A note on deformations of 2D fluid motions using 3D Born-Infeld equations

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1 Abstract

Classical fluid motions can be described either by time dependent diffeomorphisms (Lagrangian description) or by the corresponding generating vector fields (Eulerian description). We are interested in interpolating two such fluid motions. For both analytic and numerical purposes, we would like the corresponding interpolating vector fields to evolve according to some hyperbolic evolution PDEs. In the 2D case, it turns out that a convenient set of such PDEs can be deduced from the Born-Infeld (BI) theory of Electromagnetism. The 3D BI equations were originally designed as a nonlinear correction to the linear Maxwell equations allowing finite electric fields for point charges. They depend on a parameter λ. As λ goes to infinity, the classical Maxwell equations are recovered. It turns out that in the opposite case, as λ goes to zero, the BI equations provide a solution to our problem. These equations, as $\lambda = 0$, also describe classical strings (extremal surfaces) evolving in the 4D Minkowski space-time. In their static version, they can be interpreted in the framework of optimal transportation theory.

2 Deformation fields

In order to describe a deformation (or interpolation) process between two given fluid motions, it is convenient to introduce an interpolation parameter denoted by s and ranging from 0 to 1, so that the given fluid motions respectively correspond to $s = 0$ and $s = 1$. For each value of s, the interpolated fluid motion can be described in the Eulerian fashion through its density field $\rho(t, x, s) \geq 0$ and its velocity field $v(t, x, s)$ where t is the time variable

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valued in a fixed time interval $[0,T]$ and x is the space variable valued in some manifold D, typically the flat torus $\mathbb{R}^d/\mathbb{Z}^d$. The continuity equation links ρ and v through

$$\partial_t\rho + \nabla\cdot(\rho v) = 0,$$

where $\nabla = (\partial_{x_1}, ..., \partial_{x_d})$ and $\cdot$ is the inner product in $\mathbb{R}^d$. We can alternately use a Lagrangian description of the interpolated fluid through the time dependent family of diffeomorphisms $x \to X(t,x,s)$ which gives the location at time t of a particle advected by the fluid from initial position $x \in D$. Recall that X and v are linked by the ordinary differential equation

$$\partial_t X(t,x,s) = v(t, X(t,x,s), s), \quad X(0,x,s) = x,$$

where s and x are parameters. Notice that the density field ρ can be expressed in terms of the jacobian determinant of X. Namely,

$$\rho(t, X(t,x,s), s) = \frac{1}{det(\partial_x X(t,x,s))}\ .$$

Trivial interpolations are obtained by setting, first in the Eulerian style, either

$$v(t,x,s) = v(t,x,0)(1-s) + sv(t,x,1),$$

or

$$\rho(t,x,s) = \rho(t,x,0)(1-s) + s\rho(t,x,1),$$
$$(\rho v)(t,x,s) = (\rho v)(t,x,0)(1-s) + s(\rho v)(t,x,1),$$

and, next, in the Lagrangian style,

$$X(t,x,s) = X(t,x,0)(1-s) + sX(t,x,1).$$

Observe that these three trivial interpolations are not compatible. A more general setting involves a new vector field e defined by the ordinary differential equation

$$\partial_s X(t,x,s) = e(t, X(t,x,s), s),$$

where t and x are just parameters. The corresponding continuity equation reads

$$\partial_s\rho + \nabla\cdot(\rho e) = 0.$$

There is a compatibility condition between e and v obtained by setting $\partial_{st}X = \partial_{ts}X$ in the two ODEs. We get

$$\partial_t e + (v\cdot\nabla)e = \partial_s v + (e\cdot\nabla)v,$$

or, equivalently

$$\partial_t(\rho e) - \partial_s(\rho v) + \nabla\cdot(\rho(e\otimes v - e\otimes v)) = 0.$$

Provided the fields are smooth enough, these compatibility conditions enable us to fully recover diffeomorphisms X by integrating one of the two ODEs with initial condition either at $t=0$ or at $s=0$.
In the two dimensional setting $d=2$, all the equations we have used so far can be nicely recast in a $3D$ setting by defining x_3 to be the interpolation variable and using (abusive) notations

$$x = (x_1, x_2, s), \quad \nabla = (\partial_1, \partial_2, \partial_s),$$

$$B = (\rho e_1, \rho e_2, \rho), \quad E = (-\rho v_2, \rho v_1, \rho(v_2 e_1 - v_1 e_2)).$$

