Remarks on the derivation of the hydrostatic Euler equations

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ABSTRACT. L'écoulement d'un fluide parfait incompressible bidimensionnel entre deux plaques horizontales est étudié dans la limite où les deux plaques sont infiniment proches. La convergence des solutions des équations d'Euler vers celles de leur limite formelle "hydrostatique" peut être établie dans le cas où l'écoulement initial satisfait une condition de Rayleigh locale (en tout point de l'axe horizontal, le profil vertical des vitesses horizontales est supposé strictement convexe). Ce résultat a été originellement prouvé par Grenier par une méthode d'énergie à poids inspirée de celle utilisée par V.I. Arnold pour la stabilité non-linéaire des équations d'Euler. Ici, une nouvelle preuve est proposée, plus directe et plus proche encore de la méthode d'Arnold.

The motion of an inviscid incompressible fluid between two horizontal plates is studied in the limit when the plates are infinitesimally close. The convergence of the solutions of the Euler equations to those of their formal 'hydrostatic' limit can be established in the case when the initial velocity field satisfies a local Rayleigh conditions. This result, originally obtained by Grenier through weighted energy estimates based on Arnold's stability analysis of the Euler equations, is proven here by a more straightforward method even closer to Arnold's method.

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1. Introduction

Let us consider a two dimensional inviscid incompressible flow moving in a very narrow channel. More precisely, let $U = (U_1(t,x), U_2(t,x))$ be the velocity field of an inviscid incompressible fluid and let $P = P(t,x)$ be the pressure field, for $x = (x_1, x_2)$, $x_1 \in \mathbb{R}/\mathbb{Z}$, $0 < x_2 < \epsilon$, $0 \le t \le T$. These fields are subject to the Euler equations

$$(\partial_t + U.\nabla)U + \nabla P = 0, \quad \nabla.U = 0, \tag{1.1}$$

with slip boundary conditions $U_2 = 0$ at $x_2 = 0$ and $x_2 = \epsilon$. (For the mathematical analysis of the Euler equations, see [**AK**], [**Ch**], [**Li**], [**MP**], etc...) After performing the following rescaling

$$x_2 \to \epsilon x_2, \quad U_2 \to \epsilon U_2,$$

while t, x_1, U_1 and P are left unchanged, we get for the rescaled fields, still denoted by (U, P), a set of "rescaled Euler equations" in the fixed domain $x = (x_1, x_2) \in$

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$D = \mathbb{R}/\mathbb{Z}\times]0,1[$,

$$(\partial_t + U.\nabla)U_1 + \partial_1 P = 0, \tag{1.2}$$

$$\epsilon^2(\partial_t + U.\nabla)U_2 + \partial_2 P = 0, \tag{1.3}$$

$$\nabla.U = 0, \tag{1.4}$$

with slip boundary condition $U_2 = 0$ at $x_2 = 0$ and $x_2 = 1$. The rescaled vorticity Ω is defined by

$$\Omega = \partial_2 U_1 - \epsilon^2 \partial_1 U_2 \tag{1.5}$$

and solve

$$(\partial_t + U.\nabla)\Omega = 0. \tag{1.6}$$

By setting $\epsilon = 0$ in the rescaled Euler equations, we formally obtain a new set of equations, the "hydrostatic Euler" equations, following Lions' terminology [**Li**]. More precisely, we say that (u,p) is a smooth solution of the hydrostatic Euler equations, if $u = (u_1(t,x), u_2(t,x))$, $p = p(t,x)$, satisfy, for $x = (x_1,x_2) \in D$ and $t \in [0,T]$,

$$(\partial_t + u.\nabla)u_1 + \partial_1 p = 0, \tag{1.7}$$

$$\partial_2 p = 0, \tag{1.8}$$

$$\nabla.u = 0, \tag{1.9}$$

with slip boundary condition $u_2 = 0$ at $x_2 = 0$ and $x_2 = 1$. The hydrostatic vorticity ω is defined by

$$\omega = \partial_2 u_1 \tag{1.10}$$

and solve

$$(\partial_t + u.\nabla)\omega = 0. \tag{1.11}$$

The rescaled Euler equations and the hydrostatic Euler equations can be rigorously compared by the following result

