

Hamiltonian manifolds and moment map.

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July 4, 2011

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These lecture notes are a preliminary draft, based on courses taught by M. Vergne at University Paris 7 and N. Berline at Fudan University.

Introduction

By a well-known result of Issai Schur (1923) [22], the diagonal elements $(\alpha_1, \dots, \alpha_n)$ of an $n \times n$ Hermitian matrix A satisfy a system of linear inequalities involving the eigenvalues $(\lambda_1, \dots, \lambda_n)$. In geometric terms, regarding α and λ as points in \mathbb{R}^n and allowing the symmetric group S_n to act by permutation of coordinates, this result takes the form:

α is in the convex hull of the points $S_n \cdot \lambda$.

The converse was proved by A. Horn (1954)[10], so that this convex hull is exactly the set of diagonals of Hermitian matrices A with the given eigenvalues $(\lambda_1, \dots, \lambda_n)$.

B.Kostant generalized these results to any compact Lie group G in the following manner [16]. Consider the coadjoint action of G on the dual \mathfrak{g}^* of its Lie algebra \mathfrak{g} . Let $H \subseteq G$ be a maximal torus, with Lie algebra \mathfrak{h} . Restriction to \mathfrak{h} defines a projection $\mathfrak{g}^* \rightarrow \mathfrak{h}^*$. The Weyl group W acts on \mathfrak{h} and \mathfrak{h}^* . Kostant's theorem is

Theorem. *Let $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit under G . Then the projection of \mathcal{O} on \mathfrak{h}^* is the convex hull of a W -orbit.*

Schur-Horn's theorem is the particular case where G is the unitary group $U(n)$ and H is the subgroup of diagonal matrices. Then \mathfrak{g} is the Lie algebra of anti-Hermitian matrices. It is identified to its dual \mathfrak{g}^* by means of the G -invariant scalar product $\text{Tr}(AB)$. Then the projection of $A \in \mathfrak{g}^*$ on \mathfrak{h}^* is given by the diagonal of A .

This convexity theorem has been widely generalized (Atiyah [1], Guillemin-Sternberg [8], Kirwan [12], etc.). As we will see, the general relevant framework is that of a symplectic manifold M with a Hamiltonian action of a Lie group H . The projection $\mathcal{O} \rightarrow \mathfrak{h}^*$ is a particular case of the *moment map*

$$M \rightarrow \mathfrak{h}^*.$$

This moment map plays a key role in topics such as Geometric Invariant Theory, Geometric Quantization of a classical mechanical system, Moduli Varieties (which are related to infinite dimensional Hamiltonian spaces), etc..

1 Setup of Hamiltonian manifolds

To establish notations, we review some basic notions of differential geometry.

1.1 Tangent and normal vector bundle

Let M be a smooth manifold. The tangent bundle is denoted by TM . The space of smooth sections of TM is denoted by $\Gamma(M, TM)$. An element of $\Gamma(M, TM)$ is a smooth vector field on M . If X is a tangent vector at the point $m \in M$, and ϕ a smooth function on M defined near m , we denote by $(X.\phi)(m)$ the derivative of ϕ at m in the direction X . If $x(t)$ is a smooth curve on M starting at $x(0) = m$ with $\dot{x}(0) = X$, then

$$(X.\phi)(m) = \left. \frac{d}{dt} \phi(x(t)) \right|_{t=0}.$$

If X is a smooth vector field, then $X.\phi$ is again a smooth function. (From now on, we will often omit the word smooth). Thus, a vector field X on M defines a *derivation* of $C^\infty(M)$, i.e. it obeys the *Leibniz Rule*

$$X.(\phi_1\phi_2) = (X.\phi_1)\phi_2 + \phi_1(X.\phi_2).$$

Any derivation of $C^\infty(M)$ corresponds in this way to a vector field X , and it is denoted by the same letter X . The Lie *bracket* $[X, Y]$ of two vector fields is the vector field which corresponds to the derivation $X \circ Y - Y \circ X$. In this way, the space of vector fields on M is a Lie algebra, i.e. the Jacobi Identity holds

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

The Leibniz Rule also holds for the product of a scalar function ϕ with a vector field Y , i.e.

$$[X, \phi Y] = (X.\phi)Y + \phi[X, Y].$$

Let $f : M \rightarrow M'$ be a (smooth) map between two manifolds. For each point $m \in M$, the differential (also called derivative) of f at m is the linear map $df_m : T_m M \rightarrow T_{f(m)} M'$ defined by composing the curves in M starting at m with the map f .

Definition 1. Let $N \subseteq M$ be a closed submanifold of M . The normal bundle to N in M is the vector bundle over N with fiber $T_m M / T_m N$ for every point $m \in N$. The normal bundle is denoted by $\mathcal{N}(M/N)$ or simply \mathcal{N} .

The manifold N is identified with the zero section of the normal bundle \mathcal{N} . We admit the *Tubular Neighborhood theorem*, due to Jean-Louis Koszul.

Proposition 2. *Let $N \subseteq M$ be a closed submanifold of M . Let \mathcal{N} be the normal bundle to N in M . There exists a diffeomorphism from an open neighborhood of the zero section in \mathcal{N} onto an open neighborhood of N in M , which is the identity map on N .*

1.2 Calculus on differential forms

1.2.1 de Rham differential

The \mathbb{Z} -graded algebra of (smooth) differential forms on a manifold M is denoted by

$$\mathcal{A}(M) = \bigoplus_{k=0}^{\dim M} \mathcal{A}^k(M).$$

Here, we consider real-valued differential forms. Elements of $\mathcal{A}^k(M)$ will be called *homogeneous* of exterior degree k , or simply k -forms. A 0-form is just a real-valued function on M , and a k -form is a section of the vector bundle $\wedge^k TM$, the k th exterior power of TM .

If α and β are homogeneous, the exterior product $\alpha \wedge \beta$ satisfies the *sign rule* (called *super-commutativity* or *graded commutativity*)

$$\alpha \wedge \beta = (-1)^{\deg \alpha \deg \beta} \beta \wedge \alpha.$$

If α is a k -form and X_1, \dots, X_k are k vector fields, then $\alpha(X_1, \dots, X_k)$ is a function on M . It is alternate ($\alpha(X_1, \dots, X_k) = 0$ if two vectors X_i are equal), hence antisymmetric, and it is multilinear with respect to multiplication of the vector fields by scalar functions:

$$\alpha(\phi X_1, \dots, X_k) = \phi \alpha(X_1, \dots, X_k).$$

The *exterior* or *de Rham differential* d is the unique operator on $\mathcal{A}(M)$ such that

(1) If $\phi \in C^\infty(M)$ then $d\phi$ is the 1-form given by the differential of ϕ , $d\phi(X) = X.\phi$.

(2) If $\phi \in C^\infty(M)$ then $d(d\phi) = 0$.

(3) d satisfies the graded Leibniz Rule (one also says that d is a graded derivation of degree 1). If α is homogeneous then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta.$$

From these rules follows the important relation

$$d \circ d = 0.$$

It also follows that the exterior differential d increases the exterior degree by 1.

In particular, if ϕ and x_i are functions on M , one has

$$d(\phi dx_1 \wedge \cdots \wedge dx_k) = d\phi \wedge dx_1 \wedge \cdots \wedge dx_k$$

If α is a $k - 1$ -form and X_1, \dots, X_k are vector fields, then

$$\begin{aligned} d\alpha(X_1, \dots, X_k) &= \sum_{j=1}^k (-1)^{j-1} X_j \cdot \alpha(X_1, \dots, \hat{X}_j, \dots, X_k) + \\ &\quad \sum_{1 \leq i < j \leq k} (-1)^{i+j-1} \alpha([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k) \end{aligned}$$

where \hat{X}_j means that the j th term is omitted. Despite its look, this formula does define a differential form, i.e. is $C^\infty(M)$ linear. For instance, for $k = 1$, the formula reads

$$d\alpha(X, Y) = X \cdot \alpha(Y) - Y \cdot \alpha(X) - \alpha([X, Y]).$$

The $C^\infty(M)$ linearity follows from the Leibniz rule for the Lie bracket.

If $d\alpha = 0$ then the form α is called *closed*. If $\alpha = d\beta$ then the form α is called *exact*. The relation $d \circ d = 0$ implies that exact forms are closed. In other words, for every degree k , one has $d\mathcal{A}^k \subseteq \ker d|_{\mathcal{A}^k}$. The quotient vector space is called the k th space of *de Rham cohomology* and denoted by $H^k(M)$.

$$H^k(M) = (\ker d|_{\mathcal{A}^k}) / d\mathcal{A}^k$$

If U is an open ball in \mathbb{R}^n , then any closed form on U is exact. This is the Poincaré Lemma, which we will prove later in Section ??.

