

# RIGOROUS DERIVATION OF THE X-Z SEMIGEOSTROPHIC EQUATIONS <sup>\*</sup>

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**Abstract.** We prove that smooth solutions of the semigeostrophic equations in the incompressible  $x-z$  setting can be derived from the Navier-Stokes equations with the Boussinesq approximation.

**Key words.** Atmospheric sciences, fluid mechanics, asymptotic analysis

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## 1. Introduction

We consider the Navier-Stokes equations with the Boussinesq approximation (NSB):

$$\epsilon(\partial_t v + (v \cdot \nabla)v) + \alpha K v + \nabla p = y, \quad \nabla \cdot v = 0, \quad (1.1)$$

$$\partial_t y + (v \cdot \nabla)y = G(x, y), \quad (1.2)$$

where  $x \in D$ ,  $D$  being a smooth bounded domain in  $R^d$  ( $d=2,3$ ),  $v = v(t, x) \in R^d$  is the velocity field,  $p = p(t, x)$  is the pressure field,  $y = y(t, x) \in R^d$  is a vector-valued forcing term,  $G(x, y)$  is a given smooth vector-valued source term  $D \times R^d \rightarrow R^d$ ,  $\epsilon, \alpha > 0$  are scaling factors and  $K$  is the linear dissipative operator  $Kv = -\Delta v$ . We assume that the fluid sticks to the boundary:  $v = 0$  along  $\partial D$ .

We now consider the formal limit of these equations obtained by dropping the inertia term and the dissipative term (i.e. setting  $\epsilon = \alpha = 0$ ) in the NSB equations.

$$\nabla p = y, \quad \nabla \cdot v = 0, \quad v // \partial D, \quad (1.3)$$

$$\partial_t y + (v \cdot \nabla)y = G(x, y). \quad (1.4)$$

We are going to show that these equations can be justified under a strong uniform convexity assumption on the pressure field  $p$ . The situation of interest in this paper is the case when  $d=2$  and the source term

$$G(x, y) = (x_2, y_1 - x_1). \quad (1.5)$$

Then (1.3-1.4) are the semigeostrophic Eady model equations in the special incompressible “ $x-z$ ” situation. By  $x-z$ , we mean that  $D$  is part of a vertical section, the second coordinate  $x_2$  of each point  $x = (x_1, x_2) \in D$  being the vertical one. The source term in (1.5) represents the effect of the missing third dimension. In this identification,  $y$  represents the effects of rotation and stratification, and the relation  $\nabla p = y$  in (1.3) expresses geostrophic and hydrostatic balance.

The semigeostrophic model was considered by Hoskins [Ho] to model front formations in atmospheric sciences. The Eady model is defined in chapter 6 of [Cu], and models a quasi-periodic evolution in which fronts form and decay. There has been

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a lot of interest in these equations (see for instance [CNP, BB, CG, CF, Cu]), due to their beautiful geometric structure and their deep links with the Monge-Ampère equation and optimal transport theory [CM, Br1, Br2, Vi]. The rigorous derivation of the full 3 dimensional SG equations is still a challenging problem. The present short note is just the first step toward this goal.

**2. Motivation for a convexity assumption** In their study of the SG equations, Cullen and Purser have introduced a convexity assumption on the pressure field  $p$ , based on a combination of physical and mathematical arguments. Convexity is also natural in the case of the general equations (1.3-1.4), independently of the choice of the source term  $G$ , for the following reasons. At first glance, these equations look strange since there is no evolution equation for  $v$ . However,  $y$  is constrained to be a gradient. Therefore,  $v$  can be seen as a kind of Lagrange multiplier for this constraint. (Vaguely speaking, due to the presence of a source term, in order to stay a gradient, the field  $y$  needs to be continuously rearranged in a volume-preserving fashion under the action of a time-dependent divergence-free vector field  $v$ .) As a matter of fact, it is (formally) very easy to get an equation for  $v$ , once  $y = \nabla p$  is known. To do that, let us start with the 2 dimensional case and write

$$y(t, x) = (\partial_1 p, \partial_2 p)(t, x_1, x_2), \quad v(t, x) = (-\partial_2 \psi, \partial_1 \psi)(t, x_1, x_2)$$