THEOREM 1.1. *Let (U,P) and (u,p) be smooth solutions of, respectively, the rescaled Euler equations and the hydrostatic Euler equations. Assume the following "local Rayleigh condition" :*

$$\partial^2_{22} u_1(t,x_1,x_2) \geq \alpha, \tag{1.12}$$

for some constant $\alpha > 0$. Suppose that there is a constant C_0 such that at time $t = 0$

$$\int_D (u_1 - U_1)^2 + \epsilon^2(u_2 - U_2)^2 + (\Omega - \omega)^2 \leq C_0 \epsilon^2. \tag{1.13}$$

Then, for all $t \in [0,T]$,

$$\int_D (u_1 - U_1)^2 + \epsilon^2(u_2 - U_2)^2 + (\Omega - \omega)^2 \leq C_T \epsilon^2 \tag{1.14}$$

where C_T depends only on α, u, C_0 and T, but not on ϵ nor (U,P).

The existence of (local) smooth solutions of the hydrostatic equations satisfying the local Rayleigh condition (1.12) was proven in [**Br**], under two additional restrictions : 1)

$$\int u_1(t=0,x_1,x_2)dx_2 = 0, \quad \forall x_1 \in \mathbb{R}/\mathbb{Z}$$

2) there are two constants a and b

$$\partial_2 u_1(t=0,x_1,x_2=0) = a \quad \partial_2 u_1(t=0,x_1,x_2=1) = b.$$

More precisely, it is shown in [**Br**] that the new variable $Z \in [0,1]$ implicitely defined by

$$(\partial_2 u_1)(t,x_1,Z(t,x_1,\theta)) = a + \theta(b-a),$$

for $\theta \in [0,1]$, $x_1 \in \mathbb{R}/\mathbb{Z}$, $t > 0$, solves a symmetric system of first order conservation laws in (t,x), labelled by $\theta \in [0,1]$.

It is fair to say that several systems of integro-differential equations had been investigated earlier with similar methods by Teshukov [**Te**], including the Benney equations [**Za**] which correspond to the hydrostatic limit of the Euler equations with free boundary and gravity effects.

Shortly after [**Br**], Grenier [**Gr**] proved the convergence of u toward U as ϵ tends to zero. The proof of convergence was a clever blend of i) the standard energy method for evolution PDEs (see, for instance [**Ch**], in the framework of fluid equations) and ii) the "Lyapunov-Casimir method" that Arnold used for establishing the stability of some stationary solutions of the Euler equations [**AK**]. Here, we give an elementary proof, even closer to Arnold's method, which does not require any PDE machinery based on energy estimates. To prove Theorem 1.1, we just need an explicit computation that generalizes, in some sense, Arnold's argument. More precisely, let us consider a smooth real function $F(t,x_1,w)$ defined for $t \in [0,T]$, $x_1 \in \mathbb{R}/\mathbb{Z}$, $w \in \mathbb{R}$ and denote by F', F'', its first and second derivatives with respect to w. Let us now introduce

$$H_F(U,\Omega) = \int_D \left(\frac{1}{2}(U_1^2 + \epsilon^2 U_2^2) + F(t,x_1,\Omega)\right) dx_1. \tag{1.15}$$