The resulting system of equations now is

$$\partial_t B + \nabla\times E = 0, \quad \nabla\cdot B = 0, \tag{1}$$

which is very reminiscent of the Maxwell equations, combined with the non-linear constraint

$$E\cdot B = 0.$$

Of course, these PDEs do not define a complete set of evolution equations determining the fields E, B when they are known at time 0. This is not surprising since there are many ways of interpolating data as seen above. A natural idea would be to use the classical 3D Maxwell equations to get a closed system. Namely, we would add equations

$$\partial_t E = \nabla\times B, \quad \nabla\cdot E = 0, \tag{2}$$

supplemented by initial conditions at time $t=0$ and boundary conditions at $x_3=0$ and $x_3=1$ prescribing the values of $B\cdot n$ and $E\times n$, where n denotes the normal vector $n=(0,0,1)$. However, these equations are not compatible with the nonlinear constraint $E\cdot B=0$ and, therefore, do not provide a satisfactory set of hyperbolic equations to interpolate fluid motions.

3 The Born-Infeld equations

Max Born and Leopold Infeld designed in the 1930' [BI] a very interesting nonlinear theory for Electromagnetism which turns out to be well suited to our purpose. Generally speaking, nonlinear generalizations of Maxwell equations can be obtained by varying a Lagrangian of form

$$\int L(E,B)dxdt,$$

with respect to (E,B) subject to equation (1), where the 'Lagrangian density' L has to be defined and will be supposed to be (strictly) convex with respect to E. The resulting system of equations combines (1) and

$$\partial_t D = \nabla\times H, \quad \nabla\cdot D = 0, \quad D = -\frac{\partial}{\partial E}L(E,B), \quad H = \frac{\partial}{\partial B}L(E,B). \quad (3)$$

Notice that the linear Maxwell equations correspond to the special choice

$$L(E,B) = \frac{1}{2}(B^2 - E^2).$$

The resulting equations can be explicitely written as evolution equations in the variable (D,B), after introducing the 'energy (or Hamiltonian) density'

$$h(D,B) = \sup_E E\cdot D - L(E,B),$$

and setting

$$H = \frac{\partial h}{\partial B}(D,B), \quad E = \frac{\partial h}{\partial D}(D,B).$$

Born and Infeld singled out the one-parameter family

$$L_\lambda(E,B) = -\sqrt{\lambda^2 + B^2 - E^2 - \frac{(B\cdot E)^2}{\lambda^2}},$$

where the parameter $\lambda > 0$ can be interpreted as a 'maximal field intensity'. The corresponding Hamiltonian density is

$$h_\lambda(D,B) = \sqrt{(\lambda^2 + B^2)(1 + D^2) - (B\cdot D)^2}.$$

Born, Infeld and their followers found out that this (one-parameter) Lagrangian density $L = L_\lambda$ is the only one to enjoy the following properties :
1) L depends only on $B^2 - E^2$ and $B\cdot E$ which are the natural invariants for an electromagnetic theory.
2) The corresponding nonlinear Maxwell equations are hyperbolic and linearly degenerate.

3) The equations are 'self-dual', i.e. unaffected by the change of unknown $(E, B) \rightarrow (-B, E)$

We refer to the works of G. Boillat [BDLL] for a mathematical analysis of the BI equations (see also [Se], in the one-dimensional case, and [Br2]), as well as to G. Gibbons [Gi] for their relevance in high energy Physics. Clearly, the classical Maxwell theory can be recovered as λ goes to $+\infty$, since

$$\lim_{\lambda \rightarrow \infty} (L_\lambda(E, B) + \lambda)\lambda^2 = \frac{E^2 - B^2}{2}.$$

In sharp contrast, the equations we are interested in are exactly on the opposite side, as λ goes to zero! Indeed, because of the definition of L_λ, this limit precisely forces $B \cdot E$ to vanish! Of course, the definition of L_0 is quite singular with value $+\infty$ unless $E^2 \leq B^2$ and $B \cdot E = 0$, in which case :

$$L_0(E, B) = -\sqrt{B^2 - E^2}.$$

The corresponding hamiltonian density looks much nicer and is given by

$$h_0(D, B) = \sqrt{B^2(1 + D^2) - (B \cdot D)^2}.$$

Thus, the Born-Infeld equations, in the limit case $\lambda = 0$ (which are also known as Tachyonic Condensate equations in high energy Physics [Gi] and are hyperbolic only in a weak sense), provide a satisfactory set of evolutions equations for the interpolation fields, at least for two space dimensions.