(at least locally), where  $\psi$  is a “stream-function”. Then, let us “curl” equation (1.4) and get:

$$-\partial_{11}^2 p \partial_{22}^2 \psi + 2\partial_{12}^2 p \partial_{12}^2 \psi - \partial_{22}^2 p \partial_{11}^2 \psi = \partial_1(G_2(x, \nabla p)) - \partial_2(G_1(x, \nabla p)). \quad (2.1)$$

This is a linear second order elliptic equation in  $\psi$ , whenever  $p$  is a given strictly uniformly convex (or concave) function of  $x$ , i.e. when  $D_x^2 p > 0$  -in the sense of symmetric matrices- (or  $< 0$ ). In three space dimensions, we get some “magnetostatic” version of equation (2.1). Indeed, since  $v$  is divergence-free, we can (at least locally) write  $v = \nabla \times A$  for some “potential vector”  $A = A(t, x) \in \mathbb{R}^3$ , that we may assume to be itself divergence-free. Then, by curling equation (1.4), we get a linear system for  $A$  when  $p$  is convex, namely:

$$\nabla \times (M(t, x) \nabla \times A) = \nabla \times (G(x, \nabla p)). \quad (2.2)$$

This system is elliptic whenever the symmetric matrix  $M = D_x^2 p(t, x)$  is uniformly positive and bounded, which means that  $p$  is convex in a strong sense. In higher dimension,  $v$  should be viewed as a  $d-1$  form and  $p$  as a zero form. The divergence free condition (locally) means that  $v = dA$ , where  $A$  is a  $d-2$  form. Then, again taking the curl of equation (1.4), we get the multidimensional generalization of system (2.1):  $d(M(t, x) * dA) = d(G(x, dp))$  (where  $*$  denotes Hodge duality and  $M = D_x^2 p$ ) which, again, is an elliptic system in  $A$  when  $D_x^2 p$  is uniformly bounded and positive. Thus we see that requiring  $p$  to be convex is a natural solvability condition for equations (1.3-1.4).

### 3. Rigorous derivation from the Navier-Stokes equations

The generalized Cullen-Purser convexity condition plays a crucial role in the rigorous derivation of equations (1.3-1.4) from the NSB equations.

**THEOREM 3.1.** *Let  $D$  be a smooth bounded convex domain. Assume  $G$  to be smooth with bounded derivatives up to second order. Let  $(y^\varepsilon, v^\varepsilon, p^\varepsilon)$  be a Leray-type solution to the NSB equations (1.1, 1.2), where  $K = -\Delta$ , with  $\alpha = O(\varepsilon)$ . Let  $(y = \nabla p, v)$  be*

a smooth solution to equations (1.3,1.4) on a given finite time interval  $[0, T]$ . We assume  $p(t, x)$  to have a smooth convex extension for all  $x \in \mathbb{R}^d$  so that its Legendre transform

$$p^*(t, y) = \sup_{x \in \mathbb{R}^d} x \cdot y - p(t, x) \quad (3.1)$$

is also smooth for  $y \in \mathbb{R}^d$  with Hessian  $D_y^2 p^*(t, y)$  bounded away from zero and  $+\infty$ . Then, the  $L^2$  distance between  $y^\varepsilon$  and  $y$  stays uniformly of order  $\sqrt{\varepsilon}$  as  $\varepsilon$  goes to zero, uniformly in  $t \in [0, T]$ , provided it does at  $t=0$  and the initial velocity  $v^\varepsilon(t=0, x)$  stays uniformly bounded in  $L^2$ .

Notice that the theorem is meaningful, since the local existence of smooth solutions has been proven by Loeper [Lo] (in the SG case) at least for periodic boundary conditions, provided that that  $y(0, x) - x$  is not too large in some appropriate sense.

