

Topology-preserving diffusion equations for divergence-free vector fields

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Variational models and methods for evolution, Levico Terme 2012

TOPOLOGY PRESERVING DIFFUSION EQUATIONS FOR DIVERGENCE-FREE VECTOR FIELDS

① **Loop approximation of divergence-free vector fields**

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Warning: there will be only formal derivations and no rigorous analysis.

Loop decomposition of divergence-free vector fields

Every smooth loop $\mathbf{s} \in \mathbb{R}/\mathbb{Z} \rightarrow \mathbf{X}(\mathbf{s}) \in \mathbb{R}^d$ generates, in the sense of distributions, a divergence-free vector field $\mathbf{x} \in \mathbb{R}^d \rightarrow \mathbf{B}(\mathbf{x}) \in \mathbb{R}^d$

$$\mathbf{B}(\mathbf{x}) = \int_{\mathbf{s} \in \mathbb{R}/\mathbb{Z}} \mathbf{X}'(\mathbf{s}) \delta(\mathbf{x} - \mathbf{X}(\mathbf{s})) d\mathbf{s}.$$

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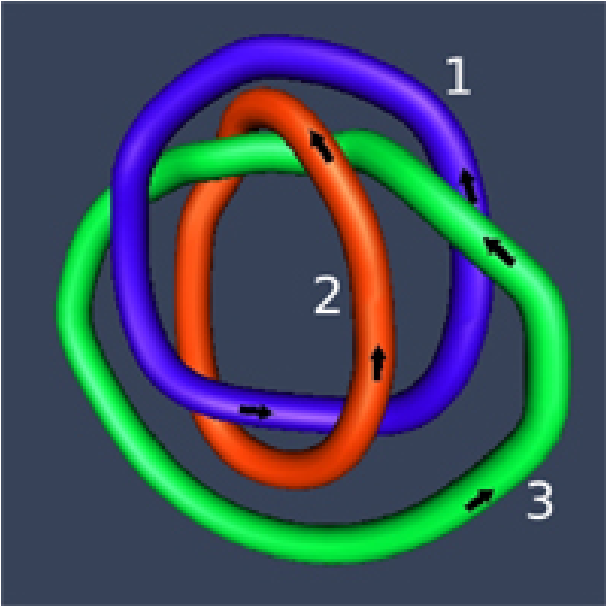
$$\mathbf{B}(\mathbf{x}) = \int_{\mathbf{s} \in \mathbb{R}/\mathbb{Z}} \mathbf{X}'(\mathbf{s}) \delta(\mathbf{x} - \mathbf{X}(\mathbf{s})) d\mathbf{s}.$$

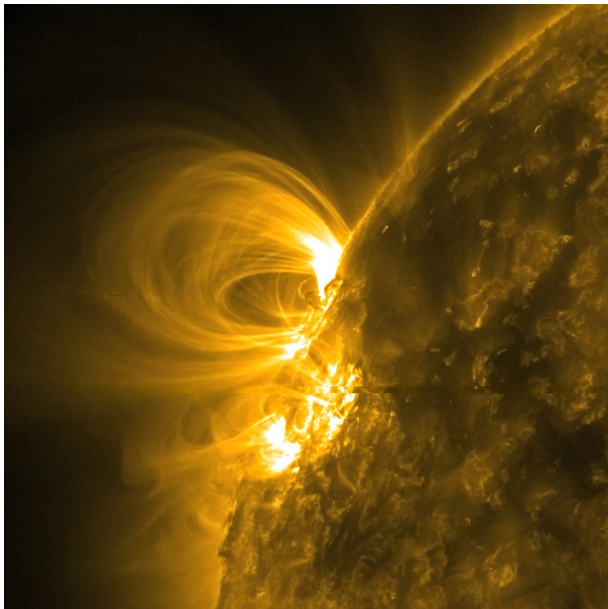
Indeed, for any smooth function q

$$\langle \mathbf{B}, r\mathbf{q} \rangle = \int_{\mathbf{s} \in \mathbb{R}/\mathbb{Z}} \mathbf{X}'(\mathbf{s}) \cdot (r\mathbf{q})(\mathbf{X}(\mathbf{s})) d\mathbf{s} = \int_{\mathbf{s} \in \mathbb{R}/\mathbb{Z}} \frac{d}{d\mathbf{s}} (q(\mathbf{X}(\mathbf{s}))) = 0.$$

Conversely, every divergence-free field can be approximated by a superposition of loops (cf. related work by S. Smirnov).

