ON A RELAXATION APPROXIMATION OF THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. We consider an hyperbolic singular perturbation of the incompressible Navier Stokes equations in two space dimensions. The approximating system under consideration, arises as a diffusive rescaled version of a standard relaxation approximation for the incompressible Euler equations. The aim of this work is to give a rigorous justification of its asymptotic limit toward the Navier Stokes equations using the modulated energy method.

1 Introduction

1 1
$$\begin{cases} \partial_t u \quad \nabla \cdot \; u \otimes u \qquad \nabla \phi, \\ \nabla \cdot u \quad 0, \\ u \;\; 0, x \qquad u_0 \;\; x \;\; , \end{cases}$$

$t, x \;\; \in \;\; 0, T \;\; \times \mathbb{T}^2$ $\mathbb{T}^2$ $\mathbb{R}^2/\mathbb{Z}^2$ u ϕ $\mathbb{R}^2$ 1

$u \otimes u$ $V \;\; \mathbb{T}^2 \to \mathbb{R}^4$

1 2
$$\begin{cases} \partial_t u \quad \nabla \cdot \;\; V \qquad \nabla \phi, \\ \partial_t V \quad a \nabla u \quad -\frac{1}{\eta} \; V - u \otimes u \;\; , \\ \nabla \cdot u \quad 0, \\ u \;\; 0, x \qquad u_0 \;\; x \;\; , V \;\; 0, x \qquad V_0 \;\; x \;\; . \end{cases}$$

η 1 1

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$\varepsilon > 0$

$$\begin{cases} u^\varepsilon \; t,x & \dfrac{1}{\sqrt{\varepsilon}} u\left(\dfrac{x}{\sqrt{\varepsilon}}, \dfrac{t}{\varepsilon}\right), \\ V^\varepsilon \; x,t & \dfrac{1}{\varepsilon} V\left(\dfrac{x}{\sqrt{\varepsilon}}, \dfrac{t}{\varepsilon}\right), \\ \phi^\varepsilon \; x,t & \dfrac{1}{\varepsilon}\phi\left(\dfrac{x}{\sqrt{\varepsilon}}, \dfrac{t}{\varepsilon}\right). \end{cases} \quad (1)$$

1 2 η 1

$$\begin{cases} \partial_t u^\varepsilon \quad \nabla \cdot \; V^\varepsilon \quad \nabla\phi^\varepsilon, \\ \sqrt{\varepsilon}\partial_t V^\varepsilon \quad \dfrac{a}{\sqrt{\varepsilon}}\nabla u^\varepsilon \quad -\dfrac{1}{\sqrt{\varepsilon}} \; V^\varepsilon - u^\varepsilon \otimes u^\varepsilon \;, \\ \nabla \cdot u^\varepsilon \quad 0, \\ u^\varepsilon \; 0,x \quad u_0^\varepsilon \; x \;, V^\varepsilon \; 0,x \quad V_0^\varepsilon \; x \;. \end{cases} \quad (1)$$

1 ε 0

$$\begin{cases} \partial_t U \quad \nabla \cdot \; U \otimes U \; - a \; U \quad \nabla\phi, \\ \nabla \cdot U \quad 0, \\ U \; 0,x \quad U_0 \; x \;. \end{cases} \quad (1)$$

$u^\varepsilon \to U$ $\varepsilon V^\varepsilon \to 0$ $u^\varepsilon \otimes u^\varepsilon \to U \otimes U$

$L^\infty \; 0,T \;, L^2 \; \mathbb{T}^2$ T

$\frac{x}{\sqrt{\varepsilon}}, \frac{t}{\varepsilon}$

11 20

1 1 1
1 1 12

1

1

1

hyperbolic
heat equation V 1

$$\begin{cases} \partial_t u^\varepsilon \quad P \; \nabla \cdot \; u^\varepsilon \otimes u^\varepsilon \quad - a \quad u^\varepsilon \quad \varepsilon \partial_{tt} u^\varepsilon \quad 0, \\ \nabla \cdot u^\varepsilon \quad 0, \end{cases} \quad (1)$$

P

21

2

2

22

2

2 Analytical backgrounds and statements

1

Theorem 2.1. *Suppose the initial data $u_0^\varepsilon x, V_0^\varepsilon x$ are smooth functions belonging to H^s for $s \geq 2$. Then, there exists a positive time T^ε, which depends only on the initial data, and a solution $u^\varepsilon, V^\varepsilon, \phi^\varepsilon \in C \; 0 T \; H^{s} \; 3$ to system (1.4). Moreover, if $T^\varepsilon < \infty$, then*