1.2.2 Contraction by vector fields

The *contraction* $\iota(X)$ of a k -form α with a vector field X is the $(k-1)$ -form defined by

$$(\iota(X).\alpha)(X_1, \dots, X_{k-1}) = \alpha(X, X_1, \dots, X_{k-1}).$$

Thus $\iota(X)$ is the unique operator on $\mathcal{A}(M)$ which is $C^\infty(M)$ linear and which satisfies the following two rules.

- If α is a 1-form

$$\iota(X)\alpha = \alpha(X).$$

- The *graded-Leibniz Rule*. if α is homogeneous, then

$$(\iota(X).(\alpha \wedge \beta)) = (\iota(X).\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge (\iota(X).\beta).$$

One has

$$\begin{aligned} \iota(X) \circ \iota(X) &= 0, \\ \iota(X_1) \circ \iota(X_2) + \iota(X_2) \circ \iota(X_1) &= 0, \end{aligned}$$

1.2.3 Lie derivative with respect to a vector field. Cartan's Homotopy Formula

Let X be a vector field on M . The *flow* $g_t(m)$ of X is the one parameter family of *local* diffeomorphisms of M defined by the differential equation with Cauchy condition:

$$g_0(m) = m$$

and

$$\frac{d}{dt}(g_t(m)) = X((g_t(m))).$$

For s, t small enough one has

$$g_s \circ g_t = g_{s+t}.$$

The local diffeomorphisms g_t act naturally on the various tensor fields on M as well as on the differential forms. By differentiating with respect to t at $t = 0$, one obtains the *Lie derivative* $\mathcal{L}(X)$. On functions, this is just the derivative with respect to the vector field itself

$$\mathcal{L}(X).\phi = X.\phi = d\phi(X).$$

On vector fields, it is given by the Lie bracket

$$\mathcal{L}(X).Y = [X, Y].$$

On tensors and differential forms, $\mathcal{L}(X)$ is a derivation (it satisfies the Leibniz rule).

$$\mathcal{L}(X)(\alpha \wedge \beta) = \mathcal{L}(X)\alpha \wedge \beta + \alpha \wedge \mathcal{L}(X)\beta$$

On differential forms, it preserves the degree and commutes with the exterior differential.

$$\mathcal{L}(X) \circ d = d \circ \mathcal{L}(X).$$

If α is a k -form, then

$$(\mathcal{L}(X)\alpha)(X_1, \dots, X_k) = X.(\alpha(X_1, \dots, X_k)) - \sum_{i=1}^k \alpha(X_1, \dots, [X, X_i], \dots, X_k).$$

Indeed, this formula can be interpreted as a Leibniz rule.

The relation between Lie derivatives and contraction is:

$$\iota(X) \circ \mathcal{L}(Y) - \mathcal{L}(Y) \circ \iota(X) = \iota([X, Y]),$$

Finally, one has the very useful Cartan Homotopy Formula

$$\mathcal{L}(X) = d \circ \iota(X) + \iota(X) \circ d.$$

Using the Leibniz rule, it suffices to check that this relation holds when applied to a function and a 1-form. For a function we have $\iota(X).\phi = 0$, $X.\phi = d\phi(X) = \iota(X).\phi$. For a 1-form, the homotopy formula is just the above formula for $d\alpha$.

1.3 Action of a Lie group on a manifold

Let G be a Lie group. The neutral element is denoted by e (or I if G is a matrix group). The Lie algebra of G is denoted by \mathfrak{g} . It is the tangent space $T_e G$. For $X \in \mathfrak{g}$, $\exp(tX)$ is the one-parameter subgroup of G with derivative at $t = 0$ equal to X .

If G acts on a set M , we will denote the action by $(g, m) \mapsto g.m$. If G acts (smoothly) on a manifold M , every element X in the Lie algebra gives rise to a vector field X^M on M , defined by

$$(X^M.\phi)(m) = \frac{d}{dt}\phi(\exp(-tX).m)|_{t=0}.$$

In other words, the flow of the vector field X^M is the one parameter group of global diffeomorphisms $m \mapsto \exp(-tX).m$. The reason for the minus sign is to make the map $X \mapsto X^M$ a Lie algebra homomorphism from \mathfrak{g} to the Lie algebra of vector fields on M

$$[X, Y]^M = [X^M, Y^M].$$

Example 3. Let V be a vector space and $A \in \text{End}(V)$, where $\text{End}(V)$ is considered as the Lie algebra of the group $\text{GL}(V)$. Then, with the above convention, for the natural action of $\text{GL}(V)$ on V , the vector field A^V is the linear vector field $A^V(v) = -A.v$.

The action of G on M gives rise naturally to a linear representation of G on each tensor space and each space $\mathcal{A}^k(M)$. For instance, on functions, the representation is

$$(g.\phi)(m) := \phi(g^{-1}m).$$

If we consider a vector field as a derivation on $C^\infty(M)$, the representation of G on vector fields is given by

$$(g.X).\phi := g.((X \circ g^{-1}).\phi).$$

On each tensor field, the Lie derivative $\mathcal{L}(X^M)$ is the operator which corresponds to X by the infinitesimal representation of \mathfrak{g} on this space.

Thus, a tensor or a differential form α is invariant under the one-parameter group $\exp tX$ if and only if $\mathcal{L}(X^M)\alpha = 0$.

1.4 Symplectic manifold. Hamiltonian action

1.4.1 Symplectic vector space

A symplectic vector space is a vector space V over R with a non degenerate alternate bilinear form B . Then V has even dimension $2n$, and there exists a basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ such that

$$B(e_i, e_j) = 0, \quad B(f_i, f_j) = 0, \quad B(e_i, f_j) = \delta_{i,j}$$

In other words, the matrix of B in the basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ is $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$.

The group of linear automorphisms of V which preserve B (*symplectic isomorphisms*) is denoted by $\text{Sp}(V, B)$ or simply $\text{Sp}(V)$. It is a closed subgroup

of $\text{GL}(V)$.

$$\text{Sp}(V) = \{g \in \text{GL}(V); B(gv, gw) = B(v, w) \text{ for all } v, w \in V\}$$

The Lie algebra of $\text{Sp}(V)$ is denoted by $\mathfrak{sp}(V)$.

$$\mathfrak{sp}(V) = \{X \in \text{End } V; B(Xv, w) = -B(v, Xw) \text{ for all } v, w \in V\}$$

Thus $X \in \text{Sp}(V)$ if and only if the bilinear form $B(Xv, w)$ on V is symmetric. In the basis $(e_1, \dots, e_n, f_1, \dots, f_n)$, the matrix of X has the form

$$\begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix},$$

where B and C are symmetric (n, n) matrices.

1.4.2 Symplectic form. Darboux coordinates

A symplectic manifold is a manifold M with a **closed** differential 2-form ω such that for every $m \in M$, the bilinear form ω_m on the tangent space $T_m M$ is non degenerate. Such a form is called a *symplectic form*. Then M has even dimension.

The simplest example is \mathbb{R}^{2n} where ω is the **constant** 2-form on V with matrix $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. If we denote the corresponding coordinates by (q_i, p_i) , we have $\omega = dq_1 \wedge dp_1 + \dots + dq_n \wedge dp_n$.

By Darboux Theorem, any symplectic manifold M is locally isomorphic to this standard symplectic vector space. A good reference for the proof (and for the whole course as well) is the revised edition of the book by Michèle Audin [2]

Theorem 4 (Darboux Theorem). *Around every $m \in M$ there exists a system of coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ such that*

$$\omega = dq_1 \wedge dp_1 + \dots + dq_n \wedge dp_n.$$

In Hamiltonian mechanics, the manifold M is a cotangent bundle T^*U , the coordinates $q = (q_i)$ parameterize a point in U (the position), and the coordinates $p = (p_i)$ a point in the cotangent space T_q^*U at q (the momentum).