4 Link with the optimal transportation theory

Going back to our original interpolation problem, let us consider the degenerate 'static' situation, where there is no time dependence and, consistently, the 'velocity' field v can be ignored. If we use the Born-Infeld model with $\lambda = 0$ for our 2 space variable interpolation problem, we are left with the following variational problem : maximize

$$\int L_0(0, B(x))dx = -\int |B(x)|dx,$$

where $B = B(x)$ is subject to $\nabla \cdot B = 0$. (The boundary conditions being periodic in x_1, x_2, with B_3 prescribed at $x_3 = 0$ and $x_3 = 1$.) This is equivalent to look for a scalar potential $\phi(x)$ such that

$$\nabla \phi = \frac{B}{|B|}.$$

Going back to earlier notations $x = (x_1, x_2, s)$, $B = (\rho e_1, \rho e_2, \rho)$, we find

$$\partial_1 \phi = \frac{e_1}{\sqrt{1+e^2}}, \quad \partial_2 \phi = \frac{e_2}{\sqrt{1+e^2}}, \quad \partial_s \phi = \frac{1}{\sqrt{1+e^2}},$$

or, equivalently

$$e_1 = \frac{\partial_1 \phi}{\sqrt{1-|\nabla\phi|^2}}, \quad e_2 = \frac{\partial_2 \phi}{\sqrt{1-|\nabla\phi|^2}},$$

$$\partial_s \phi = \sqrt{1-|\nabla\phi|^2}.$$

This has to be supplemented by the divergence free condition $\nabla \cdot B = 0$, namely :

$$\partial_s \rho + \nabla \cdot (\rho \frac{\nabla \phi}{\sqrt{1-|\nabla\phi|^2}}) = 0,$$

and the boundary conditions which amount to prescribing ρ at $s = 0$ and $s = 1$ (with periodicity in (x_1, x_2)). So, we have just recovered the optimal transportation equations corresponding to the transportation cost

$$c(x,y) = \sqrt{1+|x-y|^2},$$

in their 'Benamou-Brenier' formulation (see [BB] and Villani's textbook [Vi]). Notice that this cost function is a natural interpolant between the original Monge cost $c(x,y) = |x-y|$ and the quadratic cost $c(x,y) = |x-y|^2/2$ related to the Monge-Ampère equation [Br0], [Vi].

5 Link with extremal surfaces

Let us consider the Born-Infeld Lagrangian in the limit case $\lambda = 0$ and introduce back the interpolating flow $X(t,x,s)$ of the first section. We have

$$B = (\rho e_1, \rho e_2, \rho)$$

and, by definition of e and ρ with respect to X, we get, at $X = X(t,x,s)$,

$$B(t,X,s) = (\partial_s X_1, \partial_s X_2, 1)\rho(t,X,s).$$

Similarly,

$$E(t,X,s) = (-\partial_t X_2, \partial_t X_1, \partial_t X_2 \partial_s X_1 - \partial_s X_2 \partial_t X_1)\rho(t,X,s).$$

We know that ρ is related to the jacobian determinant of the transform $x \to X$ through :

$$\rho(t, X(t,x,s), s) det(\partial_x X(t,x,s)) = 1.$$

Thus, the Born-Infeld Action is given (in the limit $\lambda = 0$) by

$$\int L_0(E,B)dxdt =$$

$$-\int \sqrt{B^2(t,X,s) - E^2(t,X,s)}\ det(\partial_x X(t,x,s))dtdsdx_1dx_2$$

(after performing the change of variable $x \to X$)

$$= -\int \sqrt{1 + \partial_s X^2 - \partial_t X^2 - (\partial_t X \times \partial_s X)^2}\ \ dtdsdx$$

$$= -\int \sqrt{(1 + \partial_s X^2)(1 - \partial_t X^2) + (\partial_t X \cdot \partial_s X)^2}\ \ dtdsdx$$

$$= \int A[X(\cdot,\cdot,x_1,x_2)]dx_1dx_2.$$

Here, $A[Y]$ denotes the area of the 2 surface

$$(t,s) \to (t, Y_1(t,s), Y_2(t,s), s),$$

computed in $\mathbb{R}^4$ with Minkowski's metric $(-1,+1,+1,+1)$, namely

$$A[Y] = -\int \sqrt{(1 + \partial_s Y^2)(1 - \partial_t Y^2) + (\partial_t Y \cdot \partial_s Y)^2}\ \ dtds.$$

Notice that this computation is valid as long as $x \to X$ stays invertible. Then, solving the Born-Infeld equations, in the limit case $\lambda = 0$, simply amounts to finding, for each fixed value of $x = (x_1, x_2)$, an extremal surface $(t,s) \to X(t,x,s)$ for the area functional A. In physical terms, such extremal surfaces can be interpreted as classical strings, the area functional being the Nambu-Goto Action [Po]. The almost explicit resolution of the string equations is explained below in the paper's Appendix. It follows that, in the limit $\lambda = 0$, the Born-Infeld equations can be integrated, at least for short time intervals and smooth initial data.