2 1 $$\underset{t\to T^\varepsilon_-}{} || \; u^\varepsilon, V^\varepsilon \; ||_{H^2} \to \infty.$$

10

$$|u|_{H^2(\mathbb{T}^2)} \quad ||u||_{L^2(\mathbb{T}^2)} \quad || \quad u||_{L^2(\mathbb{T}^2)} \quad ||\nabla \quad u \; ||_{L^2(\mathbb{T}^2)}.$$

$\nabla\cdot u \quad 0$ H^2

C_0 $C_0 < \sqrt{a}$ K_s

$|\cdot|_{H^2(\mathbb{T}^2)}$

u^ε ε

Theorem 2.2. *Let $T \geq 0$ and U^0 be a smooth divergence free vector field on $\mathbb{T}^2$. Let also $u_0^\varepsilon, V_0^\varepsilon$ be a sequence of smooth initial data on $\mathbb{T}^2$ for problem (1.4). Assume moreover that there exists a constant C independent of ε such that*

2 2 $$||u_0^\varepsilon||_{H^1(\mathbb{T}^2)} \leq C$$

2 $$||V_0^\varepsilon||_{H^2(\mathbb{T}^2)} \leq \frac{C}{\sqrt{\varepsilon}}$$

2 $$|u_0^\varepsilon|_{H^2(\mathbb{T}^2)} < \frac{C_0}{K_s\sqrt{\varepsilon}}$$

2 $$\int_{\mathbb{T}^2} |u_0^\varepsilon \; x \; - U^0 \; x \; |^2 dx \leq C\sqrt{\varepsilon}.$$

Then, u^ε is a global solution of the relaxed system (1.4) and converges, as $\varepsilon \to 0$, in L^∞ $0, T$, L^2 $\mathbb{T}^2$ towards the (unique smooth) solution U of the incompressible Navier Stokes equations (1.5) with U^0 as initial data. In addition

$$t\in[0,T] \int_{\mathbb{T}^2} |u^\varepsilon - U|^2 dx \leq C_T \sqrt{\varepsilon},$$

where C_T depends only on T, U, C and C_0.

Proof of the theorem

1 **Preliminaries.**

L^∞ u^ε

1 1 *The energy estimate.*

Proposition 3.1. *Assume that there exists $T > 0$ such that $||u^\varepsilon||_{L^\infty} \leq \sqrt{\frac{a}{\varepsilon}}$ for all $t \leq T$. Then, setting w^ε curl u^ε, we have the following estimates*

1 $$\frac{d}{dt}\int \frac{1}{2}|u^\varepsilon \quad \varepsilon\partial_t u^\varepsilon|^2 \quad \varepsilon^2|\partial_t u^\varepsilon|^2 \quad \varepsilon a|\nabla u^\varepsilon|^2 \ dx \leq 0,$$

and

2 $$\frac{d}{dt}\int \frac{1}{2}|w^\varepsilon \quad \varepsilon\partial_t w^\varepsilon|^2 \quad \varepsilon^2|\partial_t w^\varepsilon|^2 \quad \varepsilon a|\nabla w^\varepsilon|^2 \ dx \leq 0,$$

for all $t \leq T$.

Proof. 1 u^ε $2\varepsilon\partial_t u^\varepsilon$

$$\partial_t u \partial_{tt} u \quad \partial_t \ u\partial_t u \ - \ \partial_t u \ ^2$$

$$\frac{d}{dt}\int \frac{1}{2}|u^\varepsilon \quad \varepsilon\partial_t u^\varepsilon|^2 \quad \varepsilon^2|\partial_t u^\varepsilon|^2 \quad \varepsilon a|\nabla u^\varepsilon|^2 \ dx$$

$$\varepsilon \int |\partial_t u^\varepsilon \quad \nabla\cdot \ u^\varepsilon \otimes u^\varepsilon \ |^2 dx \quad \int \left(a|\nabla u^\varepsilon|^2 - \varepsilon|\nabla\cdot \ u^\varepsilon\otimes u^\varepsilon \ |^2\right) dx \quad 0.$$

$||u^\varepsilon||_{L^\infty} \leq \sqrt{\frac{a}{\varepsilon}}$ 1

w^ε

$$w \quad \partial_2 u_1 - \partial_1 u_2$$

$$\partial_t w^\varepsilon \quad u^\varepsilon\cdot\nabla w^\varepsilon - a \quad w^\varepsilon \quad \varepsilon\partial_{tt} w^\varepsilon \quad 0.$$

w^ε $2\varepsilon\partial_t w^\varepsilon$

$$\frac{d}{dt}\int \frac{1}{2}|w^\varepsilon \quad \varepsilon\partial_t w^\varepsilon|^2 \quad \varepsilon^2|\partial_t w^\varepsilon|^2 \quad \varepsilon a|\nabla w^\varepsilon|^2 \ dx$$

$$\varepsilon \int |\partial_t w^\varepsilon \quad u^\varepsilon\cdot\nabla w^\varepsilon|^2 \quad \int a|\nabla w^\varepsilon|^2 - \varepsilon|u^\varepsilon\cdot\nabla w^\varepsilon|^2 \quad 0.$$