Observe that, if F does not depend explicitely on t and x_1, then H_F is conserved by any smooth solution (U,Ω) of the rescaled Euler equations. Let us now Taylor expand H_F about a given solution of the hydrostatic Euler equations (u,ω). We get a new functional

$$\begin{aligned} L(U,\Omega;u,\omega) &= H_F(U,\Omega) - H_F(u,\omega) - DH_F(u,\omega)(U-u,\Omega-\omega) \\ &= L_k(U,u) + L_c(\Omega,\omega), \end{aligned} \tag{1.16}$$

where

$$L_k(U,u) = \frac{1}{2}\int_D (u_1 - U_1)^2 + \epsilon^2(u_2 - U_2)^2, \tag{1.17}$$

$$L_c(\Omega,\omega) = \int_D F(t,x_1,\Omega) - F(t,x_1,\omega) - F'(t,x_1,\omega)(\Omega-\omega). \tag{1.18}$$

Notice that, if $F'' \geq \alpha$ for some constant $\alpha > 0$, then L controls the "distance" between u and U in the sense

$$L(U,\Omega;u,\omega) \geq \frac{1}{2}\int_D (u_1 - U_1)^2 + \epsilon^2(u_2 - U_2)^2 + \alpha(\omega - \Omega)^2.$$

This property will be crucial for our proof.

Remark. Estimating functionals like L is quite common in different fields of PDEs. Let us quote :
1) the "weak-strong" uniqueness principle, established by Dafermos for multidimensional systems of hyperbolic conservation laws admitting a convex entropy functionals [**BDLL**] and the related concept of dissipative solutions for the Euler equations [**Li**],
2) the "relative entropy method", frequently used for system of particles and rarefied gas dynamics, with

$$H(f) = \int f \log f, \quad L(f,g) = H(f) - H(g) - DH(g)(f-g) = \int f \log(f/g),$$

where f and g are probability densities and L is the so-called "relative entropy" [**Ya**], [**GLS**],
3) the "modulated energy" method used for different asymptotic problems [**Br2**], [**Ma**], [**Pu**], [**BMP**], [**JW**]...

We now analyse how functional L evolves in time :

PROPOSITION 1.1. *Let (U,Ω) be a smooth solution to the rescaled Euler equations and let (u,ω) be a smooth solution to the hydrostatic Euler equations. Then*

$$\frac{d}{dt}L(U,\Omega;u,\omega) = Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Z + r, \tag{1.19}$$

where

$$Q_1 = -\frac{1}{2}\int_D \partial_1 u_1 \{(u_1 - U_1)^2 - \epsilon^2 (u_2 - U_2)^2\}, \tag{1.20}$$

$$Q_2 = -\epsilon^2 \int_D \partial_2 u_1 (u_1 - U_1)(u_2 - U_2), \tag{1.21}$$

$$Q_3 = -\int_D F''(t,x_1,\omega)(u_1 - U_1)(\omega - \Omega)\partial_1 \omega, \tag{1.22}$$

$$Q_4 = \int_D (\partial_t F)(t,x_1,\Omega) - (\partial_t F)(t,x_1,\omega) - (\partial_t F')(t,x_1,\omega)(\Omega - \omega), \tag{1.23}$$

$$Q_5 = \int_D \{(\partial_1 F)(t,x_1,\Omega) - (\partial_1 F)(t,x_1,\omega) - (\partial_1 F')(t,x_1,\omega)(\Omega - \omega)\} U_1, \tag{1.24}$$

$$Z = -\int_D (F''(t,x_1,\omega)\partial_2 \omega - u_1)(u_2 - U_2)(\omega - \Omega), \tag{1.25}$$

$$r = \epsilon^2 \int_D (u_2 - U_2)(\partial_t + u_2 \partial_2) u_2. \tag{1.26}$$

Proof of Theorem 1.1. The theorem easily follows from Proposition 1.1. Let us first observe that the rescaled kinetic energy of U is preserved

$$\frac{d}{dt}\int_D \frac{1}{2}(U_1^2 + \epsilon^2 U_2^2) = 0.$$

At time $t = 0$, we have

$$\int_D \frac{1}{2}(U_1^2 + \epsilon^2 U_2^2) \leq \int_D (u_1^2 + \epsilon^2 u_2^2) + 2L_k(\Omega,\omega) \leq C$$

where C depends only on u and C_0. Thus, for all $t \in [0,T]$,

$$\int_D U_1^2 + \epsilon^2 U_2^2 \leq C. \tag{1.27}$$

Let us now assume $F'' \geq \alpha'$ for some constant $\alpha' > 0$. From this convexity assumption, we deduce the following lower bound for L defined by (1.16)

$$L \geq \frac{1}{2} \int_D (u_1 - U_1)^2 + \epsilon^2 (u_2 - U_2)^2 + \alpha'(\Omega - \omega)^2 \tag{1.28}$$

