

# Topology-preserving diffusion equations for divergence-free vector fields

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Variational models and methods for evolution, Levico Terme 2012

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## ① **Loop approximation of divergence-free vector fields**

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**Warning: there will be only formal derivations and no rigorous analysis.**

# Loop decomposition of divergence-free vector fields

Every smooth loop  $\mathbf{s} \in \mathbb{R}/\mathbb{Z} \rightarrow \mathbf{X}(\mathbf{s}) \in \mathbb{R}^d$  generates, in the sense of distributions, a divergence-free vector field  $\mathbf{x} \in \mathbb{R}^d \rightarrow \mathbf{B}(\mathbf{x}) \in \mathbb{R}^d$

$$\mathbf{B}(\mathbf{x}) = \int_{\mathbf{s} \in \mathbb{R}/\mathbb{Z}} \mathbf{X}'(\mathbf{s}) \delta(\mathbf{x} - \mathbf{X}(\mathbf{s})) d\mathbf{s}.$$



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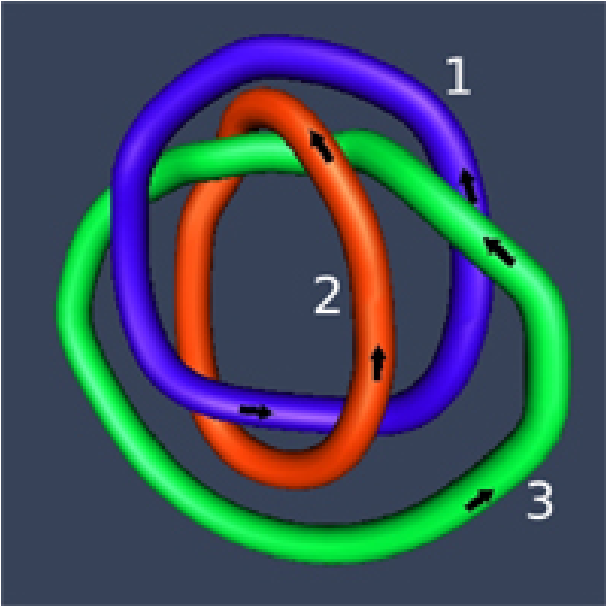
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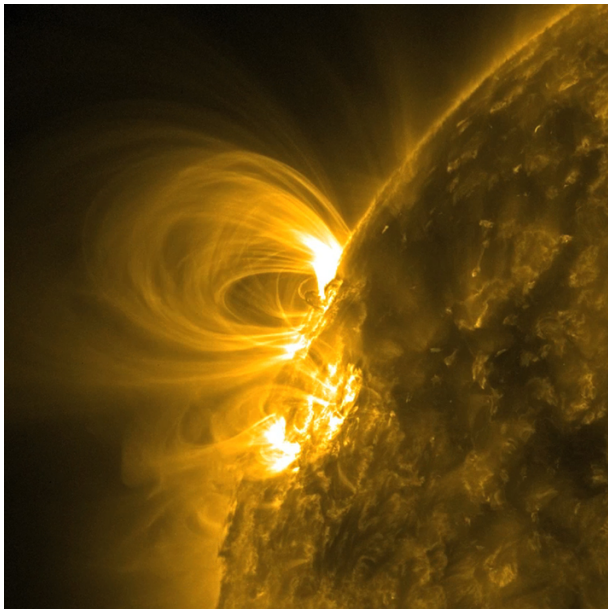
Indeed, for any smooth function  $q$

$$\langle \mathbf{B}, \nabla q \rangle = \int_{\mathbf{s} \in \mathbb{R}/\mathbb{Z}} \mathbf{X}'(\mathbf{s}) \cdot (\nabla q)(\mathbf{X}(\mathbf{s})) d\mathbf{s} = \int_{\mathbf{s} \in \mathbb{R}/\mathbb{Z}} \frac{d}{d\mathbf{s}} (q(\mathbf{X}(\mathbf{s}))) = 0.$$

**Conversely, every divergence-free field can be approximated by a superposition of loops (cf. related work by S. Smirnov).**

# SUPERPOSITION OF LOOPS





## Transport of a loop by a velocity field

Let us consider a time-dependent loop  $(t, \mathbf{s}) \rightarrow \mathbf{X}(t, \mathbf{s}) \in \mathbb{R}^d$ . moved by some velocity field  $\mathbf{v}(t, \mathbf{x}) \in \mathbb{R}^d$  so that  $\partial_t \mathbf{X}(t, \mathbf{s}) = \mathbf{v}(t, \mathbf{X}(t, \mathbf{s}))$

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the "transport" equation of  $\mathbf{B}$  by  $\mathbf{v}$  in the sense of distributions:

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$$\begin{aligned} \frac{d}{dt} \langle B, w \rangle &= \frac{d}{dt} \int w(\mathbf{X}(t, s)) \partial_s \mathbf{X}(t, s) ds = \int [(Dw)(\mathbf{X}) \partial_t \mathbf{X} \otimes \partial_s \mathbf{X} \\ &\quad + w(\mathbf{X}) \partial_{ts}^2 \mathbf{X}] ds = \int (Dw)(\mathbf{X}) [\partial_t \mathbf{X} \otimes \partial_s \mathbf{X} - \partial_s \mathbf{X} \otimes \partial_t \mathbf{X}] ds \\ &= \int (Dw)(\mathbf{x}) [\mathbf{v}(t, \mathbf{x}) \otimes \mathbf{B}(t, \mathbf{x}) - \mathbf{B}(t, \mathbf{x}) \otimes \mathbf{v}(t, \mathbf{x})] dx, \quad \forall w \in C_c^\infty. \end{aligned}$$

## Transport of divergence-free vector fields

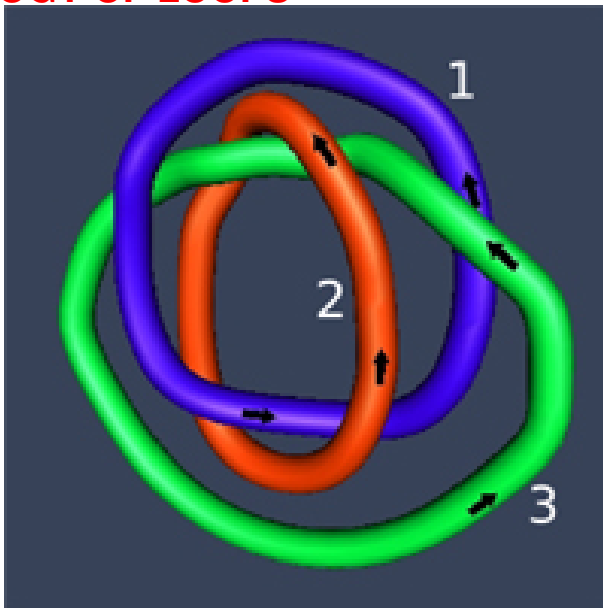
By superposition of loops, the transport of a time-dependent divergence-free vector field  $\mathbf{B}(t, \mathbf{x}) \in \mathbb{R}^d$  by some velocity field  $\mathbf{v}(t, \mathbf{x}) \in \mathbb{R}^d$ , is still described by the "topology-preserving" transport equation

$$\partial_t \mathbf{B} + \nabla \cdot (\mathbf{B} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{B}) = \mathbf{0}$$

(Of course, this equation can be derived by many other means.)



# TOPOLOGY OF LOOPS



# The usual diffusion equation is NOT topology-preserving

The usual diffusion equation for a divergence-free vector field

$\partial_t \mathbf{B} = \nabla^2 \mathbf{B}$  cannot be written in transport form

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and, therefore, is not "topology-preserving" (i.e. is not compatible with the loop decomposition).

This is in sharp contrast with the standard heat equation for density fields  $\rho > 0$

$$\partial_t \rho = \nabla^2 \rho \quad \text{which can be easily put in}$$

"transport" form

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \quad \mathbf{v} = -\nabla(\log \rho)$$

# Topology-preserving diffusion equations and turbulence theory

In the fluid mechanics literature (e.g. T. Nishiyama, 2003), one can find, for 3D divergence-free vector fields, non-linear (degenerate) diffusion equations of form

$$\partial_t \mathbf{B} + \nabla \times (\lambda(|\mathbf{B}|^2 \mathbf{I} - \mathbf{B} \otimes \mathbf{B}) \nabla \times \mathbf{B}) = \mathbf{0}$$

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Such equations are interesting because they have highly non-trivial equilibrium states, namely all fields  $\mathbf{B}$  which are colinear to their own curl. These fields are special stationary solutions of the 3D Euler equations and are believed to play a crucial role in turbulence. (They include Beltrami flows and all stationary solutions to the 2D Euler equations.)

# Design of topology-preserving diffusion equations following optimal transport ideas

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This will require a sort of generalization of the Jordan-Kinderlehrer-Otto (JKO) method, which has been a very successful way of deriving parabolic equations (in particular the regular scalar heat equation) from optimal transport considerations.