**Proof.** For the convergence, we use a relative entropy trick quite similar to the one used by the author for the hydrostatic limit of the 2D Euler equations in a thin domain [Br3]. We introduce the so-called Bregman function (or relative entropy) attached to  $p^*$ :

$$\eta_{p^*}(t, z, z') = p^*(t, z') - p^*(t, z) - (\nabla p^*)(t, z) \cdot (z' - z) \sim |z' - z|^2 \quad (3.2)$$

and the related functional:

$$e(t) = \int_D \left( \varepsilon \frac{|v^\varepsilon(t, x) - v(t, x)|^2}{2} + \eta_{p^*}(t, y(t, x), y^\varepsilon(t, x)) \right) dx. \quad (3.3)$$

Given a weak solution  $(y^\varepsilon, v^\varepsilon)$  to the NSB equations (1.1,1.2) and a solution  $y$  of (1.3-1.4), we want to get an estimate of form:

$$\frac{d}{dt}(e(t) + O(\varepsilon)) \leq (e(t) + O(\varepsilon))c, \quad (3.4)$$

where  $c$  depends only on the limit solution  $(y, v)$  on a fixed finite time interval  $[0, T]$  on which  $(y, v)$  is smooth. From this estimate (3.4), we immediately get that  $y - y^\varepsilon$  is of order  $O(\sqrt{\varepsilon})$  in  $L^\infty([0, T], L^2(D))$ . So, we are left with proving (3.4). To save time, we do calculations just as if the Leray solutions were smooth solutions. Let us compute

$$I = I_1 + I_2 + I_3 + I_4,$$

$$I_1 = \frac{d}{dt} \int_D p^*(t, y^\varepsilon(t, x)) dx,$$

$$I_2 = - \frac{d}{dt} \int_D p^*(t, y(t, x)) dx,$$

$$I_3 = - \frac{d}{dt} \int_D (\nabla p^*(t, y(t, x))) \cdot y^\varepsilon(t, x) dx,$$

$$I_4 = \frac{d}{dt} \int_D (\nabla p^*(t, y(t, x))) \cdot y(t, x) dx.$$

We first get

$$I_1 = \int_D [(\partial_t p^*)(t, y^\varepsilon(t, x)) + (\nabla p^*)(t, y^\varepsilon(t, x)) \cdot G(x, y^\varepsilon(t, x))] dx,$$

(using that  $v^\varepsilon$  is divergence free and parallel to  $\partial D$ ). Similarly,

$$I_2 = - \int_D [(\partial_t p^*)(t, y(t, x)) + (\nabla p^*)(t, y(t, x)) \cdot G(x, y(t, x))] dx.$$

Next,

$$\begin{aligned} I_3 = & - \int_D [(\partial_t \nabla p^*)(t, y(t, x)) \cdot y^\varepsilon(t, x) + (D_y^2 p^*)(t, y(t, x))(\partial_t y(t, x), y^\varepsilon(t, x))] dx \\ & - \int_D (\nabla p^*)(t, y(t, x)) \cdot G(x, y^\varepsilon(t, x)) dx + I_5, \end{aligned}$$

where

$$\begin{aligned} I_5 &= \int_D x \cdot (v^\varepsilon(t, x) \cdot \nabla) y^\varepsilon(t, x) dx \\ &= - \int_D v^\varepsilon(t, x) \cdot y^\varepsilon(t, x) dx, \end{aligned}$$

(where we have used, for the two last lines, that  $(\nabla p^*)(t, y(t, x)) = x$ , which follows from Legendre duality). Since  $v^\varepsilon$  solves the NSB equations, we find

$$\begin{aligned} I_5 &= - \int_D [\varepsilon(\partial_t + v^\varepsilon \cdot \nabla) v^\varepsilon + \nabla p^\varepsilon + \alpha K v^\varepsilon] \cdot v^\varepsilon dx \\ &= - \frac{\varepsilon d}{2dt} \int_D |v^\varepsilon|^2 dx - \int_D v^\varepsilon \cdot \alpha K v^\varepsilon dx. \end{aligned}$$