Then, we can easily get upper bounds for the different terms Q_1,...,Q_5 involved in the right hand side of identity (1.19) in terms of $L_c + L_k$. More precisely, using (1.27),

$$Q_1 + Q_2 + Q_3 + Q_4 + Q_5 \leq C \int_D (u_1 - U_1)^2 + \epsilon^2 (u_2 - U_2)^2 + (\Omega - \omega)^2.$$

(Here and below C denotes a generic constant depending only on the smoothness of u and F, or also on C_0.) Next, in (1.19), we have

$$r \leq C \int_D \epsilon^2 |U_2 - u_2| \leq C(\epsilon^2 + L_k)$$

(using assumption (1.13) and definition (1.17)). Thus, just by assuming $F'' \geq \alpha'$ for some constant $\alpha' > 0$, we have obtained

$$\frac{d}{dt} L \leq LC + Z + C\epsilon^2 \tag{1.29}$$

where Z, as defined in (1.19), is the only term that we are not able to bound without further assumptions. As a matter of fact, Z is a priori of order ϵ^{-1} with respect to L_k. However, this term can be cancelled under assumption (1.12). Indeed, for each fixed (t, x_1), $x_2 \to \omega(t, x_1, x_2)$ is strictly increasing, since $\omega = \partial_2 u_1$. So we can find a smooth function $F(t, x_1, w)$ for $t \in [0,T]$, $x_1 \in \mathbb{R}/\mathbb{Z}$, $w \in \mathbb{R}$, with $F'' \geq \alpha'$ for some constant $\alpha' > 0$, depending on u but not on U nor ϵ, so that

$$F''(t, x_1, \omega(t, x_1, x_2)) = \frac{u_1(t, x_1, x_2) - c}{\partial_{22}^2 u_1(t, x_1, x_2)}, \tag{1.30}$$

where c is a constant chosen strictly below the infimum of u_1. (More precisely, (1.30) implicitely defines $F''(t, x_1, w)$ for

$$w \in [\omega(t, x_1, 0), \omega(t, x_1, 1)].$$

Then, we can find a smooth extension of F'' for $w \in \mathbb{R}$, bounded from below by some constant $\alpha' > 0$ and we can define F by integrating F'' twice in w.) Notice that the smoothness of F depends only on u. Thanks to (1.30), we are now able to cancel Z in (1.29). It follows that

$$L \leq \exp(Ct)\epsilon^2,$$

where C depends only on u and C_0. Then, (1.14) immediately follows from the lower bound (1.28) on L. The proof of Theorem 1.1 is now complete.

Remarks. 1) In the special case when u depends only on x_2 (in which case, u is a trivial solution to both the rescaled and the hydrostatic Euler equations, with a null pressure), $F(t, x_1, w)$ does not depend on t and x_1, all terms Q_1,...,Q_5 and r vanish. Thus, after cancelling Z, we obtain

$$\frac{d}{dt}L = 0,$$

and recover Arnold's argument which shows the stability of u [**AK**].
2) As in Arnold's stablity analysis and in Grenier's convergence analysis, the Rayleigh condition can be slightly relaxed, by assuming the existence of a speed $c(t, x_1)$ such that

$$\frac{\partial_{22}^2 u_1}{u_1 - c} > 0.$$

Proof of Proposition 1.1. Let us first consider $\frac{d}{dt}L_k(U, u)$ and remind the following elementary identity

