodic in x, nicely decaying as $\xi \to \infty$, with total mass* 1*. In addition, we assume*

$$\epsilon \int |\xi|^2 f_0^\epsilon(x, \xi) d\xi dx \to 0, \tag{60}$$

$$\int |\nabla \Phi^\epsilon(0, .) -^\perp J_0(x)|^2 dx \to 0. \tag{61}$$

Then, up to the extraction of a sequence $\epsilon_n \to 0$, $-^\perp\nabla\Phi^\epsilon$ converges in $C^0([0,T], L^2(\mathbb{R}^2/\mathbb{Z}^2) - w)$ to a dissipative solution J of the Euler equations with initial condition J_0. In particular, if J_0 is smooth, the entire family converges to the unique smooth solution of the Euler equations with J_0 as initial condition.

To prove this result, we use the same technique as for the quasi-neutral limit by introducing a modulated total energy, defined in the following way. Given a smooth divergence-free vector field $v(t,x) = -^\perp\nabla\psi(t,x)$, we set

$$H_v^\epsilon(t) = \int \frac{\epsilon}{2}|\xi - v(t,x)|^2 f^\epsilon(t,x,\xi)dxd\xi + \int \frac{1}{2}|\nabla(\Phi^\epsilon - \psi)(t,x)|^2 dx. \quad (62)$$

A straightforward but lengthy calculation (using (56) in a crucial way, see the details in the appendix), leads to

$$\frac{d}{dt}H_v^\epsilon(t) = -\epsilon\int d(v)(t,x) : (\xi - v(t,x)) \otimes (\xi - v(t,x)) f^\epsilon(t,x,\xi)dxd\xi \quad (63)$$

$$+\int d(v)(t,x) : \nabla(\Phi^\epsilon - \psi)(t,x) \otimes \nabla(\Phi^\epsilon - \psi)(t,x)dx$$

$$+\epsilon\int A(v)(t,x).(\rho^\epsilon(t,x)v(t,x) - J^\epsilon(t,x))dx$$

$$+\int A(v)(t,x).(v(t,x) +^\perp \nabla\Phi^\epsilon(t,x))dx$$

where $d(v)$, $A(v)$ are still defined by (21), (22).

We also get the following bounds : $\nabla\Phi^\epsilon$ is bounded in

$$L^\infty([0,T], L^2(\mathbb{R}^2/\mathbb{Z}^2)),$$

ρ^ϵ and $\epsilon^{1/2}J^\epsilon$ are bounded in

$$L^\infty([0,T], L^1(\mathbb{R}^2/\mathbb{Z}^2))$$

(because of the conservation of charge and energy). Next, $\rho^\epsilon\nabla\Phi^\epsilon$ is bounded in

$$L^\infty([0,T], D'(\mathbb{R}^2/\mathbb{Z}^2)).$$

Indeed, for all smooth vector field $g(x)$, because of (2),

$$\int g(x).\rho^\epsilon(t,x)\nabla\Phi^\epsilon(t,x)dx = \int g.\nabla\Phi^\epsilon$$

$$+\int(\frac{1}{2}|\nabla\Phi^\epsilon|^2\nabla.g+(\nabla\Phi^\epsilon.\nabla)g.\nabla\Phi^\epsilon)dx \le C||g||_{C^1(\mathbb{R}^2/\mathbb{Z}^2)}.$$

Then, because of (57), $\rho^\epsilon - \epsilon^\perp\nabla.J^\epsilon$ is compact in

$$C^0([0,T],D'(\mathbb{R}^2/\mathbb{Z}^2)).$$

Since $\epsilon^\perp\nabla.J^\epsilon = 0(\epsilon^{1/2})$ in $L^\infty([0,T],D'(\mathbb{R}^2/\mathbb{Z}^2))$, we deduce that ρ^ϵ, and therefore $\nabla\Phi^\epsilon$, are also compact in $C^0([0,T],D'(\mathbb{R}^2/\mathbb{Z}^2))$. Thus, we conclude that, up to the extraction of a sequence $\epsilon_n \to 0$, H_v^ϵ and $\nabla\Phi^\epsilon$ converge to some limits H_v and $\nabla\Phi$, respectively in $L^\infty([0,T])$ weak-* and

$$C^0([0,T],L^2(\mathbb{R}^2/\mathbb{Z}^2)-w).$$

Then, we can pass to the limit in (62) and (63) to get

$$\int|\nabla(\Phi-\psi)(t,x)|^2dx \le H_v(t), \tag{64}$$

$$-\int H_v(t)z'(t)dt - z(0)H_{0,v} \le \int 2||d(v(t))||H_v(t)z(t)dt \tag{65}$$

$$+\int A(v).(v+^\perp\nabla\Phi)(t,x)z(t)dtdx,$$

for all smooth nonnegative $z(t)$ compactly supported in $0 \le t < T$, where

$$H_{0,v} = \int|\nabla(\Phi_0-\psi(0,.))(x)|^2dx \tag{66}$$

is, by assumption, the limit of $H_v^\epsilon(0)$. By integrating in t, we get

$$H_v(t) \le H_{0,v}\exp(\int_0^t 2||d(v(\theta))||)d\theta \tag{67}$$

$$+\int_0^t \exp(\int_s^t 2||d(v(\theta))||d\theta)(\int A(v)(s,x).(v+^\perp\nabla\Phi)(s,x))dsdx,$$

and, finally, by using (64), we conclude that $-^\perp\nabla\Phi$ is a dissipative solution of the Euler equations with initial condition $J_0 = -^\perp\nabla\Phi_0$, which concludes the proof.

