AZIZ LECTURES 2006

STRING INTEGRATION OF SOME MHD TYPE EQUATIONS

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A semi-lagrangian, string integration of MHD like equations. Outline :

1. Equations: CHAPLYGIN, Shallow water MHD, BORN-INFELD
2. Dimensional splitting —–> 1D INTEGRATION USING VIBRATING STRINGS
3. Eulerian–>Lagrangian–>Eulerian transfers needed for dimensional splitting
4. Numerical Tests ——–> SWMHD, 1D and 2D CHAPLYGIN GAS

cf. YB, preprints 2002/2006, http://www-math.unice.fr/ brenier/

THE MOST COMPLEX MODEL CONSIDERED IN THIS LECTURE:

The **augmented Born-Infeld system** (including energy-momentum conservation laws) reads:

$$\partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times \mathbf{v} + \frac{\mathbf{D}}{\mathbf{h}}) = \mathbf{0}, \quad \nabla \cdot \mathbf{B} = \mathbf{0},$$

$$\partial_t \mathbf{D} + \nabla \times (\mathbf{D} \times \mathbf{v} - \frac{\mathbf{B}}{\mathbf{h}}) = \mathbf{0}, \quad \nabla \cdot \mathbf{D} = \mathbf{0},$$

$$\partial_t (\mathbf{hv}) + \nabla \cdot (\mathbf{hv} \otimes \mathbf{v} - \frac{\mathbf{B} \otimes \mathbf{B} + \mathbf{D} \otimes \mathbf{D}}{\mathbf{h}}) = \nabla(\frac{\mathbf{1}}{\mathbf{h}}), \qquad \partial_t \mathbf{h} + \nabla \cdot (\mathbf{hv}) = \mathbf{0}.$$

cf. YB, Arch. Rat. Mech. Analysis 2003, also see: http://www-math.unice.fr/ brenier/

Particular Regimes of the ABI system: A) Galilean equations

1) No Electromagnetic field

$\mathbf{B} = \mathbf{D} = \mathbf{0}$—> **Chaplygin gas** (possible model for dark energy), for which: **pressure=-1/density and sound speed=1/density**

$$\partial_t(\mathbf{hv}) + \nabla \cdot (\mathbf{hv} \otimes \mathbf{v}) = \nabla(\frac{\mathbf{1}}{\mathbf{h}}), \quad \partial_t \mathbf{h} + \nabla \cdot (\mathbf{hv}) = \mathbf{0},$$

2) Large fields $(\mathbf{B}, \mathbf{h})$ —> **Shallow water MHD** (without gravity) (cf. solar tachocline)

$$\partial_t(\mathbf{hv}) + \nabla \cdot (\mathbf{hv} \otimes \mathbf{v} - \mathbf{hb} \otimes \mathbf{b}) = \mathbf{0}, \quad \partial_t \mathbf{h} + \nabla \cdot (\mathbf{hv}) = \mathbf{0},$$

$$\partial_t(\mathbf{hb}) + \nabla \times (\mathbf{hb} \times \mathbf{v}) = \mathbf{0}, \quad \nabla \cdot (\mathbf{hb}) = \mathbf{0},$$

with $\mathbf{B} = \mathbf{bh}$, after rescaling.

Particular Regimes of the ABI system: B) Relativistic equations

3) The **original Born-Infeld equations** (cf. Born and Infeld, Proc. Roy. Soc. London, A 144 (1934), Born, Ann. Inst. H. Poincaré, 1937) require an additional (consistent) algebraic closure:

$$\boxed{\mathbf{h} = \sqrt{\mathbf{1} + \mathbf{B^2} + \mathbf{D^2} + |\mathbf{D} \times \mathbf{B}|^\mathbf{2}}, \quad \mathbf{v} = \frac{\mathbf{D} \times \mathbf{B}}{\mathbf{h}}}$$

(The Born-Infeld theory is related to D-branes in high energy physics, cf. Polchinski's String Theory book)

4) Weak fields $\mathbf{B}, \mathbf{D} << \mathbf{1}$, $\mathbf{h} \sim \mathbf{1}$ $\mathbf{v} \sim \mathbf{0}$—> **linear Maxwell equations**;
(with Born's scaling BI fits Maxwell down to 10^{-15} meters)

The **non conservative** version of the **augmented Born-Infeld system** is even more remarkable than the original, conservative, one:

$$\boxed{\partial_{\mathbf{t}}\mathbf{b} + (\mathbf{v}\cdot\nabla)\mathbf{b} = (\mathbf{b}\cdot\nabla)\mathbf{v} - \tau\nabla\times\mathbf{d}, \quad \partial_{\mathbf{t}}\mathbf{d} + (\mathbf{v}\cdot\nabla)\mathbf{d} = (\mathbf{d}\cdot\nabla)\mathbf{v} + \tau\nabla\times\mathbf{b},}$$

$$\boxed{\partial_{\mathbf{t}}\tau + (\mathbf{v}\cdot\nabla)\tau = \tau\nabla\cdot\mathbf{v}, \quad \partial_{\mathbf{t}}\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = (\mathbf{b}\cdot\nabla)\mathbf{b} + (\mathbf{d}\cdot\nabla)\mathbf{d} + \tau\nabla\tau,}$$

where $\tau = \frac{\mathbf{1}}{\mathbf{h}}, \quad \mathbf{b} = \frac{\mathbf{B}}{\mathbf{h}}, \quad \mathbf{d} = \frac{\mathbf{D}}{\mathbf{h}}.$

The Born-Infeld algebraic constraint reads

$$\tau > \mathbf{0}, \quad \tau^{\mathbf{2}} + \mathbf{v}^{\mathbf{2}} + \mathbf{b}^{\mathbf{2}} + \mathbf{d}^{\mathbf{2}} = \mathbf{1}, \quad \tau\mathbf{v} = \mathbf{d}\times\mathbf{b}.$$