□

1 2 L^∞ *bounds.* L^∞ u^ε

Proposition 3.2. *Under the assumptions of Theorem 2.2, if*

$$|u_0^\varepsilon|_{H^2(\mathbb{T}^2)} < \frac{C_0}{K_s\sqrt{\varepsilon}},$$

where C_0 is a given positive constant such that $C_0 < \sqrt{a}$, then the solution u^ε verifies the following estimate

$$||u^\varepsilon||_{L^\infty} \leq \frac{C_0}{\sqrt{\varepsilon}},$$

for all positive t and, therefore is global.

Proof. δ $\delta < \sqrt{a} - C_0$

$$T^\delta \quad \{0 \leq t \leq T \quad_{0\leq\tau\leq t} ||u^\varepsilon \ \tau \ ||_{L^\infty} \leq \frac{C_0 \quad \delta}{\sqrt{\varepsilon}}\}.$$

$|u_0^\varepsilon|_{H^2(\mathbb{T}^2)} < \frac{C_0}{K_s\sqrt{\varepsilon}}$ $||u_0^\varepsilon||_{L^\infty} < \frac{C_0+\delta}{\sqrt{\varepsilon}}$
$u^\varepsilon \in C^0 \quad 0, T \ , L^\infty \ \mathbb{T}^2$ $T^\delta > 0$
T^δ T $T^\delta < T$

$$||u^\varepsilon \ T^\delta \ ||_{L^\infty} \quad \frac{C_0 \quad \delta}{\sqrt{\varepsilon}} < \sqrt{\frac{a}{\varepsilon}}.$$

$\mu > 0$ $t \leq T^\delta \quad \mu$ $||u^\varepsilon \ t \ ||_{L^\infty} \leq \sqrt{\frac{a}{\varepsilon}}$
$t \leq T^\delta \quad \mu$ 1 2

$$||u^\epsilon \ T^\delta \ ||_{L^2} \quad ||w^\epsilon \ T^\delta \ ||_{L^2} \leq C,$$

$$|u^\epsilon \ T^\delta \ |_{H^2} \leq \frac{C}{\sqrt{\epsilon a}}.$$

L^2 w
u H^1 u

$$||u^\epsilon \ T^\delta \ ||_{L^\infty} \leq C \ 1 \quad log^+ \ \epsilon a \quad ,$$

□

2 Convergence.

U
U^0 $\frac{1}{2}\int |u^\varepsilon - U|^2 dx \leq C_T\sqrt{\varepsilon}$

2 1 *Definition and properties of the modulated energy.*

$$E^\varepsilon \ t \qquad \int \frac{1}{2}|u^\varepsilon \quad \varepsilon\partial_t u^\varepsilon|^2 \quad \varepsilon^2|\partial_t u^\varepsilon|^2 \quad \varepsilon a|\nabla u^\varepsilon|^2 \ dx.$$

v

$$E_v^\varepsilon \ t \qquad \int \frac{1}{2}|u^\varepsilon - v \ t, x \quad \varepsilon\partial_t u^\varepsilon|^2 \quad \varepsilon^2|\partial_t u^\varepsilon|^2 \quad \varepsilon a|\nabla u^\varepsilon|^2 \ dx.$$

Proposition 3.3. *The modulated energy satisfies the identity*

$$\frac{d}{dt}E_v^\varepsilon \ t \qquad \int v \cdot \nabla \quad u^\varepsilon - v \ \otimes \ u^\varepsilon - v \qquad \int \partial_t v \quad v\cdot\nabla v - a \quad v \quad v - u^\varepsilon$$

$$-\varepsilon \int |\partial_t u^\varepsilon \quad \nabla \cdot \ u^\varepsilon \otimes u^\varepsilon \ |^2 - \varepsilon \int \partial_t v \cdot \partial_t u^\varepsilon$$

$$-a \int |\nabla \ u^\varepsilon - v \ |^2 \quad \varepsilon \int |\nabla \ u^\varepsilon \otimes u^\varepsilon \ |^2.$$

Proof.