1.4.3 Hamiltonian vector field

A function $H \in C^\infty(M)$ gives rise to a vector field X_H on M . This so called *Hamiltonian vector field* corresponds to the differential dH under the identification $TM \cong T^*M$ defined by the symplectic form ω . Thus, for every vector field Y , one has

$$dH(Y) = \omega(X_H, Y),$$

or equivalently, in terms of contraction,

$$dH = \iota(X_H)\omega.$$

In local Darboux coordinates, one has

$$\begin{aligned} dH &= \sum_{k=1}^n \frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial p_k} dp_k, \\ X_H &= \sum_{k=1}^n \frac{\partial H}{\partial q_k} \frac{\partial}{\partial p_k} - \frac{\partial H}{\partial p_k} \frac{\partial}{\partial q_k}. \end{aligned}$$

In Hamiltonian mechanics, when H (the *Hamiltonian* of the system) is the energy, the flow of the vector field X_H describes the movement in the *phase space* $\{(q_1, \dots, q_n, p_1, \dots, p_n)\}$. This flow is the solution of the system of order one differential equations

$$\begin{aligned} \dot{q}_k(t) &= \frac{\partial H}{\partial p_k} \\ \dot{p}_k(t) &= -\frac{\partial H}{\partial q_k} \end{aligned}$$

Lemma 5. *The flow of X_H preserves H and ω .*

Proof. We have immediately $X_H.H = \omega(X_H, X_H) = 0$. The invariance of ω is also proved at the infinitesimal level, using Cartan Homotopy Formula.

$$\mathcal{L}(X_H)\omega = \iota(X_H)d\omega + d(\iota(X_H)\omega) = d(dH) = 0.$$

□

This result is a particular case of Emmy Noether's Theorem [15]. It is basic to the construction of symplectic reduction (Section [?]).

1.4.4 Moment map. Hamiltonian manifold

Let G be a Lie group with Lie algebra \mathfrak{g} . The group G acts linearly on \mathfrak{g} by the *adjoint representation*. For $g \in G$ and $X \in \mathfrak{g}$,

$$\exp(g.X) := g \exp X g^{-1}.$$

The group G acts on the dual space \mathfrak{g}^* by the contragredient representation, called *coadjoint*.

$$\langle g.\xi, X \rangle = \langle \xi, g^{-1}.X \rangle,$$

for $g \in G$, $X \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$.

A G orbit in \mathfrak{g}^* is called a coadjoint orbit. We will see later that any coadjoint orbit has a canonical G -invariant symplectic structure.

Assume that G acts on M and preserves the symplectic form ω . Using again the Cartan Homotopy formula, we obtain for $X \in \mathfrak{g}$,

$$d(\iota(X^M)\omega) = \mathcal{L}(X^M)\omega = 0.$$

Thus, for each $X \in \mathfrak{g}$, the 1-form $\iota(X^M)\omega$ is closed. The action will be called Hamiltonian if this form is exact, in other words if X^M is the Hamiltonian vector field of a function $\mu_X \in C^\infty(M)$, and if the primitive μ_X satisfies an invariance condition. So we give the following definition.

Definition 6. *A moment map for the symplectic action of G on M is a G -equivariant map*

$$\mu : M \rightarrow \mathfrak{g}^*$$

such that, for every $X \in \mathfrak{g}$, the vector field X^M is the Hamiltonian vector field of the function $m \mapsto \langle \mu(m), X \rangle$. We say that (ω, μ) satisfies the Hamilton equation

$$d\langle \mu, X \rangle = \iota(X^M)\omega.$$

Note that the equivariance condition reads

$$\langle \mu(g.m), X \rangle = \langle \mu(m), g^{-1}.X \rangle.$$

Definition 7. *A G -Hamiltonian manifold is a symplectic manifold (M, ω) with an action of G which preserves the form ω , for which there exists a moment map.*

If H is Lie subgroup of G with Lie algebra \mathfrak{h} , there is a natural projection map $\mathfrak{g}^* \rightarrow \mathfrak{h}^*$ given by restriction of linear forms. If $\mu : M \rightarrow \mathfrak{g}^*$ is a moment map for G , the composed map $M \rightarrow \mathfrak{h}^*$ is a moment map for H . So if M is G -Hamiltonian, it is also H -hamiltonian for any subgroup H .

Some important families of Hamiltonian manifolds will be described in the next section. For the moment, let us just give a very simple example.

Example 8. *The manifold is \mathbb{R}^2 with symplectic form $\omega = dx \wedge dy$. The group is the one dimensional torus S^1 acting on \mathbb{R}^2 by rotations*

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \exp(\theta J)$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The corresponding vector field on $M = \mathbb{R}^2$ is $J^M = y\partial_x - x\partial_y$. Let

$$\langle \mu, J \rangle = \frac{1}{2}(x^2 + y^2).$$

Then μ is a moment map. Indeed,

$$\iota(J^M)\omega = ydy + xdx = d\langle \mu, J \rangle.$$

2 Examples of Hamiltonian manifolds

2.1 Cotangent bundle

Let G be a Lie group and let M be a G -manifold. The group G has a natural *vector bundle action* on the tangent and the cotangent bundle. By vector bundle action, one means a smooth action of G on the total space of the bundle, such that $g \in G$ maps linearly (and bijectively) the fiber above m on the fiber above $g.m$. The action on TM is given by $g.X = g_*X = dg_m(X)$, for $X \in T_mM$, where we denote also by g the diffeomorphism of M associated to an element $g \in G$. The action on the cotangent bundle T^*M is given by the contragredient action: for $Z \in T_mM$ and $\xi \in T_m^*M$,

$$\langle g.\xi, g.X \rangle = \langle \xi, X \rangle.$$

The [total space of the] cotangent bundle carries a canonical 1-form θ given by

$$\theta_{(m,\xi)}(Z, \eta) = \langle \xi, Z \rangle,$$

for $Z \in T_mM$ and $\xi, \eta \in T_m^*M$. If (q_i) is a local coordinate system on M , and (q_i, p_i) the corresponding coordinate system on T^*M , then one has

$$\theta = \sum_i p_i dq_i.$$

Being canonical, the form θ is clearly G -invariant.

Definition 9. *The two-form $\omega = -d\theta$ is called the canonical symplectic form on T^*M .*

In local coordinates, we have

$$\omega = \sum_i dq_i \wedge dp_i.$$

It follows immediately that ω is non degenerate. There is a moment map; for $X \in \mathfrak{g}$, it is given by

$$\langle \mu, X \rangle = -\iota(X^{T^*M})\theta.$$

Thus, if $X^M = \sum_i X_i(q) \frac{\partial}{\partial q_i}$ in local coordinates, we have

$$\langle \mu(q, p), X \rangle = \sum_i p_i X_i(q).$$

Remark 10. This formula is coherent with the following direct computation of the vector fields X^{TM} and X^{T^*M} .

Lemma 11. Let (q_i, u_i) (resp. (q_i, p_i)) be the system of local coordinates on TM (resp. T^*M) which extend the coordinates (q_i) on M . If $X^M = \sum_i X_i \frac{\partial}{\partial q_i}$ in local coordinates, we have

$$X^{TM} = \sum_i X_i \frac{\partial}{\partial q_i} + \sum_{i,j} \frac{\partial X_i}{\partial q_j} u_j \frac{\partial}{\partial u_i}, \quad (1)$$

$$X^{T^*M} = \sum_i X_i \frac{\partial}{\partial q_i} - \sum_{i,j} \frac{\partial X_j}{\partial q_i} p_j \frac{\partial}{\partial p_i}. \quad (2)$$

Proof. Write the action of the flow $g(t)$ of X^M on the tangent and cotangent bundles and take the derivative at $t = 0$. The minus sign in X^{T^*M} reflects the contragredient action $(g(t)^*)^{-1}$ on T^*M . \square

2.2 Symplectic and Hermitian vector spaces

Let (V, B) be a symplectic vector space of dimension $2n$. Thus V has a basis $e_i, 1 \leq i \leq 2n$ such that B is the bilinear form with matrix $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. Let ω be the symplectic form given by the constant differential 2-form equal to B .

$$\omega = dx_1 \wedge dx_{n+1} + \cdots + dx_n \wedge dx_{2n}.$$

Lemma 12. The action of the symplectic group $\mathrm{Sp}(V)$ on V is Hamiltonian. The moment map $\mu_V : V \rightarrow \mathfrak{sp}(V)^*$ is given, for $v \in V$, by the equation

$$\langle \mu_V(v), X \rangle = -\frac{1}{2} B(Xv, v). \quad (3)$$

Proof. We can write $\omega = \frac{1}{2} B(dv, dv)$, extending $B(.,.)$ to $\mathcal{A}(V) \otimes V$ by linearity. We have

$$d\left(\frac{1}{2} B(Xv, v)\right) = \frac{1}{2} B(Xdv, v) + \frac{1}{2} B(Xv, dv) = B(Xv, dv),$$

as $X \in \mathfrak{sp}(V)^*$. On the other hand, remembering the convention sign in the definition of $X^V = -Xv$, we have

$$\iota(X^M)\Omega = \frac{1}{2} B(-Xv, dv) - \frac{1}{2} B(dv, -Xv) = -B(Xv, dv).$$

\square

Example. For $n = 1$ and $X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $\langle \mu(x, y), X \rangle = \frac{1}{2}(x^2 + y^2)$.