This system is quadratic and symmetric, which helps a lot for a rigorous asymptotic analysis of the “high field regimes” $h \sim \infty$.

cf. YB, Wen-an Yong, Derivation of particle, string and membrane motions from the Born-Infeld Electromagnetism, J. Math. Physics 2005

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Let us recall the **augmented Born-Infeld system** in conservative form:

$$\partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times \mathbf{v} + \frac{\mathbf{D}}{\mathbf{h}}) = \mathbf{0}, \quad \nabla \cdot \mathbf{B} = \mathbf{0},$$

$$\partial_t \mathbf{D} + \nabla \times (\mathbf{D} \times \mathbf{v} - \frac{\mathbf{B}}{\mathbf{h}}) = \mathbf{0}, \quad \nabla \cdot \mathbf{D} = \mathbf{0},$$

$$\partial_t (\mathbf{h}\mathbf{v}) + \nabla \cdot (\mathbf{h}\mathbf{v} \otimes \mathbf{v} - \frac{\mathbf{B} \otimes \mathbf{B} + \mathbf{D} \otimes \mathbf{D}}{\mathbf{h}}) = \nabla(\frac{\mathbf{1}}{\mathbf{h}}), \quad \partial_t \mathbf{h} + \nabla \cdot (\mathbf{h}\mathbf{v}) = \mathbf{0}.$$

We want to design a numerical method based on the observation that, in 1D, this system can be integrated by solving a collection of linear 1D wave equations, describing vibrating strings.

In one space dimension (say x_1), introducing

$$\mathbf{z} = \sqrt{\mathbf{b_1^2} + \mathbf{d_1^2} + \tau^{\mathbf{2}}}\ , \quad \mathbf{u} = \left(\frac{\mathbf{b_1}}{\mathbf{z}}, \frac{\mathbf{d_1}}{\mathbf{z}}, \frac{\tau}{\mathbf{z}}\right)\ , \quad \mathbf{w} = (\mathbf{b_2}, \mathbf{b_3}, \mathbf{d_2}, \mathbf{d_3}, \mathbf{v_2}, \mathbf{v_3})\ ,$$

the **augmented Born-Infeld system** reads

$$(\partial_{\mathbf{t}} + \mathbf{v_1}\partial_{\mathbf{1}})\mathbf{z} = \mathbf{z}\partial_{\mathbf{1}}\mathbf{v_1}, \quad (\partial_{\mathbf{t}} + \mathbf{v_1}\partial_{\mathbf{1}})\mathbf{v_1} = \mathbf{z}\partial_{\mathbf{1}}\mathbf{z},$$

$$(\partial_{\mathbf{t}} + \mathbf{v_1}\partial_{\mathbf{1}})\mathbf{u} = \mathbf{0}, \quad (\partial_{\mathbf{t}} + \mathbf{v_1}\partial_{\mathbf{1}})\mathbf{w} = \mathbf{z}\ \mathbf{A}(\mathbf{u})\partial_{\mathbf{1}}\mathbf{w},$$

$$\mathbf{A}(\mathbf{u}) = \begin{pmatrix} 0 & 0 & 0 & u_3 & u_1 & 0 \\ 0 & 0 & -u_3 & 0 & 0 & u_1 \\ 0 & -u_3 & 0 & 0 & u_2 & 0 \\ u_3 & 0 & 0 & 0 & 0 & u_2 \\ u_1 & 0 & u_2 & 0 & 0 & 0 \\ 0 & u_1 & 0 & u_2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -iu_3 & u_1 \\ iu_3 & 0 & u_2 \\ u_1 & u_2 & 0 \end{pmatrix}$$

(with obvious complex notations).

Introducing a new "Lagrangian" space coordinate s and new fields $\mathbf{X}, \mathbf{U}, \mathbf{W}$:

$$\partial_{\mathbf{t}}\mathbf{X}(\mathbf{t},\mathbf{s}) = \mathbf{v_1}(\mathbf{t},\mathbf{X}(\mathbf{t},\mathbf{s})), \quad \partial_{\mathbf{s}}\mathbf{X}(\mathbf{t},\mathbf{s}) = \mathbf{z}(\mathbf{t},\mathbf{X}(\mathbf{t},\mathbf{s})),$$

$$\mathbf{U}(\mathbf{t},\mathbf{s}) = \mathbf{u}(\mathbf{t},\mathbf{X}(\mathbf{t},\mathbf{s})), \quad \mathbf{W}(\mathbf{t},\mathbf{s}) = \mathbf{w}(\mathbf{t},\mathbf{X}(\mathbf{t},\mathbf{s})),$$

the one-dimensional ABI system reduces to

$$\partial_{\mathbf{tt}}\mathbf{X} = \partial_{\mathbf{ss}}\mathbf{X}, \quad \partial_{\mathbf{t}}\mathbf{U} = \mathbf{0}, \quad \partial_{\mathbf{t}}\mathbf{W} = \mathbf{A}(\mathbf{U})\partial_{\mathbf{s}}\mathbf{W},$$

where

$$\mathbf{A}(\mathbf{U}) = \begin{pmatrix} 0 & 0 & 0 & U_3 & U_1 & 0 \\ 0 & 0 & -U_3 & 0 & 0 & U_1 \\ 0 & -U_3 & 0 & 0 & U_2 & 0 \\ U_3 & 0 & 0 & 0 & 0 & U_2 \\ U_1 & 0 & U_2 & 0 & 0 & 0 \\ 0 & U_1 & 0 & U_2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -iU_3 & U_1 \\ iU_3 & 0 & U_2 \\ U_1 & U_2 & 0 \end{pmatrix}$$

The only propagation speeds of this system are $0, +1, -1$ which makes its integration very easy.

