(14p.

A HILBERTIAN APPROACH TO (SOME !!!) HYPERBOLIC NON-LINEAR CONSERVATION LAWS

Y. BRENIER
CNRS, NICE, FRANCE

Hyperbolic Conservation Laws

$$\partial_t u + \sum_{i=1}^{d} \partial_{x_i} Q_i(u) = 0 \qquad (LC)$$

$u(t,x) \in \Omega \subset \mathbb{R}^m$, $x \in \mathbb{R}^d$, $t > 0$

$Q_i : \Omega \to \mathbb{R}^m$ given for $i = 1, \dots, d$

Hyperbolicity:

for all $\theta \in \mathbb{R}^d$, for all $v \in \Omega \subset \mathbb{R}^m$,
$\sum_{i=1}^{d} \theta_i Q_i'(v)$ is a diagonalizable $m \times m$ matrix with real eigenvalues

Main features:

- Generic solutions become singular (discontinuous) in finite time (SHOCK WAVES), at least for non linear systems
- Concept of "entropy solutions"

Def: L^p stability with respect to initial conditions: $p \in [1,\infty]$

$\forall T > 0$, $\exists C_T$ s.t.

$$\boxed{\sup_{0\le t\le T} \int |u(t,x) - \tilde{u}(t,x)|^p dx \le C_T \int |u(0,x) - \tilde{u}(0,x)|^p dx} \qquad (S_p)$$

Remark: By translation invariance of (LC), $(S_p) \Rightarrow (R_p)$

where:

$$(R_p) \qquad \boxed{\sup_{0\le t\le T} \int |\nabla u(t,x)|^p dx \le C_T \int |\nabla u(0,x)|^p dx}$$

(set $\tilde{u}(0,x) = u(0,x+h)$ for $h \in \mathbb{R}^d$
so that $\tilde{u}(t,x) = u(t,x+h)$)

Known Results

① $m=1,\ d\geq 1$ Scalar conserv. laws

Kružkov ~ 1970

(S_p) is true for $p=1$ (with $C_T=1$)

but Only for $p=1$, for non linear Q_i (*)

② $m\geq 1,\ d=1$ One dimensional Systems

Bressan, coll. ~ 2000

(S_p) is true for $p=1$

(for small bounded variation init. data)

Only for $p=1$, for non linear Q (*)

(*) If (S_p) is true for a $p>1$

then (R_p) is true which is ruled out

because of SHOCKS

(3) $m \geq 2 \quad d \geq 2$

For simple linear systems
(Linear acoustics, Maxwell equns)

(S_p) is true IF AND ONLY IF $p = 2$

Brenner 1965

Also true for generic systems
Rauch ~ 1980

THERE SEEMS TO BE A MAJOR OBSTRUCTION TO A GENERAL THEORY OF MULTID. HYPERBOLIC SYSTEMS OF CONS. LAWS

Main result

For multid SCALAR conservation laws, there is always a master equation which enjoys property (S_p) for all $p \in [1, \infty]$

in particular for $p = 2$

$\hookrightarrow$ L2 FORMULATION OF MULTID. SCALAR CONS. LAWS

Idea:

Instead of solving the initial value problem for a SINGLE initial condition, let us do it for a ONE-PARAMETER Family of ORDERED initial conditions

$$u_0(x, \underset{\substack{\uparrow \\ \text{parameter}}}{y}) \qquad \begin{array}{l} x \in \mathbb{R}^d \\ y \in \mathbb{R} \end{array} \qquad \text{with} \quad \partial_y u_0(x,y) \geq 0$$

According to Kružkov's theory, the corresponding entropy solutions are well ordered $\partial_y u(t,x,y) \geq 0$

(indeed: $u_0(x,y_1) \geq u_0(x,y_2)$, $\forall x \in \mathbb{R}^d$ implies $u(t,x,y_1) \geq u(t,x,y_2)$)

For notational simplicity, we may assume w.l.o.g $0 \leq u_0(x,y) \leq 1$ and, thus $0 \leq u(t,x,y) \leq 1$