We now see that what we have achieved, by using the BI equations with $\lambda = 0$ to interpolate fluid motions, has a simple geometric interpretation. For each values of the departure points x, we just interpolate the given trajectories $t \to X(t, x, 0)$ and $t \to X(t, x, 1)$ by an extremal surface $(t, x) \to X(t, x, s)$. (As a matter of fact, we followed exactly the opposite path in an earlier work [Br1]. We built a set of field equations for E and B starting from interpolating extremal surfaces. However, we ignored the constraint $E \cdot B = 0$ and, therefore, did not exactly get the Born-Infeld equation with $\lambda = 0$.) However, this simple geometrical interpretation is no longer valid as the mapping $x \to X(t, x, s)$ fails to be invertible, which usually happens after a finite time. In the large, as $x \to X$ is no longer one-to-one, the solutions of the BI system with $\lambda = 0$ cannot preserve their smoothness and a suitable concept of generalized solutions has to be found. We believe that such a concept should rely on some reconnection mechanism for strings, as they touch each other, somewhat similar to those observed in Magnetohydrodynamics for magnetic lines.

6 Appendix : solving the classical string equations

In this appendix, we show that the classical string equations can be essentially reduced, at least in the case when there are no space boundary conditions, to the one-dimensional linear wave equation. We say that $(t, s) \to X(t, s) \in \mathbb{R}^d$ is a classical string when X is a smooth critical point of the area functional

$$A[X] = -\int \sqrt{(1 + \partial_s X^2)(1 - \partial_t X^2) + (\partial_t X \cdot \partial_s X)^2} \ \ dtds.$$

Strictly speaking, $A[X]$ is the area of the two surface

$$(t, s) \to (t, X(t, s), s)$$

in the ambient space $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ with Minkowski metric $(-1, +1, \cdots, +1)$. We are going to solve the optimality equations under the natural 'subluminal' condition $|\partial_t X| < 1$. Let us denote respectively by Y and W the partial derivatives $\partial_s X$ and $-\partial_t X$. The optimality equations can be obtained by varying the Action $\int L(Y, W) dtds$ where Y and W are subject to

$$\partial_t Y + \partial_s W = 0,$$

and the Lagrangian density L is given by

$$L(Y,W) = -\sqrt{(1+Y^2)(1-W^2)+(W\cdot Y)^2}.$$

The resulting equations are

$$\partial_t Y + \partial_s W = 0, \quad \partial_t Z + \partial_s V = 0,$$

where Z and V are defined by

$$Z = \frac{\partial L}{\partial W}(Y,W) = \frac{(1+Y^2)W - (W\cdot Y)Y}{-L}.$$

$$V = -\frac{\partial L}{\partial Y}(Y,W) = \frac{(1-W^2)Y - (W\cdot Y)W}{-L}.$$

In order to write W and V as functions of the evolution variables Y and Z, we introduce the hamiltonian function h defined as the partial Legendre transform

$$h(Y,Z) = \sup_{W\in\mathbb{R}^d} Z\cdot W - L(Y,W) = \sqrt{1+Y^2+Z^2+(Y\cdot Z)^2}.$$

Then, we get

$$V = \frac{\partial h}{\partial Y}(Y,Z) = \frac{Y+(Y\cdot Z)Z}{h}, \quad W = \frac{\partial h}{\partial Z}(Y,Z) = \frac{Z+(Y\cdot Z)Y}{h}.$$

After this first step, we find a conservation law for h, namely

$$\partial_t h + \partial_s q = 0,$$

where the energy 'flux' q is given by

$$q = Y\cdot Z = \frac{W\cdot Y}{-L} = \frac{W\cdot Y}{\sqrt{(1+Y^2)(1-W^2)+(W\cdot Y)^2}}.$$