The numerical scheme is based on the d'Alembert formula for the one-dimensional linear wave equation written as a first order system

$$\boxed{\partial_{\mathbf{t}} \mathbf{X} = \partial_{\mathbf{s}} \mathbf{Y}, \quad \partial_{\mathbf{t}} \mathbf{Y} = \partial_{\mathbf{s}} \mathbf{X},}$$

$$\boxed{\mathbf{X}(\mathbf{t}+\delta\mathbf{t},\mathbf{s}) = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{X}(\mathbf{t},\mathbf{s}+\delta\mathbf{t}) + \mathbf{X}(\mathbf{t},\mathbf{s}-\delta\mathbf{t}) + \mathbf{Y}(\mathbf{t},\mathbf{s}+\delta\mathbf{t}) - \mathbf{Y}(\mathbf{t},\mathbf{s}-\delta\mathbf{t})),}$$

$$\boxed{\mathbf{Y}(\mathbf{t}+\delta\mathbf{t},\mathbf{s}) = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{X}(\mathbf{t},\mathbf{s}+\delta\mathbf{t}) - \mathbf{X}(\mathbf{t},\mathbf{s}-\delta\mathbf{t}) + \mathbf{Y}(\mathbf{t},\mathbf{s}+\delta\mathbf{t}) + \mathbf{Y}(\mathbf{t},\mathbf{s}-\delta\mathbf{t})).}$$

The numerical solution is *exact* on a uniform mesh if

$$\boxed{\delta\mathbf{t} = \delta\mathbf{s}.}$$

Now, a major difficulty arises: **The *linear* wave equation does not preserve the inversibility condition $\partial_s X(t,s) > 0$ in the large (large data or large times).**

A reordering step is added at each time step in order to keep $\partial_s X \geq 0$.

This can be shown (YB Methods Appl. Anal. 2004) to be equivalent to a vanishing viscosity approximation to the momentum equation

$$\boxed{\partial_{\mathbf{t}}(\mathbf{h}\mathbf{v_1}) + \frac{\partial}{\partial \mathbf{x_1}}(\mathbf{h}\mathbf{v_1^2}) + \cdots = \epsilon \frac{\partial^{\mathbf{2}}}{\partial \mathbf{x_1}^{\mathbf{2}}}\mathbf{v_1}, \quad \epsilon \to \mathbf{0},}$$

Notice that this is a *realistic* (Navier-Stokes style) viscosity not an *artificial* viscosity.

SEMI-LAGRANGIAN NUMERICAL METHOD

-Cartesian grids and dimensional splitting

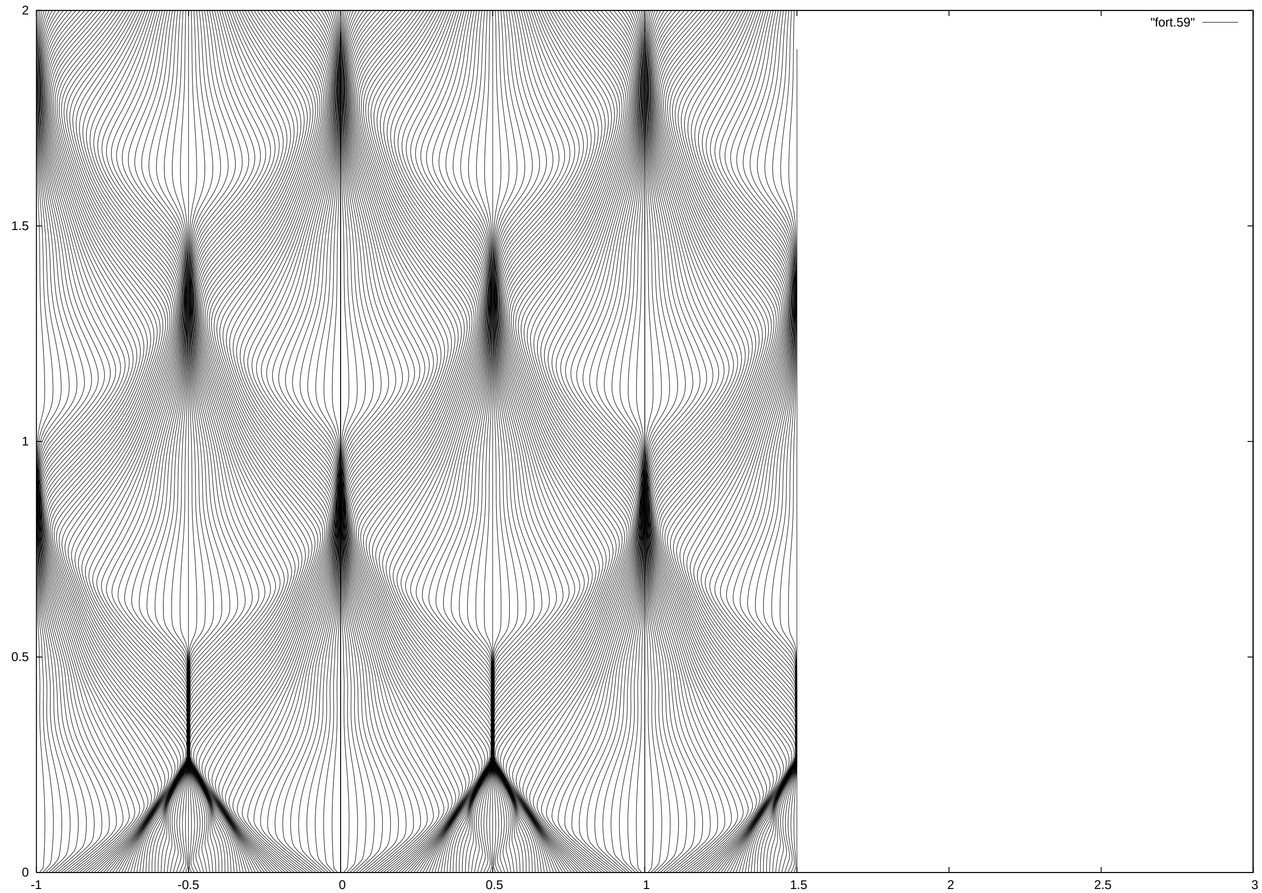
-Exact integration of the 1D equations using vibrating strings and d'Alembert formula on a Lagrangian mesh