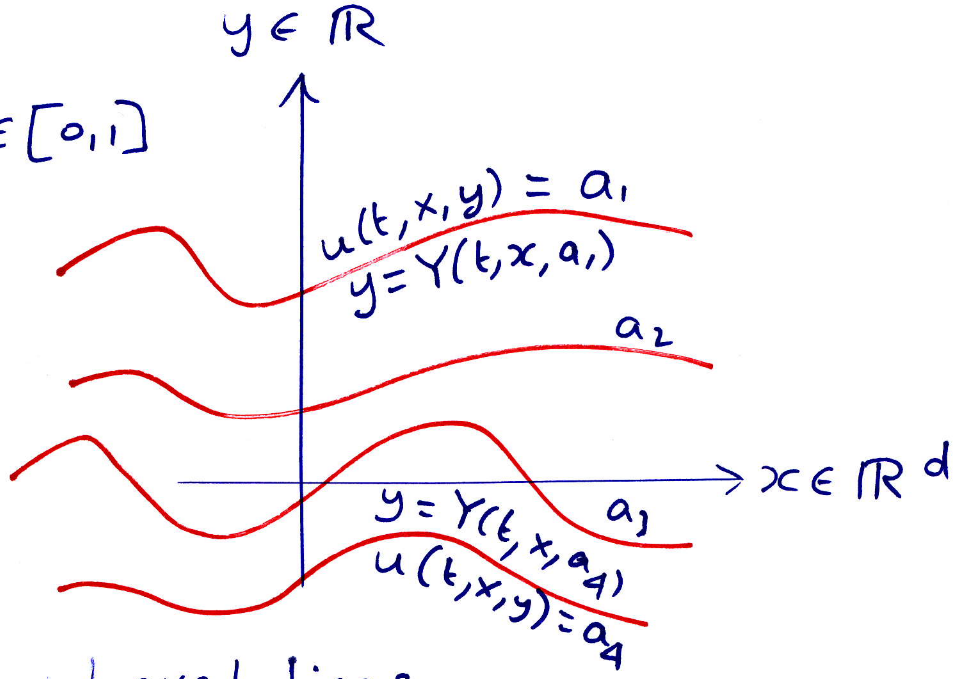
Let us look at the level lines of $u(t,x,y)$. Since $\partial_y u \geq 0$,

these lines can be written as

graphs $\boxed{y = Y(t,x,a)}$ $(x \in \mathbb{R}^d, t \geq 0, a \in [0,1])$

for (almost) all levels $a \in [0,1]$

with $\partial_a Y \geq 0$



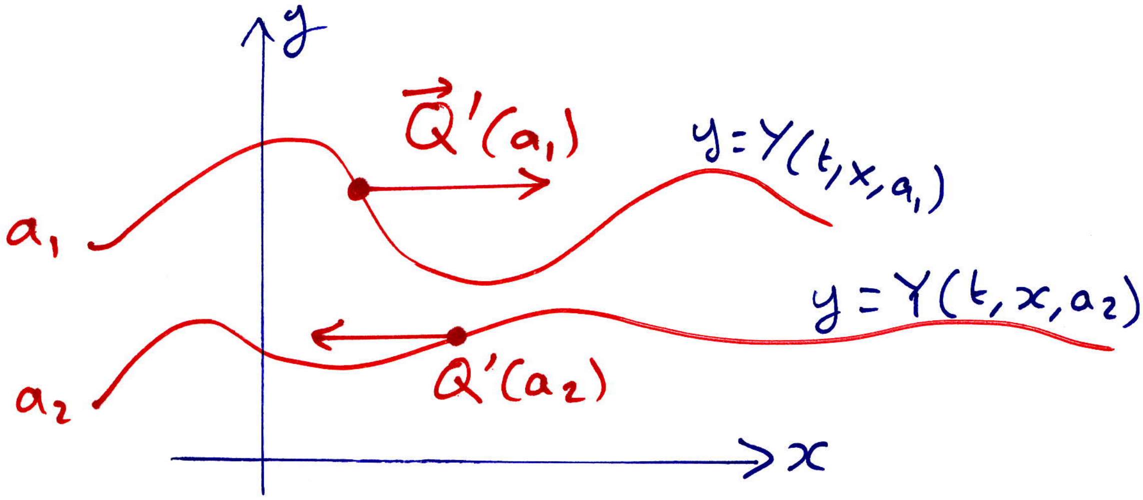
Level lines
For levels $a_4 < a_3 < a_2 < a_1$

Idea: Y is going to be the solution to the "master equation" (to be found)

Recovery of u:

$$\boxed{u(t,x,y) \underset{a.e.}{=} \int_0^1 1\{Y(t,x,a) \leq y\}\, da}$$

According to the "method of characteristics" the level sets are just slipping parallel to the x axis at constant speed $\vec{Q}'(a)$ depending only on the level $a \in [0,1]$



In other words, $Y(t,x,a)$ solves

$$\partial_t Y + Q'(a).\nabla_x Y = 0$$

However this construction cannot be global, in general,

eventually the level lines touch each other and they should'nt cross!

SHOCKS APPEAR!