A complex structure J on V is called compatible with B if it satisfies the following two conditions:

- 1) B is J -invariant, that is $B(Jv, Jw) = B(v, w)$ for all $v, w \in V$.
- 2) The bilinear form $Q(v, w) = B(v, Jw)$ is positive definite on V , i.e. $B(v, Jv) > 0$ for all $v \neq 0$.

For instance, if B is the standard symplectic form on \mathbb{R}^{2n} , the matrix $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ defines a B -compatible structure.

If J and B are compatible, the bilinear form $h = Q - iB$ is Hermitian with respect to the complex structure J . The subgroup $U(V) \subset \text{Sp}(V)$ which preserves this Hermitian product is a maximal compact subgroup of $\text{Sp}(V)$, isomorphic to the unitary group $U(n)$. We denote its Lie algebra by $\mathfrak{u}(V)$.

Conversely, let (V, h) be a Hermitian vector space of (complex) dimension n . Then the bilinear form $B(v, w) = -\Im h(v, w)$ is symplectic (\Im denotes imaginary part). The moment map $\mu_V : V \rightarrow \mathfrak{u}^*$ for the unitary group $U(V)$ is given, for $X \in \mathfrak{u}$, by

$$\langle \mu_V(v), X \rangle = \frac{1}{2} \Im h(Xv, v) = -\frac{i}{2} h(Xv, v). \quad (4)$$

Observe that $h(Xv, v)$ is pure imaginary if $X \in \mathfrak{u}$.

2.3 Complex projective space

Let V be a finite dimensional complex space. The corresponding projective space $(V \setminus \{0\})/\mathbb{C}^*$ is denoted by $\mathbb{P}(V)$. We denote the map $V \setminus 0 \rightarrow \mathbb{P}(V)$ by $u \mapsto q(u)$. If $V = \mathbb{C}^{N+1}$, we will also write $q(z) = [z_1, \dots, z_{N+1}]$.

We fix a Hermitian scalar product on V . Let $U(V)$ be the unitary group and let $\mathfrak{u}(V)$ be its Lie algebra.

We denote the unit sphere in V by $S(V)$. Let $\mathbb{T}^1 = \{z \in \mathbb{C}, |z| = 1\}$. We consider \mathbb{T}^1 as the subgroup of scalar matrices in $U(V)$. Thus $\mathbb{P}(V) = S(V)/\mathbb{T}^1$. We will also denote the projection $S(V) \rightarrow \mathbb{P}(V)$ by q . On this realization, we see that $\mathbb{P}(V)$ is compact, on the other hand we do not see the complex structure.

Let $(e_k, k = 1, \dots, N+1)$ be an orthonormal basis of V and let $z_k = x_k + iy_k$ be the corresponding coordinates on $V \simeq \mathbb{C}^N$. The symplectic form

on V associated to the Hermitian scalar product is

$$\Omega = \sum_k dx_k \wedge dy_k = \frac{i}{2} \sum_k dz_k \wedge d\bar{z}_k = \frac{i}{2}(dz, d\bar{z}),$$

where $(u, v) := \sum_k u_k \wedge v_k$ denotes the scalar product of vectors whose entries are differential forms.

Lemma 13. *There exists a unique 2-form ω on $\mathbb{P}(V)$ such that*

$$q^*\omega = \Omega|_{S(V)}. \quad (5)$$

The form ω is symplectic and invariant under the action of $U(V)$ on $\mathbb{P}(V)$.

Remark: this is a particular case of symplectic reduction which will be described in Section 3. Consider the following proof as an exercise on Section 3!

Proof. The form Ω is invariant under the unitary group, in particular it is invariant under the torus. Let us show that $\Omega|_{S(V)}(Z, Z') = 0$ if Z or Z' in $T_v S(V)$ is "vertical", meaning tangent to a \mathbb{T}^1 orbit. (A differential form on $S(V)$ with these two properties is called *basic* with respect to the action of \mathbb{T}^1 .)

We observe that the orthogonal of the tangent space to the sphere at a point z with respect to Ω_z is i times the orthogonal with respect to the Euclidean scalar product, that is the line $i\mathbb{R}z$, which is precisely the tangent space to the orbit under \mathbb{T}^1 .

Thus, if Z and Z' in $T_v S(V)$ project onto $q_*(Z)$ and $q_*(Z')$ in the tangent space of $\mathbb{P}(V)$ at the point $q(v)$, then $\Omega(Z, Z')$ depends only on the projections $q(v)$, $q_*(Z)$ and $q_*(Z')$, thus $\omega_{q(v)}(q_*(Z), q_*(Z'))$ is well-defined. It follows also from the above observation, that ω is non-degenerate. It is clearly $U(V)$ invariant, since Ω is. Finally, we have $q^*(d\omega) = d(q^*\omega) = d\Omega|_{S(V)} = 0$, thus ω is closed. \square

This symplectic form on $\mathbb{P}_N(\mathbb{C})$ is sometimes called the Fubini-Study symplectic two-form, as it is related to the Fubini-Study *metric* on $\mathbb{P}_N(\mathbb{C})$.

Let us compute ω in the coordinates $z \in \mathbb{C}^N$ defined by the chart $z \mapsto [z, 1] = q(z, 1)$. We factor this map through the unit sphere in order to use (5). Thus we consider the embedding $w : \mathbb{C}^N \hookrightarrow S(V)$ given by

$$w(z) = \frac{1}{\rho}(z, 1),$$

where $\rho(z) = \sqrt{1 + \|z\|^2}$. The image W is a submanifold of the sphere and we have $q^*\omega|_W = \Omega|_W$. By a straightforward computation in the coordinates $z \in \mathbb{C}^N$, we obtain the two-form ω .

$$\begin{aligned}\omega = w^*(\Omega|_W) &= \frac{i}{2}(dw, d\bar{w}) \\ &= \frac{i}{2}(1 + \|z\|^2)^{-2}((1 + \|z\|^2)(dz, d\bar{z}) - (dz, \bar{z}) \wedge (z, d\bar{z})).\end{aligned}$$

Remark 14 (Kähler potential). *On the open subset $\mathbb{C}^N \subset \mathbb{P}_N(\mathbb{C})$ we have*

$$\omega = \frac{i}{2}\partial\bar{\partial}\ln(\|z\|^2 + 1).$$

The function $\ln(\|z\|^2 + 1)$ is called the Kähler ¹ potential.

On the open subset of $\mathbb{P}_N(\mathbb{C})$ defined by $z_{N+1} \neq 0$, a system of Darboux coordinates is given by the following map from the unit open ball in \mathbb{C}^N to $\mathbb{P}_N(\mathbb{C})$

$$z \mapsto q(z_1, \dots, z_n, \sqrt{1 - \|z\|^2}).$$

Thus this open subset is isomorphic to \mathbb{C}^N with its usual symplectic structure, but beware, the isomorphism does not preserve the complex structure.

Let us compute moment maps under the action of the compact group $U(V)$ on $V = \mathbb{C}^N$ and on $\mathbb{P}(V)$. For $X \in \mathfrak{u}(V)$, the vector field X^V on V generated by X is $X^V(v) = -X.v$. It follows easily that the moment map $\mu_V : V \rightarrow \mathfrak{u}(V)^*$ is given by

$$\langle \mu_V(v), X \rangle = -\frac{i}{2}(X.v, v). \quad (6)$$

¹In 1934 Chern came to Hamburg. He writes in "A tribute to Herrn Erich Kähler", in E.Kähler, Mathematical works (R.Berndt and O.Riemenschneider (eds.), de Gruyter, Berlin, 2003), 1-2.

I arrived in Hamburg in the summer of 1934. The University began in November and I attended, among other classes, Kähler's seminar on exterior differential systems. He had just published his booklet entitled *Einführung in die Theorie der Systeme von Differentialgleichungen*, which gives a treatment of the theory developed by Elie Cartan. [For my thesis I] received much advice from Kähler, from whom I learned the subject of exterior differential calculus and what is now known as the Cartan-Kähler theory. We had frequent lunches together at the restaurant *Curio Haus* near the Seminar. He told me many things, mathematical or otherwise. My gratitude to him cannot be overstated.