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**This will require a sort of generalization of the Jordan-Kinderlehrer-Otto (JKO) method, which has been a very successful way of deriving parabolic equations (in particular the regular scalar heat equation) from optimal transport considerations.**

**Unfortunately, there will be no rigorous analysis, due to the highly non-linear and degenerate structure of this kind of equations.**



## An example of transportation cost

For every elementary loop moving in the Minkowski space  $\mathbf{R} \times \mathbf{R}^d$ , we define our cost function as the area spanned by the loop

$$\int \int \sqrt{(1 - |\partial_t \mathbf{X}|^2)|\partial_s \mathbf{X}|^2 + (\partial_t \mathbf{X} \cdot \partial_s \mathbf{X})^2} \, ds dt$$

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In terms of fields  $(\mathbf{B}, \mathbf{v})$ , this generalizes (by superposition) as

$$\int \int \sqrt{(1 - |\mathbf{v}|^2)|\mathbf{B}|^2 + (\mathbf{v} \cdot \mathbf{B})^2} \, dx dt$$

which will be our definition of the transport cost of  $\mathbf{B}$  by  $\mathbf{v}$ .

## Optimal velocity field

Let us optimize in  $\mathbf{v} = \mathbf{v}(\mathbf{t}, \mathbf{x}) \in \mathbf{R}^d$  the convex cost function

$$- \int \int \sqrt{(1 - |\mathbf{v}|^2)|\mathbf{B}|^2 + (\mathbf{v} \cdot \mathbf{B})^2} \, d\mathbf{x}d\mathbf{t}$$

under the linear differential constraint

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Introducing Lagrange multiplier  $\mathbf{A} = \mathbf{A}(\mathbf{t}, \mathbf{x}) \in \mathbb{R}^d$  for the differential constraint, we find the structure condition

$$\mathbf{v} = \lambda \, \mathbf{dA} \cdot \mathbf{B}, \quad \mathbf{dA}_{ij} = \partial_j \mathbf{A}_i - \partial_i \mathbf{A}_j$$

where  $\lambda = 1 / \sqrt{|\mathbf{B}|^2 + |\mathbf{dA} \cdot \mathbf{B}|^2}$ .

## A generalized (BB-)JKO scheme

Let  $\mathbf{B} \rightarrow \mathbf{E}[\mathbf{B}] \in \mathbf{R}$  be a given functional on divergence-free vector fields, say, for simplicity,  $\mathbf{E}[\mathbf{B}] = \int |\mathbf{B}|^2 / 2 dx$ .

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We want to define the gradient flow of this functional according to the "optimal transport metric". Let us do that using the JKO (Jordan-Kinderlehrer-Otto) scheme, in BB (Benamou-Brenier) style:

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Given a time step  $h > 0$ ,  $\mathbf{B}(t, \mathbf{x})$ , supposed to be already known for  $0 \leq t \leq (n-1)h$ , is obtained for  $nh \geq t \geq (n-1)h$  as a critical point of the functional

$$\mathbf{E}[\mathbf{B}(nh, \cdot)] - \int_{(n-1)h}^{nh} dt \left\{ \int \sqrt{(1 - |\mathbf{v}|^2)|\mathbf{B}|^2 + (\mathbf{v} \cdot \mathbf{B})^2} dx \right\}$$

where  $\mathbf{v}$  is optimized under the linear differential constraint

$$\partial_t \mathbf{B} + \nabla \cdot (\mathbf{B} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{B}) = 0$$

## Resulting diffusion equation

We find, as necessary conditions:

$$\mathbf{A}(\mathbf{nh}, \cdot) = \mathbf{E}'[\mathbf{B}(\mathbf{nh}, \cdot)], \quad \mathbf{v} = \lambda \mathbf{dA} \cdot \mathbf{B}$$

where  $\lambda > 0$  is an explicit function of  $\mathbf{B}$  and  $\mathbf{dA} \cdot \mathbf{B}$ .



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where  $\lambda > 0$  is an explicit function of  $\mathbf{B}$  and  $\mathbf{dA} \cdot \mathbf{B}$ .  
Combined with the transport equation

$$\partial_t \mathbf{B} + \nabla \cdot (\mathbf{B} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{B}) = 0$$

the closure equations formally obtained as  $h \rightarrow 0$ ,

$$\mathbf{v} = \lambda \mathbf{dA} \cdot \mathbf{B}, \quad \mathbf{A} = \mathbf{E}'[\mathbf{B}]$$

provide a self-consistent evolution equation for  $\mathbf{B}$ , which is the desired "topology-preserving" diffusion equation (up to the precise definition of  $\lambda > 0$  as a function of  $\mathbf{B}$  and  $\mathbf{dA} \cdot \mathbf{B}$ ).

# THANKS FOR YOUR ATTENTION...

## SOME REFERENCES

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- 2 **V.I. Arnold, B. Khesin, Topological methods in hydrodynamics, Springer 1998.**
- 3 **J.-D. Benamou, Y. Brenier, Num. Math 2000.**
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- 7 **C. Villani, Topics in Optimal Transportation, AMS 2003.**

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