Discrete Approximation

δt = time step

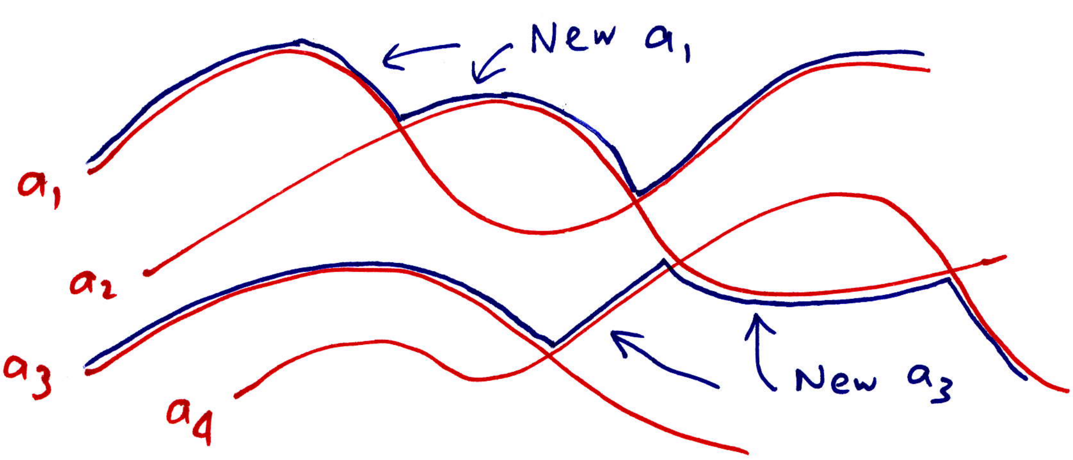
$Y(n\delta t, x, a) \simeq Y_n(x,a), \quad n = 0, 1, 2, \ldots$

Predictor: $Y^{\#}_{n+1}$ = sol. at time $(n+1)\delta t$ of $\partial_t Y + Q'(a).\nabla_x Y = 0$ with $Y = Y_n$ at time $n\delta t$

Namely: $Y^{\#}_{n+1}(x,a) = Y_n(x - \delta t\, Q'(a), a)$

Notice that $\partial_a Y^{\#}_{n+1} \geq 0$ is NOT GUARANTEED!

Corrector: For each fixed x, $[0,1] \ni a \to Y_{n+1}(x,a)$ is the increasing rearrangement of $a \longrightarrow Y^{\#}_{n+1}(x,a)$



RECONNECTION OF LINES

KEY PROPERTY OF THE DISCRETE SCHEME : TWO SOLUTIONS $Y, \tilde{Y}$

$$\int_0^1 \int_{\mathbb{R}^d} |Y_n(x,a) - \tilde{Y}_n(x,a)|^p \, dx \, da$$

$$= \int_0^1 \left(\int_{\mathbb{R}^d} |\underbrace{Y_n(x - \delta t\, Q'(a), a)}_{= Y^{\#}_{n+1}(x,a)} - \underbrace{\tilde{Y}_n(x - \delta t\, Q'(a), a)}_{= \tilde{Y}^{\#}_{n+1}(x,a)}|^p \, dx \right) da$$

$$= \int_{\mathbb{R}^d} \underbrace{\left(\int_0^1 |Y^{\#}_{n+1}(x,a) - \tilde{Y}^{\#}_{n+1}(x,a)|^p \, da \right)}_{\geq} dx$$

$$\geq \int_{\mathbb{R}^d} \left(\int |Y_{n+1}(x,a) - \tilde{Y}_{n+1}(x,a)|^p \, da \right) dx \qquad (C_p)$$

CONTRACTION PROPERTY OF REARRANGEMENT IN a, FOR EACH FIXED x

Define $\boxed{u_n(x,y) = \int_0^1 1\{Y_n(x,a) \leq y\} \, da}$

Then (C_p) exactly means:

$$\iint |u_n(x,y) - \tilde{u}_n(x,y)|^p \, dx\,dy \geq \iint |u_{n+1}(x,y) - \tilde{u}_{n+1}(x,y)|^p \, dx\,dy$$

ONLY for $p = 1$ (coarea formula)

Th

As $\delta t \to 0$

$n\delta t \to t \in \mathbb{R}_+$

1) $u_n(x,y) \xrightarrow[L^1]{} u(t,x,y)$ entropy solutions of $\partial_t u + \nabla . Q(u) = 0$

2) $Y_n(x,a) \xrightarrow[L^p]{} Y(t,x,a)$ solution of the

MASTER EQUATION

$$\boxed{\partial_t Y + \underbrace{Q'(a).\nabla_x Y + \partial 1_K(Y)}_{} \ni 0 \quad \text{where } 1_K(Y) = \begin{cases} 0 & \text{if } \partial_a Y \geq 0 \\ +\infty & \text{otherwise} \end{cases}}$$

and $$\boxed{u(t,x,y) = \int_0^1 1\{Y(t,x,a) \leq y\}\, da}$$

MAXIMAL MONOTONE OPERATOR IN L^2,

NON EXPANSIVE IN ALL L^p

cf. Y.B. arxiv 2006

"L2 formulation of multid...."