From (5), it follows that a moment map $\mu : \mathbb{P}(V) \rightarrow \mathfrak{u}(V)^*$ on the projective space is given by

$$\langle \mu([v]), X \rangle = -\frac{i}{2} \frac{(X.v, v)}{\|v\|^2}. \quad (7)$$

Let us write these formulas in the case $N = 1$. On $\mathbb{C} \subset \mathbb{P}_1(\mathbb{C})$, the symplectic two-form is

$$\omega = \frac{i}{2} \frac{dz \wedge d\bar{z}}{(|z|^2 + 1)^2} = \frac{dx \wedge dy}{(|z|^2 + 1)^2},$$

and the moment map is

$$\mu(z) = \frac{1}{2} \frac{|z|^2}{|z|^2 + 1}.$$

2.4 Coadjoint orbits

Let G be a Lie group with Lie algebra \mathfrak{g} and let $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit. The tangent space at $\xi \in \mathcal{O}$ is the image of \mathfrak{g} under the map $X \mapsto X_\xi^\mathcal{O}$. The kernel of this map is the infinitesimal stabilizer of ξ ,

$$\mathfrak{g}(\xi) = \{X \in \mathfrak{g}; \langle \xi, [X, Y] \rangle = 0 \text{ for every } Y \in \mathfrak{g}\}.$$

Therefore, there is a well defined 2-form ω on \mathcal{O} such that, for every $\xi \in \mathcal{O}$,

$$\omega_\xi(X^\mathcal{O}, Y^\mathcal{O}) = -\langle \xi, [X, Y] \rangle. \quad (8)$$

By construction, ω is non degenerate.

Remark 15. *It follows that coadjoint orbits have even dimension.*

One checks easily that ω is G -invariant and that the inclusion map $\mathcal{O} \subset \mathfrak{g}^*$ is a moment map for ω . Note that the minus sign is needed because of the minus sign in the definition of the vector field $X^\mathcal{O}$. Indeed, setting $\langle \mu(\xi) = \xi \rangle$, we have

$$\begin{aligned} d\langle \mu, X \rangle(Y^\mathcal{O})(\xi) &= Y^\mathcal{O} \cdot \langle \xi, X \rangle = \frac{d}{dt} \langle \exp(-tY) \cdot \xi, X \rangle|_{t=0} \\ &= \langle \xi, [Y, X] \rangle. \end{aligned}$$

Let us show that $d\omega = 0$. It is enough to show that $\iota(X^\mathcal{O})d\omega = 0$. By Cartan Homotopy Formula and the relation $\mathcal{L}(X^\mathcal{O})\omega = 0$, we have $\iota(X^\mathcal{O})d\omega = -d(\iota(X^\mathcal{O})\omega) = -d(d\langle \mu, X \rangle) = 0$.

This canonical 2-form is sometimes called the *Kirillov-Kostant-Souriau* symplectic form on the coadjoint orbit.

3 Reduced spaces

3.1 Fiber bundles

3.1.1 Fibration

A fiber bundle over a manifold M with typical fiber E is a manifold which is **locally** the product $M \times E$ of M with a fixed manifold E .

Definition 16. Let \mathcal{E} , E and M be manifolds. A (smooth) map $\pi : \mathcal{E} \rightarrow M$ is called a fibration over M with typical fiber E if there exists a covering of M with open sets U_i for $i \in I$, and for each $i \in I$, a diffeomorphism $\phi_i : \pi^{-1}(U_i) \rightarrow U_i \times E$, such that $\pi : \pi^{-1}(U_i) \rightarrow U_i$ is the composition of ϕ_i with the first projection $U_i \times E \rightarrow U_i$.

The manifold \mathcal{E} is called a *fiber bundle* over M . For every $m \in M$, the fiber $\pi^{-1}(m)$ is isomorphic to E , hence the terminology "typical fiber".

Let U_i and U_j be two open sets in the covering with non empty intersection. Then the restriction of ϕ_j to $\pi^{-1}(U_i \cap U_j)$ can be composed with ϕ_i^{-1} restricted to $(U_i \cap U_j) \times E$. The composed map, abusively denoted by $\phi_j \circ \phi_i^{-1}$, is a diffeomorphism of $(U_i \cap U_j) \times E$ onto itself of the form

$$(m, x) \mapsto (m, g_{i,j}(m, x)),$$

where for fixed m , the map $x \mapsto g_{i,j}(m, x)$ is a diffeomorphism of E onto itself.

Definition 17. A fiber bundle $\pi : \mathcal{E} \rightarrow M$ is called a *vector bundle* if E is a vector space and the ϕ_i can be chosen in such a way that the maps $x \mapsto g_{i,j}(x, m)$ are linear (hence linear automorphisms of E).

A *section* s of a fiber bundle $\pi : \mathcal{E} \rightarrow M$ is a map $M \rightarrow \mathcal{E}$ such that $s(m)$ belongs to the fiber $E_m := \pi^{-1}(m)$ for all m . In the case of a vector bundle, the space of sections $\Gamma(M, \mathcal{E})$ is a vector space for pointwise operations. The section $s(m) = 0 \in E_m$ identifies M with a subset of \mathcal{E} called the zero section.

Let $\phi : N \rightarrow M$ be a smooth map. A point $m \in M$ is called a *regular value* of ϕ is, for every $a \in \phi^{-1}(m)$, the differential $d\phi_a$ maps $T_a N$ **onto** $T_m M$. Then $E = \phi^{-1}(m)$ is a closed submanifold of N . If moreover ϕ is a *proper* map (the preimage of any compact subset is compact) then, by the implicit function theorem, there exists an open neighborhood $U \subseteq M$

of m and an isomorphism $\phi^{-1}(U) \rightarrow U \times E$ such that ϕ becomes the first projection. Thus the open subset $\phi^{-1}(U) \subseteq N$ is a fiber bundle over U with typical fiber $\phi^{-1}(m)$.

Note that a fibration $\pi : \mathcal{E} \rightarrow M$ is a proper map if and only if the typical fiber is compact.

3.1.2 Actions of compact Lie groups, linearization.

Let G be a **compact** Lie group acting on a manifold M . The compactness of G implies properties which do not hold necessarily hold if G is not compact. In particular, there exists a G -invariant Riemannian metric on M . Using geodesics for an invariant metric, one proves the existence of an equivariant tubular neighborhood.

Proposition 18 (Koszul). *Let G be a compact Lie group acting on a manifold M , and let $N \subset M$ be a G -invariant closed manifold. Let \mathcal{N} be the normal bundle to N in M . There exists a G -equivariant diffeomorphism from an open G -invariant neighborhood of N in M onto an open G -invariant neighborhood of N (identified with the zero section) in \mathcal{N} which is the identity on N .*

In particular, if m is a fixed point for G , the action of G is linearizable around m . Let m be an **isolated** fixed point for a compact one parameter group $\exp tX$. Then the Lie derivative $\mathcal{L}(X)$ induces a linear map in the tangent space $T_m M$, and this map is invertible. This result is not true if the group is \mathbb{R} .

Example 19. *Consider the one parameter group $g(t)$ acting on the circle $S = \{z \in \mathbf{c}; |z| = 1\}$ by*

$$g(t)(z) = \frac{(1 + it)z - it}{itz + 1 - it}.$$

The point $z = 1$ is fixed, and it is the only fixed point. The complement $\{z \neq 1\}$ is an orbit. The Cayley transform

$$z = C(x) = \frac{x - i}{x + i}$$

is an isomorphism of $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \infty$ on S which maps \mathbb{R} onto the orbit $\{z \neq 1\}$ and ∞ on $z = 1$, and conjugates $g(t)$ to the group of translations $x \mapsto x + \frac{t}{2}$.

When t increases starting from $t = 0$, then $g(t).z$ moves towards 1 if z is in the upper half circle ($x \geq 0$), while $g(t).z$ first moves away from 1 if z is in the lower half circle ($x < 0$).

This movement is not possible for the flow of a vector field which is linearizable near $z = 1$, because there would be a diffeomorphism $z = h(x)$ from an interval $] -a, a[$ to a neighborhood of 1 in S , such that $g(t)h(x) = h(e^{\lambda t}x)$.

The vector field on S generated by $g(t)$ is $\frac{d}{dt}(g(-t)(z))|_{t=0} = i(z - 1)^2$. It vanishes at order 2 at the point $z = 1$, therefore its Lie derivative $\mathcal{L}(X)$ acts by 0 in $T_{(1)}S$.

If G is compact, the set M^G of G -fixed points is a closed submanifold of M , with tangent space $T_m M^G = (T_m M)^G$. Indeed, if $Z \in T_m M$ is fixed by G , then the geodesic starting at m with tangent vector Z remains in M^G .