$$\frac{d}{dt}\frac{1}{2}\int_D |v - V|^2 = -\int_D \sum_{i,j} \partial_j v_i (v_i - V_i)(v_j - V_j) + \int \sum_i (v_i - V_i)(\partial_t + v.\nabla)v_i, \tag{1.31}$$

which is valid for all smooth solution V of the Euler equations and all test divergence free vector field v. (This identity is the starting point of the concept of dissipative solutions introduced by Lions [**Li**].) After rescaling this identity (or, directly from the rescaled equations), we get for u and U smooth solutions of the respectively hydrostatic and rescaled Euler equations

$$\frac{d}{dt}L_k(U, u) = A_1 + A_2 + A_3 + A_4 + A_5,$$

where

$$A_1 = -\int_D (u_1 - U_1)^2 \partial_1 u_1$$

$$A_2 = -\int_D (u_1 - U_1)(u_2 - U_2)\partial_2 u_1$$

$$A_3 = -\epsilon^2 \int_D (u_1 - U_1)(u_2 - U_2)\partial_1 u_2$$

$$A_4 = -\epsilon^2 \int_D (u_2 - U_2)^2 \partial_2 u_2 = \epsilon^2 \int_D (u_2 - U_2)^2 \partial_1 u_1$$

$$A_5 = \epsilon^2 \int_D (u_2 - U_2)(\partial_t + u \cdot \nabla)u_2.$$

(To get A_5, we have used that u solves the hydrostatic equations.) Notice that $A_3 = Q_2$, according to definition (1.21). After integrating by part in x_2, we get $A_2 = A_6 + A_7$, where

$$A_6 = \int_D u_1(u_1 - U_1)\partial_2(u_2 - U_2)$$

and

$$A_7 = \int_D u_1(u_2 - U_2)\partial_2(u_1 - U_1).$$

We have

$$A_6 = -\int_D u_1(u_1 - U_1)\partial_1(u_1 - U_1) = \frac{1}{2}\int_D (u_1 - U_1)^2\partial_1 u_1$$

(using that $u - U$ is divergence free). We have $A_7 = A_8 + A_9 + A_{10}$, where

$$A_8 = \int_D u_1(u_2 - U_2)(\partial_2 u_1 - \partial_2 U_1 + \epsilon^2\partial_1 U_2),$$

$$A_9 = \epsilon^2\int_D u_1(u_2 - U_2)\partial_1(u_2 - U_2) = -\epsilon^2\frac{1}{2}\int_D (u_2 - U_2)^2\partial_1 u_1$$

and

$$A_{10} = -\epsilon^2\int_D u_1(u_2 - U_2)\partial_1 u_2.$$

Introducing

$$Y = \int_D u_1(u_2 - U_2)(\omega - \Omega), \tag{1.32}$$

where ω and Ω are defined by (1.5), (1.10), we get $A_8 = Y$. Combining these identities, we have a new expression for A_2 :

$$A_2 = \frac{1}{2}\int_D ((u_1 - U_1)^2 - \epsilon^2(u_2 - U_2)^2)\partial_1 u_1 + Y - \epsilon^2\int_D (u_2 - U_2)u_1\partial_1 u_2.$$

Thus, we have $A_1 + A_2 + A_3 + A_4 + A_5 = Q_1 + Q_2 + Y + r$, where Q_1, Q_2, r and Y are defined by (1.20), (1.21), (1.26), (1.32), which leads to :

$$\frac{d}{dt}L_k(U,u) = Q_1 + Q_2 + Y + r. \tag{1.33}$$

Let us now compute $\frac{d}{dt}L_c(\Omega,\omega)$. We immediately get from definitions (1.18) and (1.23)

$$\frac{d}{dt}L_c(\Omega,\omega) = Q_4 + I,$$

where,

$$I = \int_D F'(t,x_1,\Omega)\partial_t\Omega - \int_D F'(t,x_1,\omega)\partial_t\omega$$
$$-\int_D F''(t,x_1,\omega)\partial_t\omega(\Omega - \omega) - \int_D F'(t,x_1,\omega)(\partial_t\Omega - \partial_t\omega).$$