REMARK

$$1\{Y(t,x,a) \leq y\} = 1\{a \leq u(t,x,y)\}$$
$$= j(t,x,y,a) \in \{0,1\}$$

solves the kinetic formulation

$$\boxed{\begin{array}{l} \partial_t j + Q'(a).\nabla_x j = \partial_a \mu \\ \text{with } \mu(t,x,y,a) \geq 0 \quad \text{LPT} \end{array}}$$

and $j(t,x,y,a) = 1\{a \leq u(t,x,y)\}$

to be compered with the

"L2 formulation"

$$\partial_t Y + Q'(a).\nabla_x Y + \partial 1_K(Y) \ni 0$$

which means

$$\boxed{\begin{array}{l} \partial_t Y + Q'(a).\nabla_x Y = -\partial_a \nu \\ \partial_a Y \geq 0 \quad \text{and} \quad \ll \nu \partial_a Y = 0 \gg \\ \nu = \nu(t,x,a) \geq 0 \end{array}}$$

OTHER EXAMPLES OF L2 FORMULATIONS FOR:

Born-Infeld-Chaplygin : (YB, MAA 2004)

$$\begin{cases} \partial_t(\rho v) + \partial_x(\rho(v^2 - b^2)) = \partial_y(\rho b) & + 0_+ \partial_x(\mu \partial_x v) \\ \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho b) = \partial_y(\rho v) \end{cases}$$

vanishing viscosity

($\simeq$ pressureless MHD)

modelling sticky strings

and, as a subcase, pressureless gases in one space dimension (sticky particles)

Pressureless 1d Euler-Poisson

with vanishing viscosity

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho v) + \partial_x(\rho v^2) + \rho\, \partial_x \varphi = 0 + 0_+ \partial_x(\mu \partial_x v) \\ \beta\, \partial_{xx} \varphi = \rho - \langle \rho \rangle \end{cases}$$

electric potential

coupling constant

space average

BIC

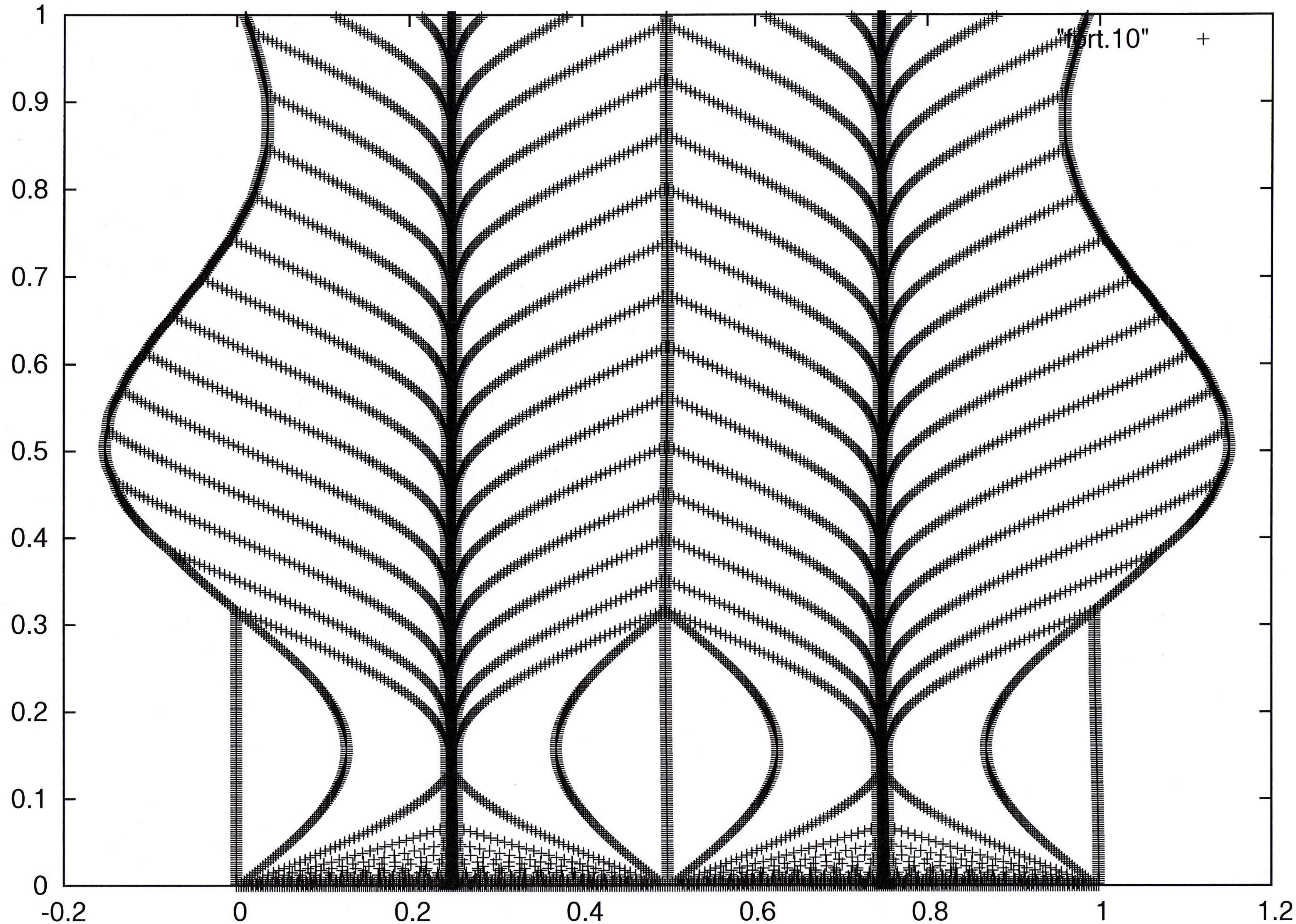
$$-\partial_t X \in \partial_y U + \partial 1_K(X)$$
$$\partial_t U = \partial_y X$$
$$X = X(t,y,a) \qquad 1_K(X) = \begin{cases} 0 & \text{if } \partial_a X \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