Consider as above a proper map $\phi : N \rightarrow M$, where N is also a G -manifold and ϕ is G -equivariant. Let $m \in M$ be a regular value of ϕ . Let $G(m) \subseteq G$ be the stabilizer of m . Observe that the fiber $\phi^{-1}(m)$ is preserved by $G(m)$. Then there exists a $G(m)$ -invariant neighborhood $U \subseteq M$ of m such that $\phi : \phi^{-1}(U) \rightarrow U$ is a fibration, with a $G(m)$ -equivariant diffeomorphism $\phi^{-1}(U) \rightarrow U \times \phi^{-1}(m)$ which carries the action of $G(m)$ on $\phi^{-1}(U)$ to the product action $h.(u, p) = (h.u, h.p)$ for $h \in G(m)$, $u \in U$ and $p \in \phi^{-1}(m)$. In particular, if p is fixed by an element $h \in G(m)$, then any curve $c(t)$ starting at m and contained in M^h (the set of points in M which are fixed by h) lifts to a curve starting at p and contained in N^h .

3.1.3 Free action of a Lie group

Let G be a Lie group acting on a manifold M . For $g \in G$ we denote the fixed points set of g by M^g . We denote $M^G = \bigcap_{g \in G} M^g$ the set of points which are fixed by the whole group. For X in the Lie algebra \mathfrak{g} of G , we denote by $M^0(X)$ the set of zeroes of the vector field X^M . Thus $M^0(X)$ is the set of points which are fixed by the one-parameter subgroup $\exp(tX) \subseteq G$ acting on M .

For $m \in M$, we denote the stabilizer of m by $G(m)$. It is a closed subgroup of G . Its Lie algebra is $\mathfrak{g}(m) = \{X \in \mathfrak{g}; X^M(m) = 0\}$ and is called the infinitesimal stabilizer of m . We will use the notations

$$M_{\text{free}/G} = \{m \in M; G(m) = \{e\}\},$$

$$M_{\text{free}/\mathfrak{g}} = \{m \in M; \mathfrak{g}(m) = \{0\}\}.$$

Lemma 20. *Let G be a compact Lie group acting on a connected manifold M . Then the subsets $M_{\text{free}/G}$ and $M_{\text{free}/\mathfrak{g}}$ are either empty or connected dense open subsets of M .*

Proof. Let $m \in M$. The compactness of G implies that the G -orbit $G.m$ is a closed subvariety of M . Therefore there exists a G -equivariant tubular neighborhood U of $G.m$. If $m' \in U$ corresponds to $(g.m, u) \in \mathcal{N}$, then $G(m') \subseteq gG(m)g^{-1}$. Hence, the subsets $M_{\text{free}/G}$ and $M_{\text{free}/\mathfrak{g}}$ are open. To be completed. Cf. Bredon "Introduction to compact transformation groups" \square

The action of G is called *free* if $G(m) = \{e\}$ for every $m \in M$, thus $M_{\text{free}/G} = M$. The action is called *infinitesimally free* if $\mathfrak{g}(m) = \{0\}$ for every $m \in M$, thus $M_{\text{free}/\mathfrak{g}} = M$. In that case, all the orbits in M have the same dimension, equal to the dimension of G .

Lemma 21. *Let G be a compact Lie group acting freely on a manifold M . Then the space of orbits is a quotient manifold.*

Proof. Consider an orbit $\mathcal{O} = G.m$. There exists a G -invariant neighborhood of \mathcal{O} in M which is diffeomorphic to a product $G \times V$, where V is a neighborhood of 0 in the normal space $\mathcal{N}_m = T_m M / T_m \mathcal{O}$, with G acting on itself by left translations. Then the map $V \rightarrow M/G$ is a system of coordinates on M/G around the point $\mathcal{O} \in M/G$. \square

If the action is only infinitesimally free, then every point has a finite stabilizer, and locally the space of orbits M/G is a quotient \mathbb{R}^n/Γ where Γ is a finite group of linear transformations. Such a structure is called an orbifold.

3.1.4 Principal bundles. Basic differential forms

Let G be a **compact** Lie group.

A principal bundle P with structure group G is just a manifold P with a **free** G -action. It is usual to write the action on the right: $(g, p) \mapsto p.g$, so that $(p.g_1).g_2 = p.(g_1.g_2)$. Thus for $X \in \mathfrak{g}$, the corresponding vector field X^P on P is now

$$X_p^P = \frac{d}{dt} p. \exp(tX).$$

We denote the quotient map by $q : P \rightarrow P/G$.

Let V be a G -manifold where now the action is written on the left. The product action of G on $P \times V$ is also free. Note that if we write it as right action, it is given by the formula

$$(p, v).g = (p.g, g^{-1}v).$$

The quotient $\mathcal{V} = (P \times V)/G$ is a fiber bundle over P/G with typical fiber V . It is called the *associated fiber bundle*. In particular, if V is vector space with a linear action of G , then the associated fiber bundle is actually a vector bundle.

For an open subset $U \subseteq P/G$, the space $C^\infty(U)$ is identified with the space $c^\infty(q^{-1}(U))^G$ of smooth functions on $q^{-1}(U)$ which are G -invariant. There is a useful similar description of differential forms on U . Let $\alpha \in \mathcal{A}(U)$ be a differential form on $U \subseteq P/G$. Its pull-back $q^*\alpha$ is a differential form on $q^{-1}(U) \subseteq P$. It is clear that $q^*\alpha$ is G -invariant. Moreover, if $X \in \mathfrak{g}$, then the projection $q_*(X^P)$ is 0, therefore $\iota(X^P)q^*\alpha = 0$. (The tangent vector X^P to the orbit $p.G$ is called *vertical*). Thus $q^*\alpha$ is *basic* in the sense of the following definition.

Definition 22. Let β be a differential form on P .

(i) β on P is called *horizontal* if $\iota(X^P)\beta = 0$ for every $X \in \mathfrak{g}$.

(ii) β is called *basic* if it is horizontal and G -invariant.

The subspace of basic differential forms is denoted by $\mathcal{A}_{bas}(P)$.

Proposition 23. The pull-back map q^* induces an isomorphism of $\mathcal{A}(P/G)$ onto $\mathcal{A}_{bas}(P)$ which preserves the degree and commutes with d .

Proof. Let $\beta \in \mathcal{A}_{bas}^k(P)$. Let $m \in P/G$ and let $V_1, \dots, V_k \in T_m P/G$. Let $q \in P$ such that $m = q.G$ and let $W_i \in T_p P$ such that $q_* W_i = V_i$. Then the horizontality of β implies that the value $\beta_p(W_1, \dots, W_k)$ does not depend on the choice of W_i 's. Moreover, the G -invariance of β implies that this value does not depend on the choice of p in the fiber of m . Indeed, another point of this fiber has the form $p' = p.g$, and we can take $W'_i = g.W_i$ to lift V_i . Then $\beta_p(W_1, \dots, W_k) = \beta_{p'}(W'_1, \dots, W'_k)$. If β is basic, then $d\beta$ is also basic. This follows from Cartan Homotopy formula. \square

Remark. On the contrary, the exterior differential d does not preserve horizontality.

Often it will be easier to compute "upstairs", in $\mathcal{A}_{bas}^k(P)$ rather than $\mathcal{A}(P/G)$. Furthermore, if the action of G is only locally free, so that the

quotient P/G is no longer a manifold, we will pretend that it is, by considering functions and differential forms "upstairs".

3.2 Pre-Hamiltonian manifold

It is useful to consider manifolds with a closed two-form ω **without** the assumption that the bilinear form ω_m on the tangent space T_mM is non degenerate. Such a manifold as well as the two-form ω are called *presymplectic*. Although a function on M does not define a unique Hamiltonian vector field any more, the notion of moment map still makes sense, when a Lie group G acts on a presymplectic manifold M and preserves the form ω . If a moment map exists, we say that the action and the G -manifold M are G -pre-Hamiltonian.

Definition 24. *A G -pre-Hamiltonian manifold is a manifold with a G action, a closed G -invariant two-form ω and a G -equivariant map $\mu : M \rightarrow \mathfrak{g}^*$ such that, for every $X \in \mathfrak{g}$,*

$$d\langle\mu, X\rangle = \iota(X^M)\omega. \quad (9)$$

3.2.1 Examples of pre-Hamiltonian manifolds

For example, pre-Hamiltonian manifolds arise in the two following situations. First, let M be a G -manifold with a G -invariant 1-form θ . The form $\omega = d\theta$ is closed, but it may be degenerate. A moment map μ is defined by

$$\langle\mu, X\rangle = \iota(X^M)\theta.$$

Indeed, as $d\omega = 0$, the Hamilton equation $d\langle\mu, X\rangle = \iota(X^M)\omega$ follows from the Cartan Homotopy Formula.