So, using equations (1.6), (1.11),

$$I = I_1 + I_2 + I_3,$$

where

$$I_1 = -\int_D F'(t,x_1,\Omega)U.\nabla\Omega,$$

$$I_2 = \int_D F''(t,x_1,\omega)(u.\nabla\omega)(\Omega - \omega),$$

$$I_3 = \int_D F'(t,x_1,\omega)U.\nabla\Omega.$$

We have

$$I_1 = -\int_D \nabla(F(t,x_1,\Omega)).U + \int_D (\partial_1 F)(t,x_1,\Omega)U_1 = \int_D (\partial_1 F)(t,x_1,\Omega)U_1$$

(since U is divergence free). Next,

$$I_2 = I_{2a} + I_{2b}$$

where

$$I_{2a} = \int_D F''(t, x_1, \omega) U.\nabla\omega(\Omega - \omega),$$

$$I_{2b} = \int_D F''(t, x_1, \omega)(u - U).\nabla\omega(\Omega - \omega) = Q_3 + Z - Y$$

(Q_3, Y, Z being defined by (1.22), (1.25), (1.32)) Since

$$I_3 = -\int_D \nabla(F'(t, x_1, \omega)).U\Omega$$

$$= -\int_D F''(t, x_1, \omega)\nabla\omega.U\Omega - \int_D (\partial_1 F')(t, x_1, \omega)U_1\Omega,$$

we deduce

$$I_2 + I_3 = Q_3 + Z - Y - \int_D F''(t, x_1, \omega)U.\nabla\omega\omega - \int_D (\partial_1 F')(t, x_1, \omega)U_1\Omega$$

$$= Q_3 + Z - Y - \int_D \nabla(F'(t, x_1, \omega)\omega).U + \int_D F'(t, x_1, \omega)\nabla\omega.U$$

$$+ \int_D (\partial_1 F')(t, x_1, \omega)\omega U_1 - \int_D (\partial_1 F')(t, x_1, \omega)U_1\Omega$$

$$= Q_3 + Z - Y + \int_D \nabla(F(t, x_1, \omega)).U - \int_D (\partial_1 F)(t, x_1, \omega))U_1$$

$$- \int_D (\partial_1 F')(t, x_1, \omega)(\Omega - \omega)U_1.$$

Thus,

$$I = I_1 + I_2 + I_3 = Q_3 + Z - Y - \int_D (\partial_1 F)(t, x_1, \omega))U_1$$

$$- \int_D (\partial_1 F')(t, x_1, \omega)(\Omega - \omega)U_1 + \int_D (\partial_1 F)(t, x_1, \Omega)U_1 = Q_3 + Z - Y + Q_5,$$

according to definition (1.24) and, therefore, we have obtained

$$\frac{d}{dt}L_c(\Omega, \omega) = Q_3 + Z - Y + Q_4 + Q_5.$$

Combined with (1.33), this identity shows that

$$\frac{d}{dt}(L_k(U, u) + L_c(\Omega, \omega)) = Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Z + r$$

as expected. The proof of Proposition 1.1 is now complete.

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Added in Proof. As suggested by both Nader Masmoudi and a referee, the error estimate obtained in Theorem 1.1 can be improved after integrating by part the error term r defined by (1.26). More precisely, introducing

$$q(t,x_1,x_2) = \frac{u_2^2}{2}(t,x_1,x_2) + \partial_t \int_0^{x_2} u_2(t,x_1,y)dy,$$

we have

$$r = \int_D \epsilon^2(u_2 - U_2)\partial_2 q = -\int_D \epsilon^2(u_1 - U_1)\partial_1 q,$$

using that $u - U$ is divergence free. Thus we can substitute ϵ^β for ϵ^2 in the right hand side of both (1.13) and (1.14), for all $\beta \in [2,4]$.

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