SUPERPOSITION OF LOOPS





Transport of a loop by a velocity field

Let us consider a time-dependent loop $(t, \mathbf{s}) \mapsto \mathbf{X}(t, \mathbf{s}) \in \mathbb{R}^d$. moved by some velocity field $\mathbf{v}(t, \mathbf{x}) \in \mathbb{R}^d$ so that $\partial_t \mathbf{X}(t, \mathbf{s}) = \mathbf{v}(t, \mathbf{X}(t, \mathbf{s}))$

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$$\mathbf{B}(t, \mathbf{x}) = \int \partial_s \mathbf{X}(t, s) \delta(\mathbf{x} - \mathbf{X}(t, s)) ds$$

the "transport" equation of \mathbf{B} by \mathbf{v} in the sense of distributions:

$$\partial_t \mathbf{B} + \operatorname{div}(\mathbf{B} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{B}) = 0$$

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We find for the corresponding divergence-free vector field

$$B(t, x) = \int \partial_s X(t, s) \delta(x - X(t, s)) ds$$

the "transport" equation of B by v in the sense of distributions:

$$\partial_t B + \operatorname{div} (B \otimes v - v \otimes B) = 0$$

$$\begin{aligned} \frac{d}{dt} \langle B, w \rangle &= \frac{d}{dt} \int w(X(t, s)) \partial_s X(t, s) ds = \int [(Dw)(X) \partial_t X - \partial_s X \\ &\quad + w(X) \partial_{ts}^2 X] ds = \int (Dw)(X) [\partial_t X - \partial_s X \otimes \partial_s X - \partial_t X] ds \\ &= \int (Dw)(x) [v(t, x) \otimes B(t, x) - B(t, x) \otimes v(t, x)] dx, \quad \delta w \in C_c^\infty. \end{aligned}$$

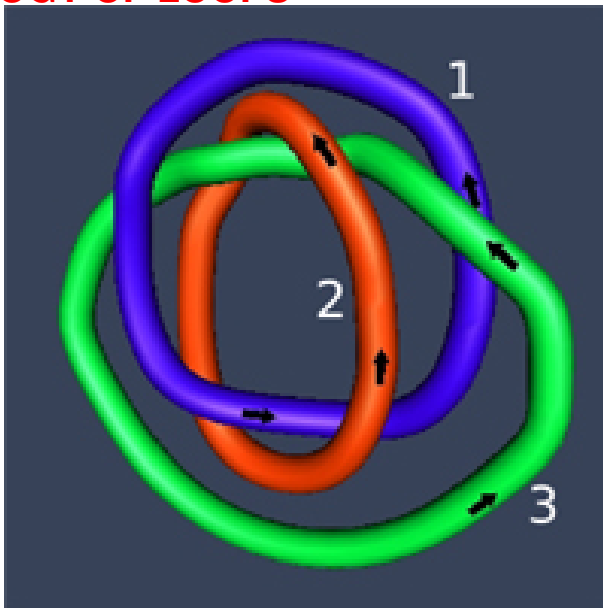
Transport of divergence-free vector fields

By superposition of loops, the transport of a time-dependent divergence-free vector field $\mathbf{B}(t, \mathbf{x}) \in \mathbb{R}^d$ by some velocity field $\mathbf{v}(t, \mathbf{x}) \in \mathbb{R}^d$, is still described by the "topology-preserving" transport equation

$$\partial_t \mathbf{B} + \mathbf{v} \cdot \nabla \mathbf{B} - \nabla \mathbf{v} \cdot \mathbf{B} = \mathbf{0}$$

(Of course, this equation can be derived by many other means.)

TOPOLOGY OF LOOPS



The usual diffusion equation is NOT topology-preserving

The usual diffusion equation for a divergence-free vector field

$\partial_t \mathbf{B} = r^2 \mathbf{B}$ cannot be written in transport form

$$\partial_t \mathbf{B} + r (\mathbf{B} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{B}) = 0$$

and, therefore, is not "topology-preserving" (i.e. is not compatible with the loop decomposition).

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The usual diffusion equation for a divergence-free vector field

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and, therefore, is not "topology-preserving" (i.e. is not compatible with the loop decomposition).

This is in sharp contrast with the standard heat equation for density fields $\rho > 0$

$$\partial_t \rho = r^2 \rho \quad \text{which can be easily put in}$$

"transport" form

$$\partial_t \rho + r (\rho \mathbf{v}) = 0, \quad \mathbf{v} = r (\log \rho)$$

Topology-preserving diffusion equations and turbulence theory

In the fluid mechanics literature (e.g. T. Nishiyama, 2003), one can find, for 3D divergence-free vector fields, non-linear (degenerate) diffusion equations of form

$$\partial_t \mathbf{B} + r \left(\lambda \|\mathbf{B}\|^2 \mathbf{I} - \mathbf{B} \mathbf{B} \right) r \mathbf{B} = \mathbf{0}$$

(where λ is some positive constant or function). They can be written in "transport-form" and are topology-preserving.

Design of topology-preserving diffusion equations following optimal transport ideas

The goal of this talk is the (formal) derivation of such a topology-preserving diffusion equation for divergence-free vector fields, in any dimensions, following "optimal transport" ideas.

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This will require a sort of generalization of the Jordan-Kinderlehrer-Otto (JKO) method, which has been a very successful way of deriving parabolic equations (in particular the regular scalar heat equation) from optimal transport considerations.

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This will require a sort of generalization of the Jordan-Kinderlehrer-Otto (JKO) method, which has been a very successful way of deriving parabolic equations (in particular the regular scalar heat equation) from optimal transport considerations.

Unfortunately, there will be no rigorous analysis, due to the highly non-linear and degenerate structure of this kind of equations.