Next, let (M, ω, μ) be a G -Hamiltonian manifold and let $N \subset M$ be a G -invariant submanifold. Then N is G -pre-Hamiltonian for the restrictions of ω and μ to N . Even if ω itself is non-degenerate the restriction of ω_m to T_mN may be degenerate. More generally, if $f : N \rightarrow M$ is a G -equivariant map and M is G -Hamiltonian (or only pre-Hamiltonian), then N is G -pre-Hamiltonian for the two-form and the moment map obtain by pulling back those of M .

Example 25. *Consider C^n with its natural Hermitian scalar product. The group G is the unitary group $U(n)$. The two-form $\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge$*

dy_n on \mathbb{C}^n is symplectic and $U(n)$ -invariant. A moment map is given by

$$\langle \mu(v), X \rangle = -\frac{i}{2}h(Xv, v)$$

for $X \in \mathfrak{u}(n)$, i.e. X is an anti-Hermitian matrix.

Now, let $S \subset \mathbb{C}^n$ be the unit sphere $\{\|v\|^2 = 1\}$. As S has dimension $2n - 1$, the restriction of ω to S must be degenerate. Indeed, the kernel of $\omega_v|_{T_v S}$ is easy to compute. It is one-dimensional, with basis the vector field which generates the group of homotheties $v \mapsto e^{it}v$.

3.2.2 Consequences of Hamilton equation. Homogeneous manifolds and coadjoint orbits

One can go one step further and drop also the assumption that the two-form ω is closed in Definition 24. Some consequences of Hamilton equation (9) hold without assuming that ω is closed.

Lemma 26 (Infinitesimal equivariance). *Let $f : M \rightarrow \mathfrak{g}^*$ be a G -equivariant map. Then for every $X, Y \in \mathfrak{g}$, we have*

$$X^M \cdot \langle \mu, Y \rangle = \langle \mu, [X, Y] \rangle,$$

Proof. This relation is obtained by differentiating the G -equivariance equation $\langle f((\exp -tY).m), X \rangle = \langle f(m), (\exp tY).X \rangle$. \square

Lemma 27. *Let M be a G -manifold with a G -invariant two-form ω and a moment map $\mu : M \rightarrow \mathfrak{g}^*$. Then*

- (i) *The restriction of ω to any G -orbit is a closed two-form.*
- (ii) *For every $X, Y \in \mathfrak{g}$, we have*

$$\omega(X^M, Y^M) = -\langle \mu, [X, Y] \rangle.$$

(iii) *If $m \in M$ is such that the differential $d_m \mu : T_m M \rightarrow \mathfrak{g}^*$ is surjective, then the infinitesimal stabilizer $\mathfrak{g}(m)$ is 0.*

(iv) *For $m \in M$, we have*

$$\ker(d_m \mu) = \{Z \in T_m M; \omega(X^M, Z) = 0 \text{ for every } X \in \mathfrak{g}\}.$$

In particular, if $X \in \mathfrak{g}$, then $d_m \mu(X^M) = 0$ if and only if X is in the infinitesimal stabilizer $\mathfrak{g}(\mu(m))$ of the element $\mu(m) \in \mathfrak{g}^$*

Proof. For every $X \in \mathfrak{g}$, we write once again the Cartan Homotopy Formula. As ω is G -invariant, we get $\iota(X^M)d\omega + d(\iota(X^M)\omega) = \mathcal{L}(X^M)\omega = 0$. Hence

$$\iota(X^M)d\omega = 0,$$

Any tangent vector at m to the orbit $G.m$ has the form X^M for some $X \in \mathfrak{g}$, hence (i).

From Hamilton equation, we get $\omega(X^M, Y^M) = Y^M \cdot \langle \mu, X \rangle$, hence (ii) follows from the infinitesimal equivariance of the map μ (Lemma 26).

Let us prove (iii). Let $X \in \mathfrak{g}$. For every $Z \in T_m M$, we have, (Hamilton equation), $\langle (d_m \mu)(Z), X \rangle = \omega(X_m^M, Z)$. Assume that $X \in \mathfrak{g}(m)$. Then $X_m^M = 0$. So, if the image of $d_m \mu$ is the whole of \mathfrak{g}^* , this implies $X = 0$.

Let us prove (iv). The first statement follows immediately from Hamilton equation. For $X \in \mathfrak{g}$, the equation $d\mu_m(X_m^M) = 0$ is equivalent to $\langle \mu(m), [X, Y] \rangle = 0$ for every $Y \in \mathfrak{g}$. By Lemma 26, this equality holds if and only if X is in the infinitesimal stabilizer of the point $\mu(m) \in \mathfrak{g}^*$. \square

As a consequence of this lemma, one gets a description of all homogeneous G -Hamiltonian manifolds. Let M be a G -manifold which is homogeneous under G , and let $m \in M$. Then a G -equivariant map $\mu : M \rightarrow \mathfrak{g}^*$ exists if and only if there exists $\xi \in \mathfrak{g}^*$ such that $G(m) \subseteq G(\xi)$.

Theorem 28 (Kostant). *Let M be a G -manifold which is homogeneous under G . Let $m \in M$. Assume that there exists $\xi \in \mathfrak{g}^*$ such that $G(m) \subseteq G(\xi)$. Let μ be the G -equivariant map $M \rightarrow \mathfrak{g}^*$ such that $\mu(m) = \xi$.*

Then there exists a unique two-form ω on M such that μ is a moment map. This two-form ω is closed, and it is the pull back of the KKS two-form of the coadjoint orbit $G.\xi \subset \mathfrak{g}^$.*

Moreover, ω is symplectic (i.e. non degenerate) if and only if $\mathfrak{g}(m) = \mathfrak{g}(\xi)$ that is to say, if and only if $G(m)$ has finite index in $G(\xi)$, so that M is a finite covering of a coadjoint orbit.

Proof. By Lemma 27(ii), the map μ determines the restriction of the two-form ω to any G -orbit $G.m \subseteq M$, and this restriction is closed and non-degenerate by the lemma. \square

Example 29. *Consider \mathbb{R}^2 with the alternate bilinear form $B(v, w) = v_1 w_2 - v_2 w_1$. The group $G = SL(2, \mathbb{R})$ acts on \mathbb{R}^2 . The open subset $M := \mathbb{R}^2 \setminus \{0\}$ is an orbit. The map $\mu : M \rightarrow \mathfrak{g}^*$ defined by $\langle \mu(v), X \rangle = -\frac{1}{2}B(Xv, v)$ is a moment map for the symplectic form $\omega = dv_1 \wedge dv_2$.*

Let us identify \mathfrak{g} with \mathfrak{g}^* by means of the pairing $K(X, Y) = \text{Tr}(XY)$. Then $M := \mathbb{R}^2 \setminus \{0\}$ is a covering of the orbit of the (nilpotent) matrix

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

3.3 Hamiltonian reduction

In this section G is a compact group.

Consider again (M, ω, μ) where M is a G -manifold with a G -invariant two-form ω and $\mu : M \rightarrow \mathfrak{g}^*$ is a moment map. We do not assume that ω is closed and non degenerate. Let $\xi \in \mathfrak{g}^*$ be a regular value of μ . Then $\mu^{-1}(\xi)$ is a submanifold of M which is stable under $G(\xi)$. Moreover the action of $G(\xi)$ on $\mu^{-1}(\xi)$ is infinitesimally free.

Lemma 30. *The restriction of ω to $\mu^{-1}(\xi)$ is basic with respect to $G(\xi)$.*

Proof. Let $P = \mu^{-1}(\xi)$. The two-form $\omega|_P$ is G -invariant. Let us show that it is horizontal. The tangent space $T_m P$ is $T_m P = \{Z \in T_m M; d_m \mu(Z) = 0\}$. Let $X \in \mathfrak{g}(\xi)$. Then $X_m^P = X_m^M$.

$$\iota(X_m^P) \cdot \omega|_{T_m P} = (\iota(X_m^M) \cdot \omega)|_{T_m P} = d\langle \mu, X \rangle|_{T_m P} = 0$$

□

Let us assume that the action of $G(\xi)$ on $\mu^{-1}(\xi)$ is free. (If it is only infinitesimally free, we will compute "upstairs" as explained previously). Then there is a smooth quotient map $q : \mu^{-1}(\xi) \rightarrow \mu^{-1}(\xi)/G(\xi)$. There exists a unique two-form $\omega_{red, \xi}$ on $\mu^{-1}(\xi)/G(\xi)$ whose pull-back to $\mu^{-1}(\xi)$ is the restriction of ω .