Dichionary
$$\begin{cases} v(t, X(t,y,a), y) = \partial_t X(t,y,a) \\ b(t, X(t,y,a), y) = \partial_y X(t,y,a) \\ \rho(t, X(t,y,a), y) = 1/\partial_a X(t,y,a) \end{cases}$$

Pressureless Euler-Poisson

$$-\partial_t X \in Y + \partial 1_K(X)$$
$$\partial_t Y = a - X$$
$$X = X(t,a) \qquad 1_K(X) = \begin{cases} 0 & \text{if } \partial_a X \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

$$v(t, X(t,a)) = \partial_t X(t,a)$$
$$\rho(t, X(t,a)) = 1/\partial_a X(t,a)$$
$$\partial_x \varphi(t, X(t,a)) = X(t,a) - a$$



"fort.10"
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0.9
0.8
0.7
0.6
0.5
0.4
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0.2
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0
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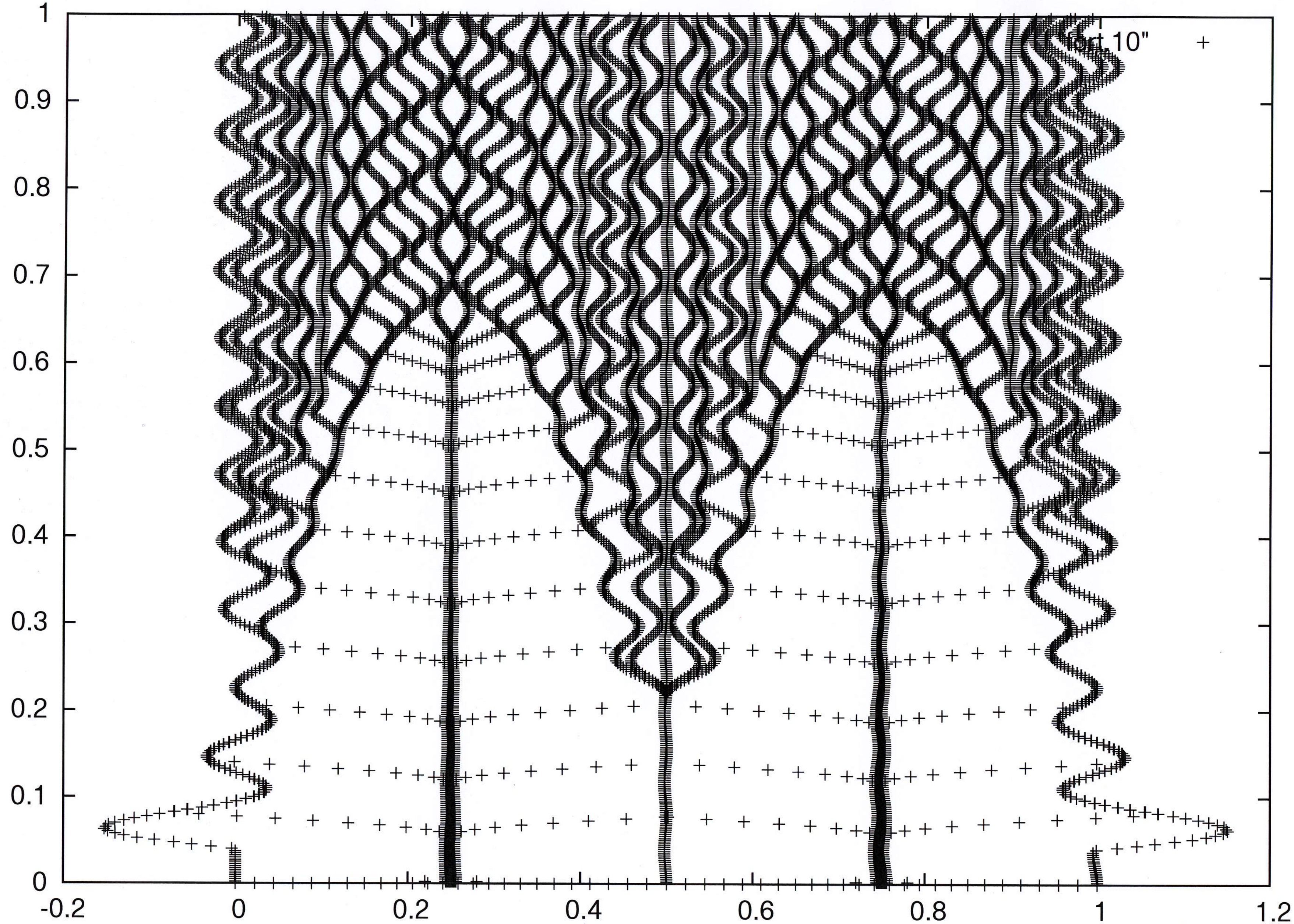
0
0.2
0.4
0.6
0.8
1
1.2