$$\frac{d}{dt}E_v^\varepsilon\ t \qquad \frac{d}{dt}E\ t\ -\int v\cdot\partial_t u^\varepsilon - \int \partial_t v\cdot u^\varepsilon - \varepsilon\int\partial_t v\cdot\partial_t u^\varepsilon - \varepsilon\int v\cdot\partial_{tt}u^\varepsilon \quad \int v\partial_t v.$$

1

$$\int v\cdot\nabla \quad u^\varepsilon\otimes u^\varepsilon \qquad \int v\cdot\nabla\quad u^\varepsilon - v\ \otimes\ u^\varepsilon - v \qquad \int v\cdot\nabla\quad u^\varepsilon\otimes v$$

$$\int v\cdot\nabla\quad v\otimes u^\varepsilon\ -\int v\cdot\nabla\quad v\otimes v$$

$$\frac{d}{dt}E_v^\varepsilon\ t \qquad \int v\cdot\nabla\quad u^\varepsilon - v\ \otimes\ u^\varepsilon - v \qquad \int\partial_t v\quad v\cdot\nabla v\quad v-u^\varepsilon$$

10 $$-\varepsilon\int|\partial_t u^\varepsilon\quad\nabla\cdot\ u^\varepsilon\otimes u^\varepsilon\ |^2 - \varepsilon\int\partial_t v\cdot\partial_t u^\varepsilon$$

$$a\int\quad u^\varepsilon\ u^\varepsilon - v \qquad \varepsilon\int|\nabla\ u^\varepsilon\otimes u^\varepsilon\ |^2.$$

$$a\int\quad u^\varepsilon\ u^\varepsilon - v \qquad a\int\quad u^\varepsilon - v\quad u^\varepsilon - v \qquad a\int\quad v\ u^\varepsilon - v\ ,$$

□

2 2 *Proof of Theorem 2.2.*

$$\int|u^\varepsilon|^2 \le CE^\varepsilon\ t\ \le CE^\varepsilon\ \mathbf{0}\ \le C.$$

$$\int|u^\varepsilon - v|^2dx \le CE_v^\varepsilon\ t\ .$$

v U U
$\mathbf{1}$ U^0

$$\tfrac{d}{dt}E_v^\varepsilon\ t\ \le CE_v^\varepsilon\ t\ -\varepsilon\int\partial_t v\cdot\partial_t u^\varepsilon - a\int|\nabla\ u^\varepsilon - v\ |^2\quad\varepsilon|u^\varepsilon\cdot\nabla u^\varepsilon|^2.$$

v

CE_v^ε v

$$-\varepsilon\int\partial_t v\cdot\partial_t u^\varepsilon \qquad -\varepsilon\frac{d}{dt}\int\partial_t v\cdot u^\varepsilon \qquad \varepsilon\int\partial_{tt}v\cdot u^\varepsilon,$$

ε

$$A^\varepsilon \qquad -a\int|\nabla\ u^\varepsilon - v\ |^2 \qquad \varepsilon\int|u^\varepsilon\cdot\nabla u^\varepsilon|^2$$

$\varepsilon\to\mathbf{0}$

$$\varepsilon\int|u^\varepsilon\cdot\nabla u^\varepsilon|^2 \le \varepsilon\ \mathbf{1}\quad\theta\ \int|u^\varepsilon\cdot\nabla\ u^\varepsilon - v\ |^2 \quad \varepsilon\ \mathbf{1}\quad\frac{\mathbf{1}}{\theta}\ \int|u^\varepsilon\cdot\nabla v|^2.$$

$||u^\varepsilon||_{L^\infty} \le \frac{\sqrt{a}}{\sqrt{\varepsilon}}$

$$A^\varepsilon \le \theta a\int|\nabla\ u^\varepsilon - v\ |^2 \quad \varepsilon\ \mathbf{1}\quad\frac{\mathbf{1}}{\theta}\ \int|u^\varepsilon\cdot\nabla v|^2.$$

$$A^{\varepsilon} \le \theta a||\nabla u^{\varepsilon}||^2_{L^2(\mathbb{T}^2)} \quad \varepsilon \;\; 1 \quad \frac{1}{\theta} \; c||u^{\varepsilon}||^2_{L^2(\mathbb{T}^2)}.$$

1 2

$$\int |u^{\varepsilon}|^2 \qquad \int |\nabla \;\; u^{\varepsilon} - v \;\; |^2 \le C.$$

θ $\sqrt{\varepsilon}$ A^{ε} $O \; \sqrt{\varepsilon}$ ε

$$\frac{d}{dt} \; E_v \;\; t \qquad O \;\; \varepsilon \quad \le C E_v \;\; t \qquad O \;\; \sqrt{\varepsilon} \;\; .$$

$$E_v \;\; 0 \qquad O \;\; \sqrt{\varepsilon} \;\; .$$

$$_{t\in[0,T]} \int |u^{\varepsilon} - v|^2 dx \le C E_v^{\varepsilon} \le C_T \sqrt{\varepsilon},$$

C_T T v □

References

[1] D. Aregba-Driollet, R. Natalini, and S.Q. Tang. Diffusive kinetic explicit schemes for nonlinear degenerate parabolic systems. Quaderno IAC 26/2000, to appear in Math. Comp..