-Lagrangian->Eulerian->Lagrangian steps used to allow dimensional splitting, not necessarily performed at each time step (in order to reduce numerical dissipation: cf. large time step Godunov schemes à la LeVeque, mid 80')

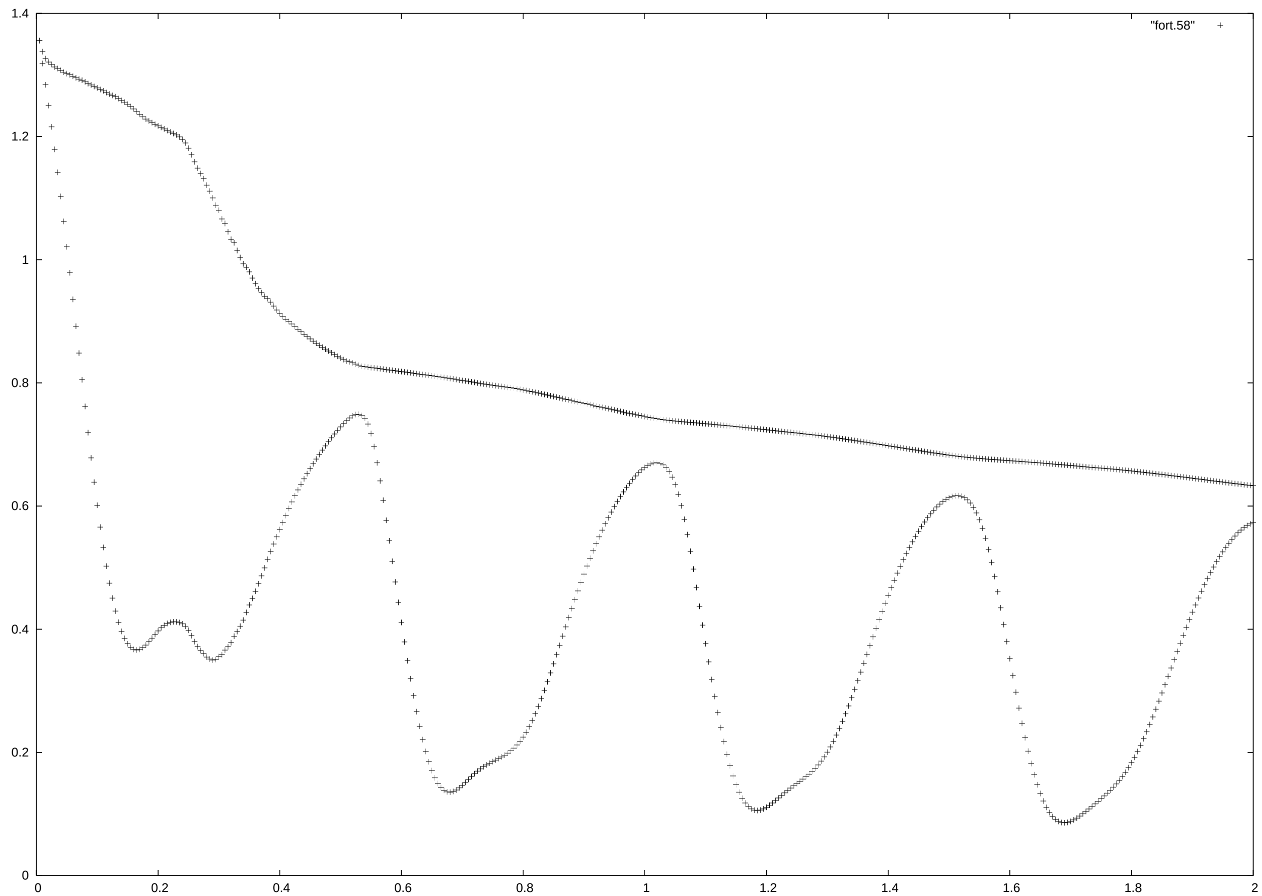
Typically, CFL=5 allows good energy conservation

-Numerical tests (see next slides): 1D and 2D Chaplygin equations, and also on SW-MHD; No tests yet on the full augmented Born-Infeld system.