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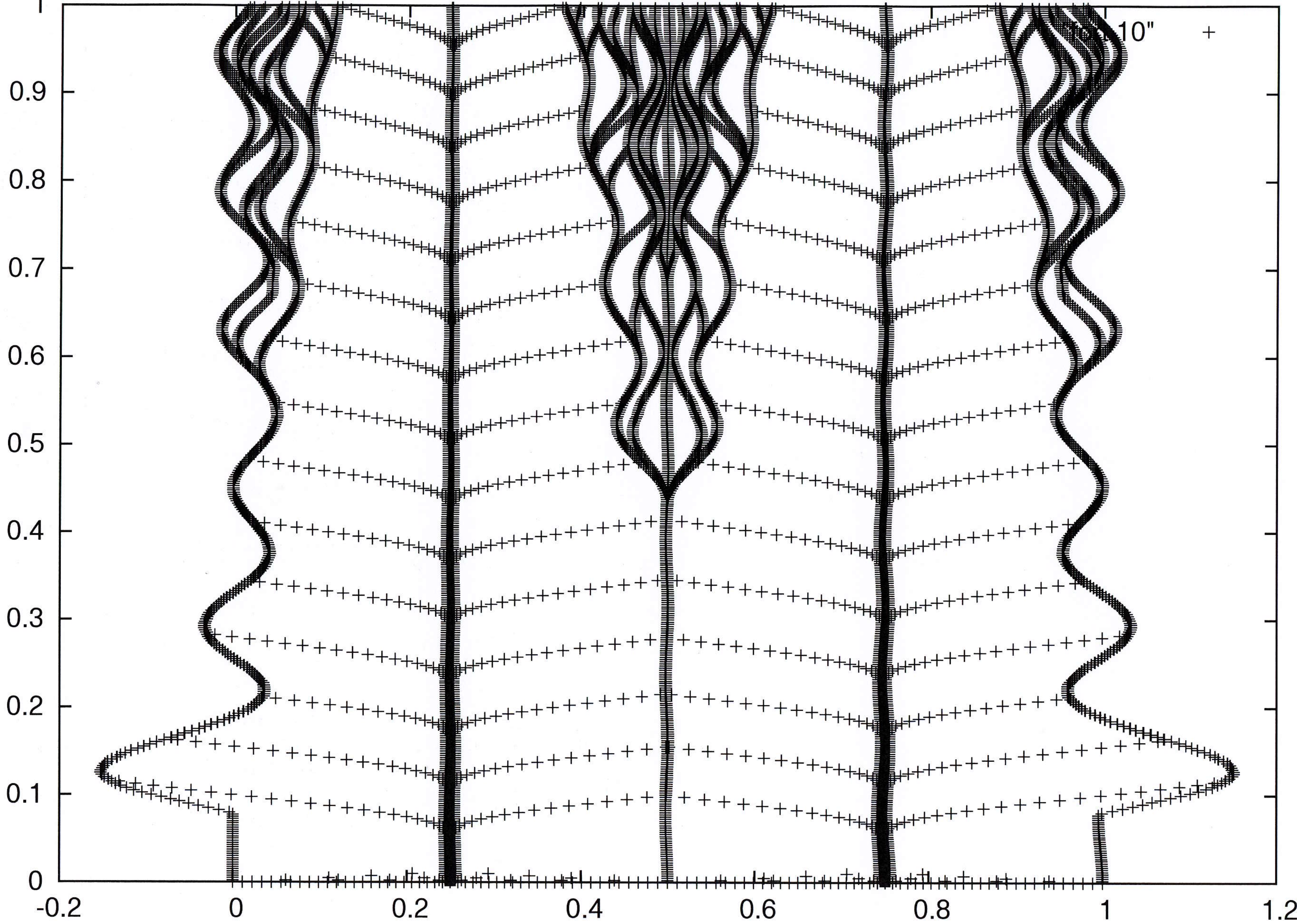