[2] M.K.Banda, A.Klar, L.Pareschi, M.Seaid. Lattice-Boltzmann type relaxation systems and high order relaxation schemes for the incompressible Navier-Stokes equations Preprint (2001).

[3] Y. Brenier, Convergence of the Vlasov-Poisson system to the incompressible Euler equations. *Comm. Partial Differential Equations* 25 (2000), no. 3-4, 737–754.

[4] F. Bouchut, F. Guarguaglini, and R. Natalini. Discrete kinetic approximation to multidimensional parabolic equations. *Indiana Univ. Math. J.*, 49:723–749, 2000.

[5] H. Brezis, T. Gallouet. Nonlinear Schrödinger evolution equations. Nonlinear Anal. 4 (1980), no. 4, 677-681.

[6] C. Cattaneo. Sulla conduzione del calore. Atti Sem. Mat. Fis. Univ. Modena 3, (1949). 83–101.

[7] D. Donatelli and P. Marcati. Convergence of singular limits for multi-D semilinear hyperbolic systems to parabolic systems. Technical Report 12, Scuola Normale Superiore, Pisa, 2000.

[8] F. Bouchut and F. Golse and M. Pulvirenti. Kinetic equations and asymptotic theory. Series in Appl. Math., Gauthiers-Villars, 2000.

[9] E. Grenier. Boundary layers of 2D inviscid fluids from a Hamiltonian viewpoint. *Math. Res. Lett.* 6 (1999), 257–269.

[10] S. Klainerman. Global existence for nonlinear wave equations. *Comm. Pure Appl. Math.* 33 (1980), no. 1, 43–101.

[11] T. Kurtz. Convergence of sequences of semigroups of nonlinear operators with an application to gas kinetics. *Trans. Amer. Math. Soc.*, 186:259–272, 1973.

[12] S. Jin, H.L. Liu, Diffusion limit of a hyperbolic system with relaxation. *Meth. and Appl. Anal.* 5 (1998), 317-334.

[13] S. Jin, L. Pareschi, G. Toscani. Diffusive relaxation schemes for multiscale discrete-velocity kinetic equations. *SIAM J. Numer. Anal.* 35 (1998), 2405–2439.

[14] S. Jin and Z. Xin. The relaxation schemes for systems of conservation laws in arbitrary space dimensions. *Comm. Pure Appl. Math.* 48 (1995), 235–277.

[15] P.L. Lions, G. Toscani. Diffusive limit for finite velocity Boltzmann kinetic models, *Rev. Mat. Iberoamericana* 13 (1997), 473–513.

[16] H. Liu and R. Natalini. Long-Time Diffusive Behavior of Solutions to a Hyperbolic Relaxation System. *Asymptot. Anal.* 25 (2001), no. 1, 21–38.

[17] P. Marcati and A. Milani. The one-dimensional Darcy's law as the limit of a compressible Euler flow. *J. Differential Equations*, 84:129–147, 1990.

[18] P. Marcati, A. Milani, and P. Secchi. Singular convergence of weak solutions for a quasilinear nonhomogeneous hyperbolic system. *Manuscripta Math.*, 60:49–69, 1988.

[19] P. Marcati and B. Rubino. Hyperbolic to Parabolic Relaxation Theory for Quasilinear First Order Systems. *J. Differential Equations*, 162:359–399, 2000.

[20] H.P. McKean. The central limit theorem for Carleman's equation. *Israel J. Math.*, 21(1):54–92, 1975.

[21] J.C. Saut. Some remarks on the limit of viscoelastic fluids as the relaxation time tends to zero. Trends in applications of pure mathematics to mechanics (Bad Honnef, 1985), 364–369, Lecture Notes in Phys., 249, Springer, Berlin, 1986.

[22] A.E. Tzavaras. Materials with internal variables and relaxation to conservation laws. *Arch. Ration. Mech. Anal.* 146 (1999) 129–155.

[23] H.T. Yau. Relative entropy and hydrodynamics of Ginzburg-Landau models. *Lett. Math. Phys.* 22 (1991), 63–80.