1D Chaplygin gas
Eulerian scheme



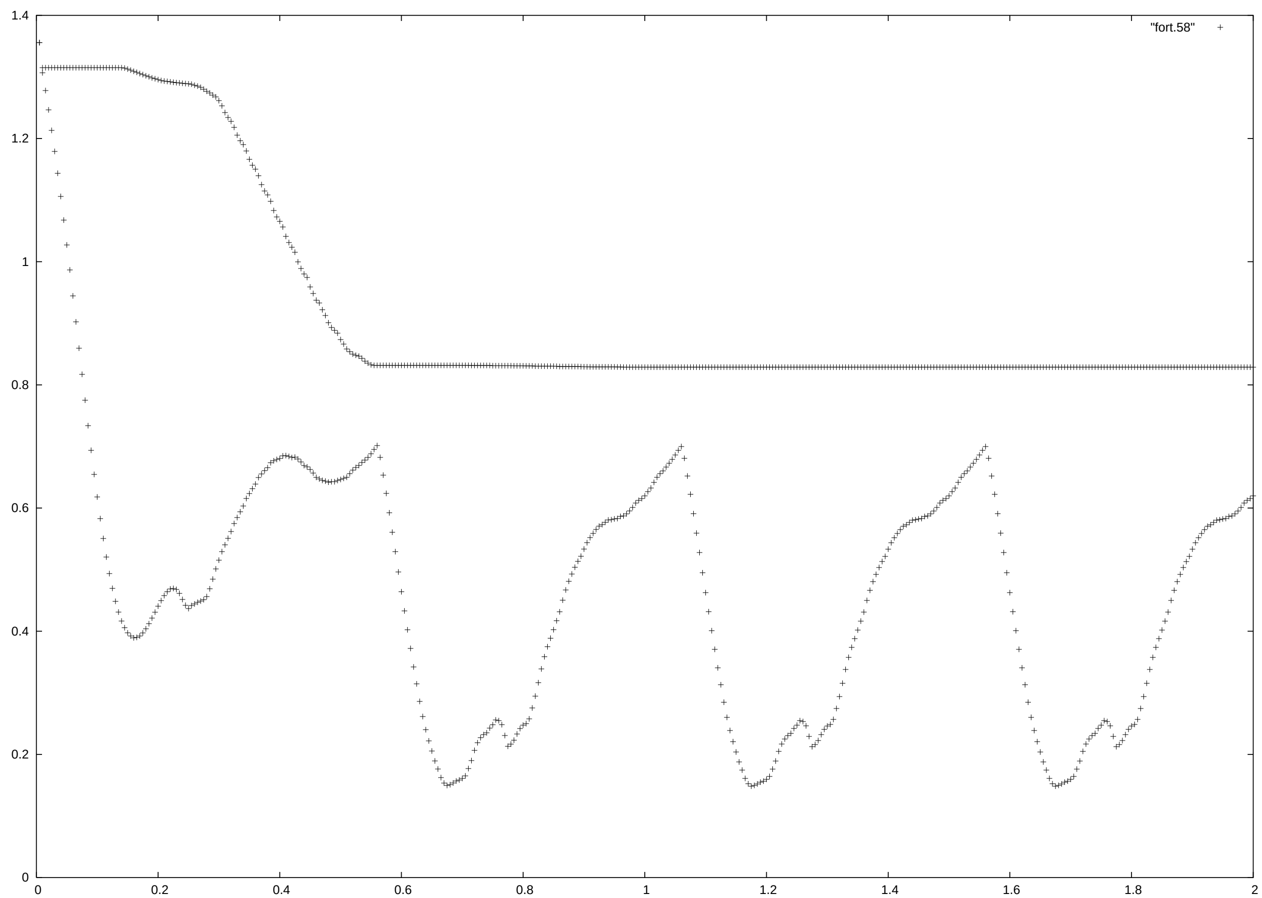
1D Chaplygin gas
Eulerian scheme



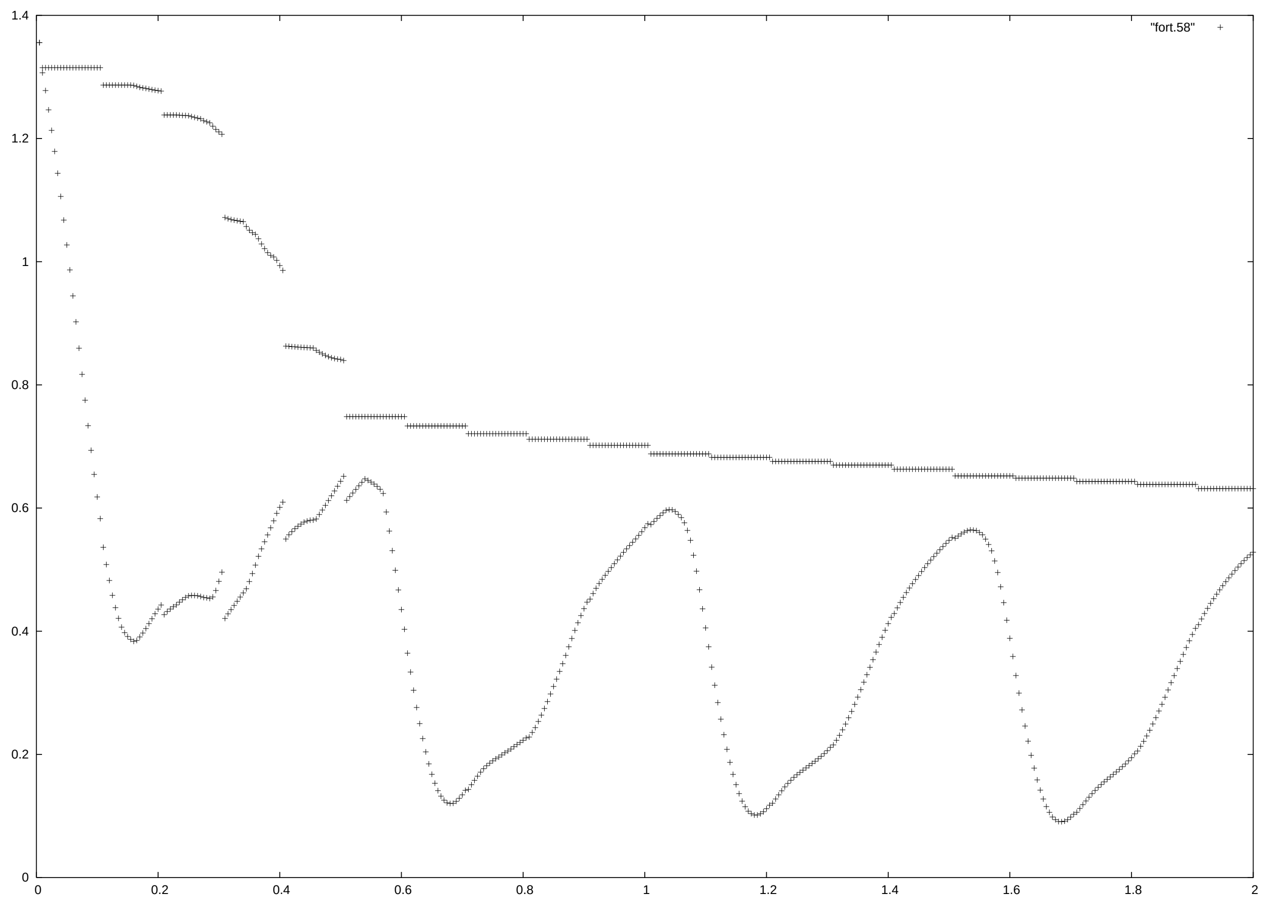
kinetic/total
energy vs time

1D Chaplygin gas
Lagrangian scheme

kinetic/total
energy vs time



1D Chaplygin gas
Semi-Lagrangian sch.