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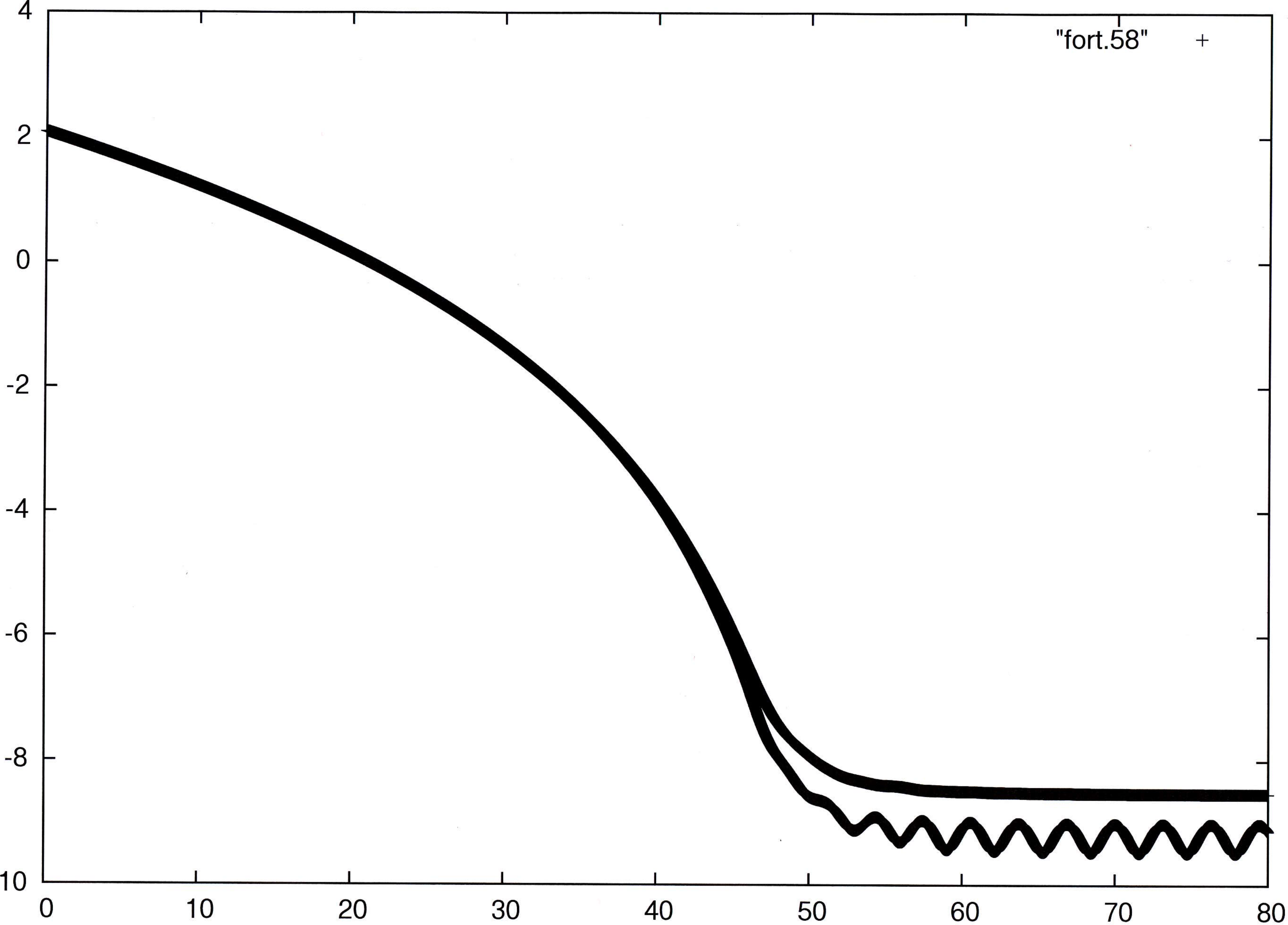