An example of transportation cost

For every elementary loop moving in the Minkowski space \mathbb{R}^d , we define our cost function as the area spanned by the loop

$$\int \int \sqrt{(1 - |\partial_t \mathbf{X}|^2) |\partial_s \mathbf{X}|^2 + (\partial_t \mathbf{X} \cdot \partial_s \mathbf{X})^2} \, ds dt$$

(named Nambu-Goto action in string theory).

An example of transportation cost

For every elementary loop moving in the Minkowski space \mathbb{R}^d , we define our cost function as the area spanned by the loop

$$\int \int \sqrt{(1 - j\partial_t \mathbf{X} j^2) j\partial_s \mathbf{X} j^2 + (\partial_t \mathbf{X} - \partial_s \mathbf{X})^2} \, ds dt$$

(named Nambu-Goto action in string theory).

In terms of fields (\mathbf{B}, \mathbf{v}) , this generalizes (by superposition) as

$$\int \int \sqrt{(1 - j\mathbf{v} j^2) j\mathbf{B} j^2 + (\mathbf{v} - \mathbf{B})^2} \, dx dt$$

which will be our definition of the transport cost of \mathbf{B} by \mathbf{v} .

Optimal velocity field

Let us optimize in $\mathbf{v} = \mathbf{v}(t, \mathbf{x}) \in \mathbb{R}^d$ the convex cost function

$$\int \int \sqrt{(1 - \nu^2) \mathbf{B}^2 + (\mathbf{v} \cdot \mathbf{B})^2} \, dx dt$$

under the linear differential constraint

$$\partial_t \mathbf{B} + r (\mathbf{B} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{B}) = 0$$

Introducing Lagrange multiplier $\mathbf{A} = \mathbf{A}(t, \mathbf{x}) \in \mathbb{R}^d$ for the differential constraint, we find the structure condition

$$\mathbf{v} = \lambda \, d\mathbf{A} \cdot \mathbf{B}, \quad dA_{ij} = \partial_j A_i - \partial_i A_j$$

where $\lambda = 1 / \sqrt{\mathbf{B}^2 + |d\mathbf{A} \cdot \mathbf{B}|^2}$.

A generalized (BB-)JKO scheme

Let $B : E[B] \rightarrow \mathbb{R}$ be a given functional on divergence-free vector fields, say, for simplicity, $E[B] = \int |\mathbf{B}|^2 / 2 dx$.

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We want to define the gradient flow of this functional according to the "optimal transport metric". Let us do that using the JKO (Jordan-Kinderlehrer-Otto) scheme, in BB (Benamou-Brenier) style:

A generalized (BB-)JKO scheme

Let $B \in E[B] \subset \mathbb{R}$ be a given functional on divergence-free vector fields, say, for simplicity, $E[B] = \int jBj^2/2dx$.

We want to define the gradient flow of this functional according to the "optimal transport metric". Let us do that using the JKO (Jordan-Kinderlehrer-Otto) scheme, in BB (Benamou-Brenier) style:

Given a time step $h > 0$, $B(t, x)$, supposed to be already known for $0 \leq t \leq (n-1)h$, is obtained for $nh \leq t \leq (n+1)h$ as a critical point of the functional

$$E[B(nh, \cdot)] = \int_{(n-1)h}^{nh} dt \int \sqrt{(1 - jvj^2)jBj^2 + (v \cdot B)^2} dx g$$

where v is optimized under the linear differential constraint

$$\partial_t B + r \cdot (B \cdot v - v \cdot B) = 0$$

Resulting diffusion equation

We find, as necessary conditions:

$$\mathbf{A}(\mathbf{nh}, \cdot) = \mathbf{E}'[\mathbf{B}(\mathbf{nh}, \cdot)], \quad \mathbf{v} = \lambda \mathbf{dA} \mathbf{B}$$

where $\lambda > 0$ is an explicit function of \mathbf{B} and $\mathbf{dA} \mathbf{B}$.

Resulting diffusion equation

We find, as necessary conditions:

$$\mathbf{A}(\mathbf{h}, \cdot) = \mathbf{E}'[\mathbf{B}(\mathbf{h}, \cdot)], \quad \mathbf{v} = \lambda \mathbf{dA} \mathbf{B}$$

where $\lambda > 0$ is an explicit function of \mathbf{B} and $\mathbf{dA} \mathbf{B}$.
Combined with the transport equation

$$\partial_t \mathbf{B} + r(\mathbf{B}, \mathbf{v}, \mathbf{v}, \mathbf{B}) = 0$$

the closure equations formally obtained as $h \neq 0$,

$$\mathbf{v} = \lambda \mathbf{dA} \mathbf{B}, \quad \mathbf{A} = \mathbf{E}'[\mathbf{B}]$$

provide a self-consistent evolution equation for \mathbf{B} , which is the desired "topology-preserving" diffusion equation (up to the precise definition of $\lambda > 0$ as a function of \mathbf{B} and $\mathbf{dA} \mathbf{B}$).

THANKS FOR YOUR ATTENTION...

SOME REFERENCES

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- 7 **C. Villani, Topics in Optimal Transportation, AMS 2003.**

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