7 Appendix

In this appendix, we prove the crucial identity (63). Because of the conservation of energy, we get from definition (62) :

$$\frac{d}{dt}H_v = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7,$$

where index ϵ has been dropped and

$$2I_1 = \epsilon \int |v|^2 \partial_t \rho, \quad 2I_2 = \epsilon \int \rho \partial_t |v|^2,$$

$$I_3 = -\epsilon \int v.\partial_t J, \quad I_4 = -\epsilon \int J.\partial_t v,$$

$$2I_5 = \int \partial_t |v|^2, \quad I_6 = -\int \nabla\Phi.\partial_t \nabla\psi, \quad I_7 = -\int \nabla\psi.\partial_t \nabla\Phi.$$

We have

$$I_7 = \int \partial_t \Delta\Phi\psi = -\int \partial_t \rho\psi$$
$$= \int \nabla.J\psi = -\int J.\nabla\psi$$

$$I_3 = \epsilon \int v.(\nabla : \int \xi \otimes \xi f) + \int \rho v.\nabla\Phi - \int v.^{\perp} J$$
$$= -\epsilon \int d(v) : \int \xi \otimes \xi f - \int \Delta\Phi v.\nabla\Phi + \int J.\nabla\psi$$
$$= -\epsilon \int d(v) : \int \xi \otimes \xi f + \int d(v) : \nabla\Phi \otimes \nabla\Phi - I_7.$$

Thus

$$I_3 + I_7 = Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6 + Q_7$$

where

$$Q_1 = -\epsilon \int d(v) : \int (\xi - v) \otimes (\xi - v) f + \int d(v) : \nabla(\Phi - \psi) \otimes \nabla(\Phi - \psi),$$

$$Q_2 = -\epsilon \int Dv : J \otimes v, \quad Q_3 = -\epsilon \int Dv : v \otimes J$$

$$Q_4 = \int Dv : \nabla\psi \otimes \nabla\Phi, \quad Q_5 = \int Dv : \nabla\Phi \otimes \nabla\psi$$

$$Q_6 = \epsilon \int \rho Dv : v \otimes v, \quad Q_7 = -\int Dv : \nabla\psi \otimes \nabla\psi.$$

Then,

$$Q_3 = -\epsilon \int \partial_j v_i J_j v_i$$

$$= \frac{1}{2}\epsilon \int \nabla . J|v|^2 = -\frac{1}{2}\epsilon \int \partial_t \rho |v|^2 = -I_1.$$

Next, we observe that

$$I_2 + I_4 + Q_2 + Q_6 = \epsilon \int A(v).(v - J)$$

and

$$I_5 + I_6 + Q_4 + Q_5 + Q_7 = \int A(v).(v +^{\perp} \nabla\Phi) + R,$$

where

$$R = R_1 + R_2 + R_3 + R_4,$$

$$R_1 = \int Dv : (-v \otimes v) = -\int v_i v_j \partial_j v_i = 0,$$

$$R_2 = \int Dv : \nabla(\Phi - \psi) \otimes \nabla\psi,$$

$$R_3 = \int Dv : (\nabla\psi \otimes \nabla\Phi -^{\perp} \nabla\Phi \otimes v).$$

Since $v = -^{\perp}\nabla\psi$ and $(^{\perp})^2 = -1$, , we get

$$R_3 = \int (\nabla \otimes v) : (-v \otimes^{\perp} \nabla\Phi + \nabla\Phi \otimes^{\perp} v)$$

$$= \int (\nabla \otimes^{\perp} v) : (v \otimes \nabla\Phi - \nabla\Phi \otimes v) = \int (\nabla \otimes \nabla\psi) : (v \otimes \nabla\Phi - \nabla\Phi \otimes v)$$

$$= \int (\nabla \otimes \nabla)\psi : (v \otimes \nabla\Phi - \nabla\Phi \otimes v) = 0.$$

Similarly, after setting $\theta = \Phi - \psi$,

$$R_2 = \int (\nabla \otimes v) : \nabla\psi \otimes \nabla\theta$$

$$= \int (-\nabla \otimes^{\perp} \nabla\psi) : \nabla\psi \otimes \nabla\theta = \int (\nabla \otimes \nabla\psi) : \nabla\psi \otimes^{\perp} \nabla\theta$$

$$= \int (\nabla \otimes \nabla)\psi : \nabla\psi \otimes^{\perp} \nabla\theta = \int \nabla(\frac{1}{2}|\nabla\psi|^2).^{\perp}\nabla\theta = 0.$$

Thus, $R = 0$ and we finally get

$$\frac{d}{dt}H_v = Q_1 + \int A(v).[v +^{\perp} \nabla\Phi + \epsilon(\rho v - J)],$$

which is the desired result.

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