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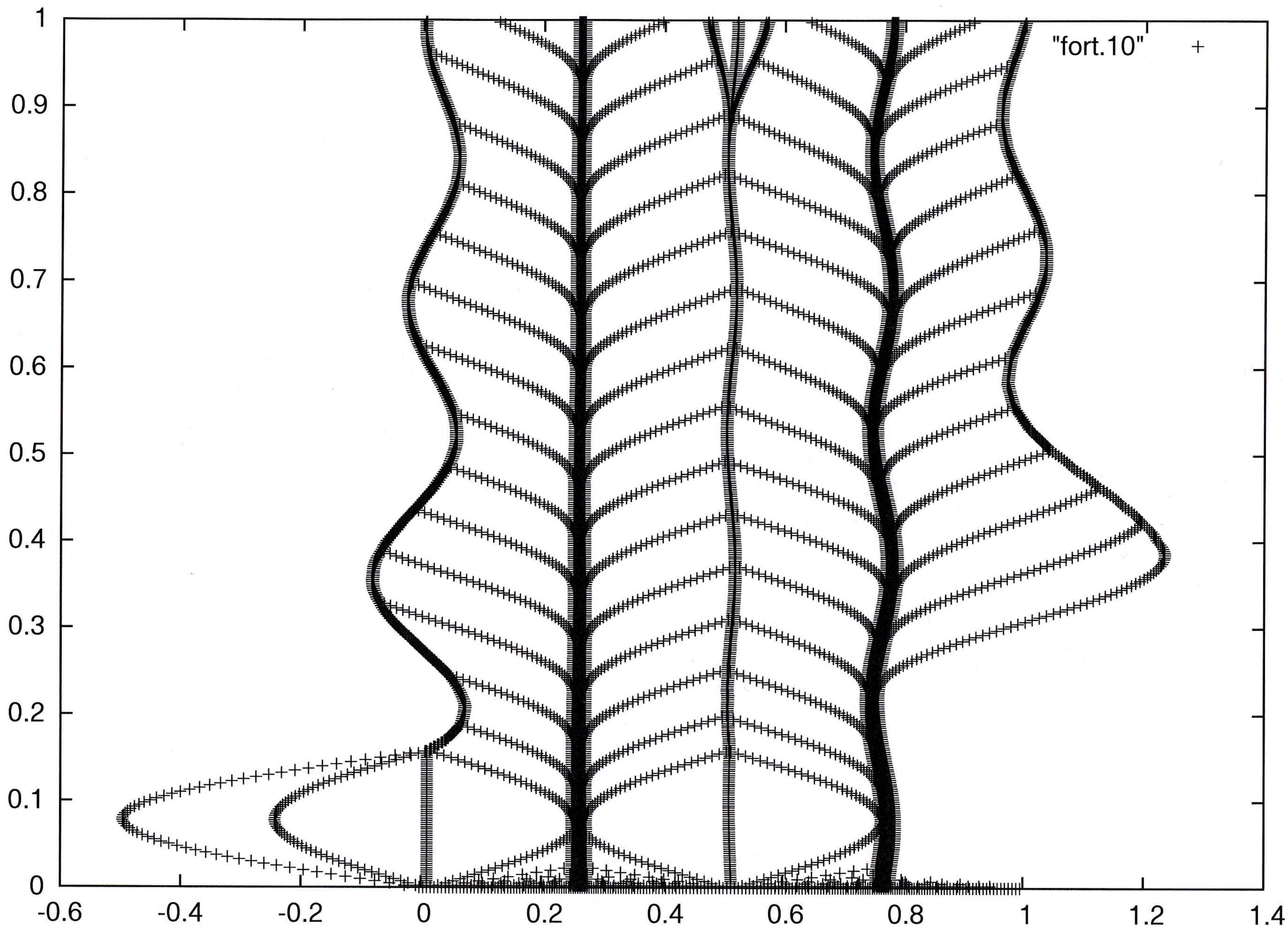
10
20
30
40
50
60
70
80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