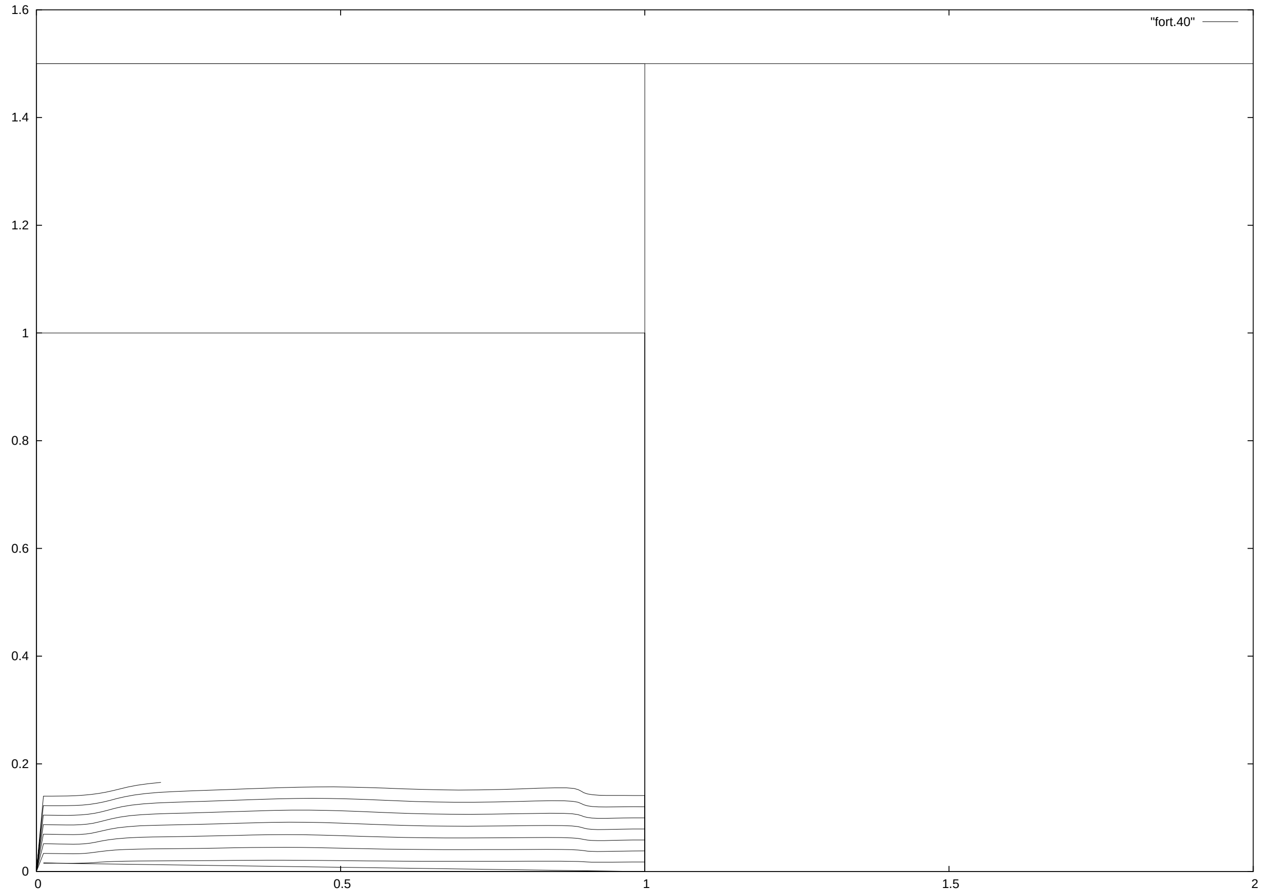


kinetic/total
energy vs time

Chaplygin gas

2D square

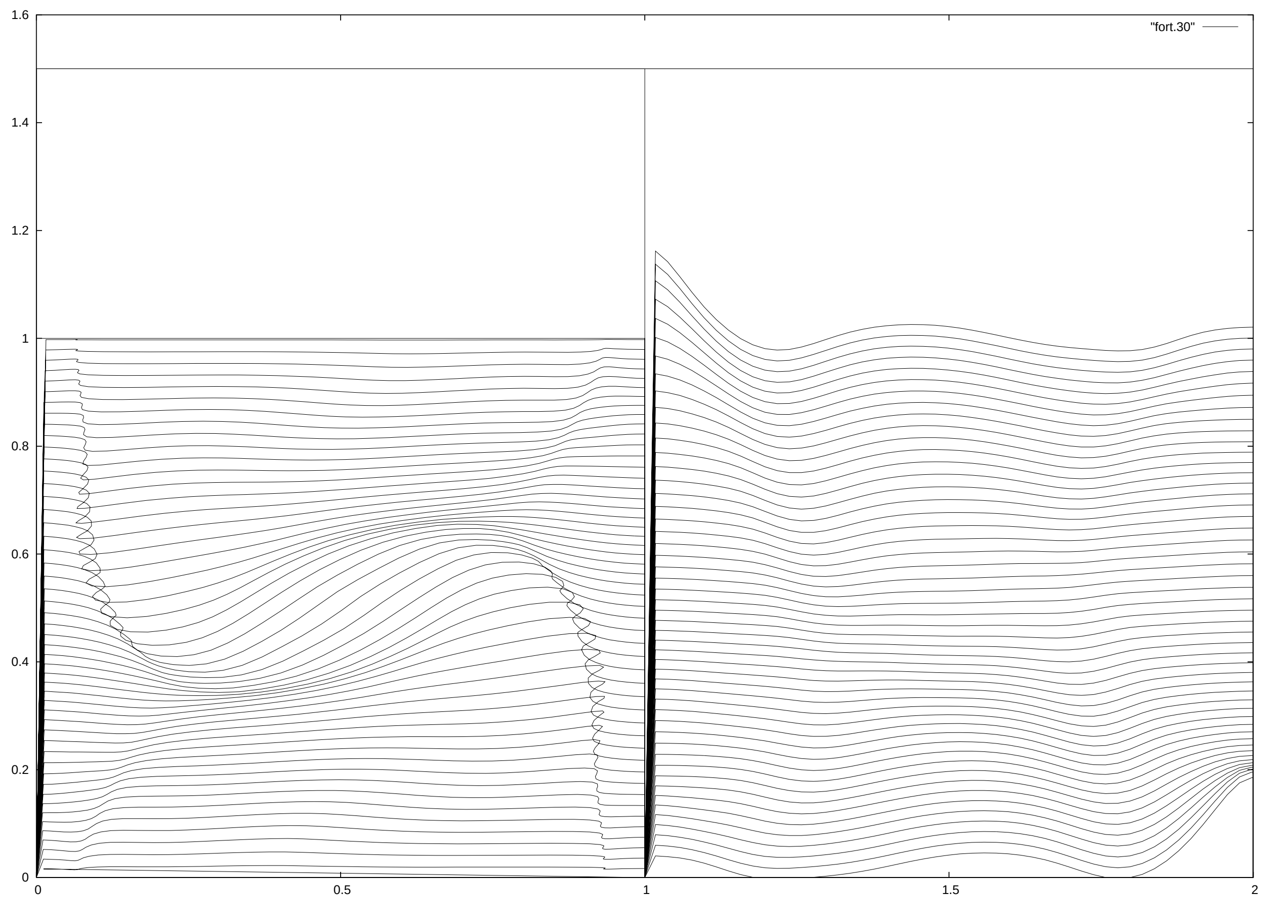
Velocity/Density



Chaplygin gas

2D square

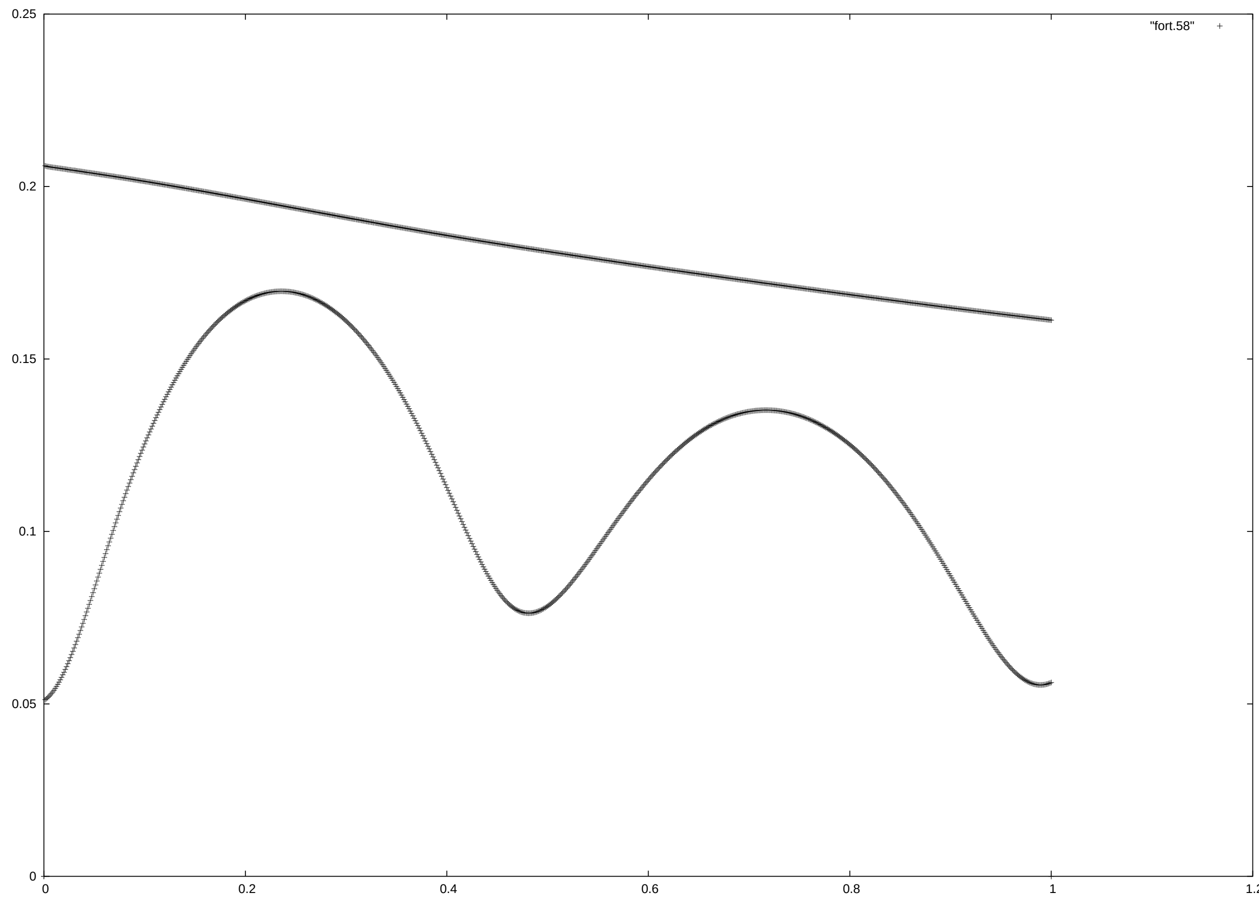
Velocity/Density



String integrator

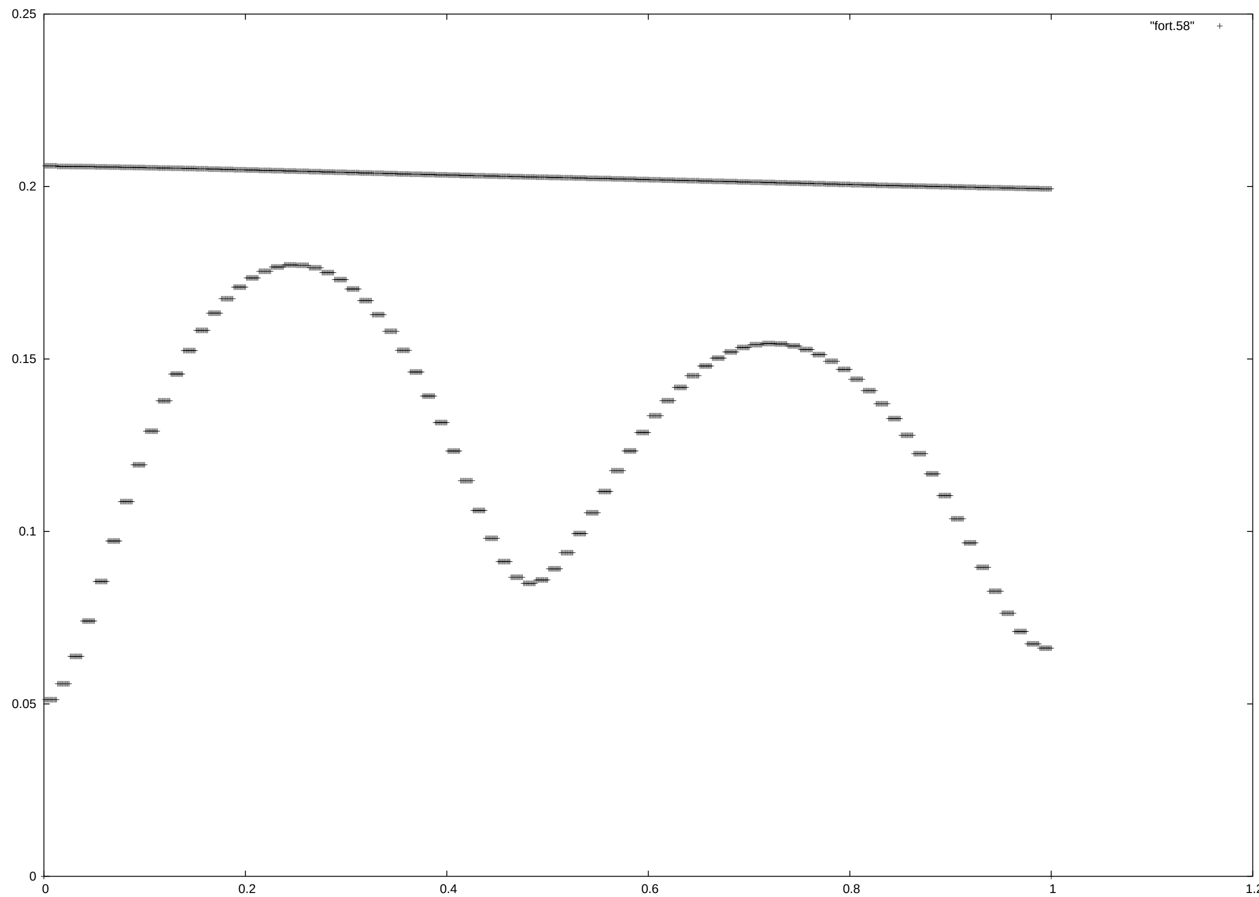
Eulerian CFL=10

Chaplygin gas
2D square