Similarly

$$\begin{aligned} I_4 &= \int_D [(\partial_t \nabla p^*)(t, y(t, x)) \cdot y(t, x) + (D_y^2 p^*)(t, y(t, x))(\partial_t y(t, x), y(t, x))] dx + \\ &+ \int_D \nabla p^*(t, y(t, x)) \cdot G(x, y(t, x)) dx. \end{aligned}$$

Collecting all terms, we get

$$I = I_5 + I_6 + I_7 + I_8 + I_9,$$

where

$$I_6 = \int_D \eta_{\partial_t p^*}(t, y(t, x), y^\varepsilon(t, x)) dx,$$

(which involves the Bregman functional associated with  $\partial_t p^*$  and therefore is bounded by  $e(t)c$  where  $c$  is a constant depending only on the limit solutions  $y = \nabla p$ ),

$$\begin{aligned} I_7 &= - \int_D [(\nabla p^*)(t, y) - (\nabla p^*)(t, y^\varepsilon)] \cdot G(x, y^\varepsilon) dx, \\ I_8 &= \int (D_y^2 p^*)(t, y)(G(x, y), y - y^\varepsilon) dx, \\ I_9 &= \int (D_y^2 p^*)(t, y)(\partial_t y - G(x, y), y - y^\varepsilon) dx, \\ &= \int (D_y^2 p^*)(t, y)((v \cdot \nabla)y, y^\varepsilon - y) dx. \end{aligned}$$

We easily see that

$$\begin{aligned} I_7 + I_8 &= \int_D \eta_{\nabla p^*}(t, y(t, x), y^\varepsilon(t, x)) \cdot G(x, y) dx \\ &+ \int_D [(\nabla p^*)(t, y) - (\nabla p^*)(t, y^\varepsilon)] \cdot (G(x, y) - G(x, y^\varepsilon)) dx, \end{aligned}$$

(which is again bounded by  $e(t)c$  where  $c$  is a constant depending only on the limit solutions  $y = \nabla p$ ). Let us finally consider the most delicate term  $I_9$ . We can write  $I_9$  in index notations as:

$$\begin{aligned} I_9 &= \int \sum_{ijk} \partial_{ij}^2 p^*(t, y) v_k \partial_k y_i (y^\varepsilon - y)_j, \\ &= \int \sum_{ijk} \delta_{jk} v_k (y^\varepsilon - y)_j = \int v \cdot (y^\varepsilon - y), \end{aligned}$$

(indeed,  $p^*$  is the Legendre transform of  $p$  and  $y = \nabla p$ , thus  $D^2 p^*(y) D y = D^2 p^*(\nabla p) D^2 p = Id$ )

$$= \int v \cdot y^\varepsilon,$$

(since  $y$  is a gradient and  $v$  is divergence free and parallel to  $\partial D$ )

$$= \int v \cdot (\varepsilon(\partial_t + v^\varepsilon \cdot \nabla)v^\varepsilon + \alpha K v^\varepsilon),$$

(using the NSB equations)

$$= J_1 + J_2,$$

where

$$J_1 = \frac{d}{dt} \varepsilon \int v^\varepsilon \cdot v,$$

and

$$|J_2| \leq \varepsilon \left( \int |v^\varepsilon|^2 + 1 \right) c \leq \varepsilon \left( \int |v^\varepsilon - v|^2 + 1 \right) c \leq (e(t) + \varepsilon) c,$$

where  $c$  are constants only depending on the limit solution  $v$ . Thus, again collecting all terms, and using that

$$I_5 = -\frac{\varepsilon d}{2dt} \int_D |v^\varepsilon|^2 dx - \alpha \int v^\varepsilon \cdot K v^\varepsilon dx,$$

we have obtained

$$I + \frac{d}{dt} \left( \frac{\varepsilon}{2} \int |v^\varepsilon - v|^2 + O(\varepsilon) \right) + \alpha \int v^\varepsilon \cdot K v^\varepsilon dx \leq (e(t) + O(\varepsilon)) c,$$

which leads to the desired inequality (3.4) and completes the proof.

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