Indeed,

$$\begin{aligned}\partial_t h &= \frac{\partial h}{\partial Y}\cdot\partial_t Y + \frac{\partial h}{\partial Z}\cdot\partial_t Z\\ &= -V\cdot\partial_s W - W\cdot\partial_s V = -\partial_s(W\cdot V)\end{aligned}$$

where

$$W \cdot V = \frac{(Z + (Y \cdot Z)Y)(Y + (Y \cdot Z)Z)}{h^2} = Z \cdot Y.$$

Notice that, by definition, q satisfies $|q| < 1$ if and only if $|W| < 1$, i.e. if and only if $|\partial_t X| < 1$, which is a subluminal speed condition for the string. Next, let us look for an evolution equation for the flux q. On one hand, we have

$$-\partial_t q = Z \cdot \partial_s W + Y \cdot \partial_s V = Z \cdot \partial_s(\frac{Z + (Y \cdot Z)Y}{h}) + Y \cdot \partial_s(\frac{Y + (Y \cdot Z)Z}{h}).$$

$$= Z \cdot \partial_s(\frac{(Y \cdot Z)Y}{h}) + Y \cdot \partial_s(\frac{(Y \cdot Z)Z}{h}) + h\partial_s \frac{Z^2 + Y^2}{2h^2},$$

and, on the other hand,

$$\partial_s \frac{q^2}{h} = \partial_s \frac{(Y \cdot Z)^2}{h} = Z \cdot \partial_s \frac{Y(Y \cdot Z)}{h} + \partial_s Z \cdot \frac{Y(Y \cdot Z)}{h}$$

$$= Z \cdot \partial_s \frac{Y(Y \cdot Z)}{h} + Y \cdot \partial_s \frac{Z(Y \cdot Z)}{h} - Y \cdot Z \partial_s \frac{Y \cdot Z}{h}$$

$$= Z \cdot \partial_s \frac{Y(Y \cdot Z)}{h} + Y \cdot \partial_s \frac{Z(Y \cdot Z)}{h} - h\partial_s \frac{(Y \cdot Z)^2}{2h^2}.$$

Thus

$$\partial_t q + \partial_s \frac{q^2}{h} = -h\partial_s \frac{Z^2 + Y^2 + (Y \cdot Z)^2}{2h^2} = -h\partial_s \frac{h^2 - 1}{2h^2} = \partial_s \frac{1}{h}.$$

So, we have obtained

$$\partial_t q + \partial_s(\frac{q^2 - 1}{h}) = 0.$$

The self-consistent system governing h and q, the so-called Chaplygin gas equations, is integrable and has global smooth solutions precisely as $h > 0$ and $|q| < 1$, as discussed below. Once, $h > 0$ and q are smooth and known, we easily recover, for all times, Y and Z from their initial values, by solving the *linear* variable coefficient wave system :

$$\partial_t Y + \partial_s(\frac{Z - qY}{h}) = 0, \quad \partial_t Z + \partial_s(\frac{Y - qZ}{h}) = 0.$$

So we are just left with solving the Chaplygin system. Let us assume that both $q(0, \cdot)$ and $h(0, \cdot)$ are smooth and satisfy

$$\forall s, \quad |q(0, s)| < 1, \quad h(0, s) > 0.$$

(Notice that, from the string equations, we already have $h \geq 1$.) First, notice that formula

$$y = \int_0^{\xi_0(y)} h(0,s)ds$$

implicitely defines $y \to \xi_0(y)$ as a diffeomorphism of the real line. Next, introduce

$$v_0(y) = \frac{q(0,\xi_0(y))}{h(0,\xi_0(y)}$$

and solve the linear wave equation

$$\partial_{tt}\xi = \partial_{yy}\xi,$$

with initial conditions

$$\xi(t=0,y) = \xi_0(y), \quad \partial_t\xi(t=0,y) = v_0(y).$$

We have, by d'Alembert's formula,

$$\xi(t,y) = \frac{1}{2}(\xi_0(y+t) + \xi_0(y-t)) + \frac{1}{2}\int_{y-t}^{y+t} v_0(\sigma)d\sigma.$$

This formula defines $\xi(t,\cdot)$ as a diffeomorphism for *all* real t if and only if

$$\inf_y v_0(y) + \xi_0'(y) > \sup_y v_0(y) - \xi_0'(y)$$

which, in terms of q, exactly means $|q(0,s)| < 1$, using

$$h(0,\xi_0(y)) = \frac{1}{\xi_0'(y)}.$$

Then, $|q(t,s)| < 1$ holds true for all t and s. So, the solution (h,q) to the Chaplygin system is now implicitely given by

$$h(t,\xi(t,y)) = \frac{1}{\partial_y\xi(t,y)}, \quad q(t,\xi(t,y)) = h(t,\xi(t,y))\partial_t\xi(t,y).$$

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