Kinetic/Total
Energy vs time



String integrator
CFL=1

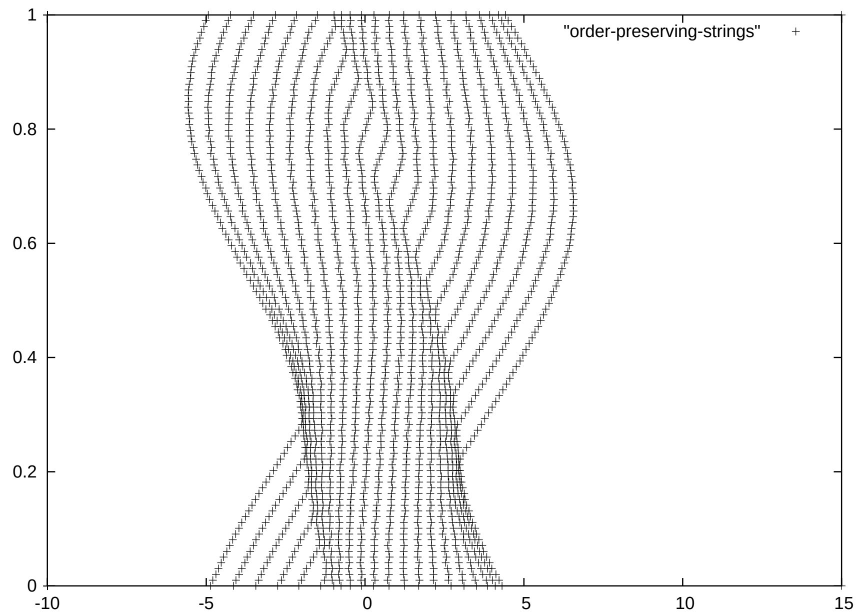
Chaplygin gas
2D square
Kinetic/Total
Energy vs time



String integrator
CFL=10

SW-MHD

Magnetic lines



String integrator

Purely Lagrangian

CONCLUSION

A numerical method has been introduced for MHD type equations derived from the Born-Infeld model, based on their exact integration in 1D, using vibrating strings and dimensional splitting.
Numerical calculations have been peformed on some simple SWMHD tests, 1D and 2D Chaplygin equations on cartesian grids. A good numerical control of the energy dissipation can be achieved by reducing the number of Eulerian<->Lagrangian steps