Definition 31. *The quotient manifold $\mu^{-1}(\xi)/G(\xi)$ is called the reduced manifold of M at ξ . It is denoted by $M_{red, \xi}$. The two-form $\omega_{red, \xi}$ is called the reduced two-form.*

If ω is closed, then $\omega_{red, \xi}$ is also closed. In other words, $(M_{red, \xi}, \omega_{red, \xi})$ is a presymplectic manifold.

Lemma 32. *If ω is non degenerate, then $\omega_{red, \xi}$ is also non degenerate.*

Proof. We denote again $P = \mu^{-1}(\xi)$. Let $m \in P$ and $Z \in T_m P$. Assume that $\omega(Z, W) = 0$ for every $W \in T_m P$. We have seen that $T_m P$ is the orthogonal of the tangent space to the orbit $G.m$ with respect to ω_p . As ω_p is non degenerate, we must have $Z = X_m^P$ for some $X \in \mathfrak{g}$. By Lemma 27 (iv), we see that $X \in \mathfrak{g}(\xi)$. \square

Let us now consider the case where a second group H acts on M and the actions of H and G commutes. The following proposition follows from the previous discussions.

Proposition 33. *Let M be a $G \times H$ -manifold with a $G \times H$ -invariant two-form ω , and $\mu = (\mu^G, \mu^H) : M \rightarrow \mathfrak{g}^* \times \mathfrak{h}^*$ a $G \times H$ equivariant map which satisfy the Hamilton equation with respect to $G \times H$. Let $\xi \in \mathfrak{g}^*$ be a regular value of μ^G . Denote by $M_{red,\xi}$ the manifold reduced with respect to G . Then $M_{red,\xi}$ still carries an action of H . The reduced two-form $\omega_{red,\xi}$ is H -invariant. The map μ^H restricted to $(\mu^G)^{-1}(\xi)$ descends to a map $\mu_{red,\xi}^H : M_{red,\xi} \rightarrow \mathfrak{h}^*$ which is H -equivariant.*

The pair $(\omega_{red,\xi}, \mu_{red,\xi}^H)$ satisfies the Hamilton equation with respect to the group H .

If (M, ω, μ) is $G \times H$ -pre-Hamiltonian, then $(M_{red,\xi}, \omega_{red,\xi}, \mu_{red,\xi}^H)$ is H -pre-Hamiltonian. If (M, ω, μ) is $G \times H$ -Hamiltonian, then $(M_{red,\xi}, \omega_{red,\xi}, \mu_{red,\xi}^H)$ is H -Hamiltonian.

4 Duistermaat-Heckman measure and volumes of reduced spaces

4.1 Poincaré Lemma

A subset $U \subseteq \mathbb{R}^n$ is called star-shaped (centered at 0) if for every $x \in U$ and $0 \leq t \leq 1$, we have $tx \in U$. For instance, if U is convex and $0 \in U$ then U is star-shaped. The well-known Poincaré Lemma states that a closed differential form on a star-shaped open set is exact. Moreover the homotopy operator gives a particular primitive. We are going to recall the proof in order to include parameters and also to extend the Poincaré Lemma to the case of equivariant differential forms.

Since U is star-shaped, it is stable under the homothety mapping

$$h : U \times [0, 1] \rightarrow U; h(x, t) = tx.$$

Let P (the parameter set) be a manifold. We denote by p the projection $P \times U \rightarrow P$ and by j the inclusion map of the zero-section $P \times \{0\}$ identified with P .

$$j : P \simeq P \times \{0\} \rightarrow P \times U).$$

Thus if ϕ is a differential form on $P \times U$, then $j^*\phi$ is the restriction of ϕ to the zero-section and $p^*j^*\phi$ is a form on $P \times U$ such that $j^*(\phi - p^*j^*\phi) = 0$. More explicitly, let y be local coordinates on P . We write

$$\phi = \sum_{J,I} \phi_{J,I}(y, x) dy_J \wedge dx_I.$$

Then

$$p^*j^*\phi = \sum_J \phi_{J,\emptyset}(y, 0) dy_J.$$

Lemma 34 (Parametric G -equivariant Poincaré Lemma). *Let G be a Lie group with a linear action on \mathbb{R}^n . (Here we do not assume that G is compact). Let U be a star-shaped open subset of \mathbb{R}^n . Let P be a G -manifold. Let p be the projection $P \times U \rightarrow P$ and let j be the inclusion map of the zero-section $P \simeq P \times \{0\} \rightarrow P \times U$.*

(i) *Let ϕ be a closed k -form on $P \times U$. Let α be the $(k-1)$ -form on $P \times U$ defined by*

$$\alpha = \int_0^1 \iota\left(\frac{\partial}{\partial t}\right).h^*\beta dt \tag{10}$$

where $\beta = \phi - p^*j^*\phi$. Thus $j^*\alpha = 0$. We have

$$\phi = p^*j^*\phi + d\alpha. \quad (11)$$

(ii) Assume that U is G -invariant. If ϕ is G -invariant, then α is G -invariant. If ϕ is G -basic, then α is G -basic.

Proof. Let us first explain the meaning of (10). The pull-back $h^*\beta$ is a differential form on $P \times U \times \mathbb{R}$. After contraction with the vector field $\frac{\partial}{\partial t}$, it can be considered as a differential form on $P \times U$ with coefficients depending on t . Thus it makes sense to integrate it on $[0, 1]$.

The result relies on the Cartan Homotopy Theorem applied to the vector field $\frac{\partial}{\partial t}$ on $P \times U \times [0, 1]$. Let us prove directly this particular case in the form which we need. A differential form φ on $U \times [0, 1]$ can be written

$$\varphi = \tau + dt \wedge \sigma,$$

where σ and τ are differential forms on $P \times U$ whose coefficients depend on t . Thus $\tau = \sum_{I,J} \tau_{I,J}(t, y, x) dy_I \wedge dx_J$. We write $\tau = \tau(t)$ to indicate this dependance. Taking the exterior differential, we have

$$d\varphi = d_M\tau + dt \wedge \dot{\tau}(t) - dt \wedge d_M\sigma,$$

where d_M is the partial exterior differential with respect to the variable $(y, x) \in P \times U$ and $\dot{\tau}(t) = \sum_{I,J} \frac{\partial \tau_{I,J}(t, y, x)}{\partial t} dy_I \wedge dx_J$. Contracting with the vector field $\frac{\partial}{\partial t}$, and substituting $\sigma = \iota(\frac{\partial}{\partial t}).\varphi$, we get

$$\iota(\frac{\partial}{\partial t})d\varphi + d_M\iota(\frac{\partial}{\partial t})\varphi = \dot{\tau}(t).$$

Assume that φ is closed. Integrating with respect to the variable t , we obtain

$$\tau(1) - \tau(0) = d_M \int_0^1 \iota(\frac{\partial}{\partial t}).\varphi(t) dt.$$

We apply this relation to the form $\varphi = h^*\beta$. With the previous notations, we have $\tau(1) = \beta = \phi - p^*j^*\phi$ and $\tau(0) = 0$. Thus (11) is proven. The other statements follow immediately from the definition of α . \square

More generally, instead of just a product $P \times U$, we can consider a vector bundle $p : V \rightarrow P$ with fiber \mathbb{R}^n and a starshaped open neighborhood $\mathcal{U} \subseteq V$ of the zero-section.

Theorem 35. *Let $\phi \in \mathcal{A}^k(\mathcal{U})$ a closed differential form. Then there exists $\alpha \in \mathcal{A}^{k-1}(\mathcal{U})$ such that*

$$\phi - p^* j^* \phi = d\alpha.$$

Assume that the fiber bundle is G -equivariant and that \mathcal{U} is invariant. Then if ϕ is invariant, we can choose α to be G -invariant; if ϕ is G -basic, we can choose α to be G -basic.

4.2 Pre-Hamiltonian structures on $P \times \mathfrak{g}^*$.

Let G be a Lie group acting on a manifold P . Let $M = P \times \mathfrak{g}^*$, with G acting on \mathfrak{g}^* by the coadjoint action. We will determine all the pre-Hamiltonian structures on M such that the moment map μ is the projection $P \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$. Then all points of \mathfrak{g}^* are regular values. By ??, a necessary condition is that G acts infinitesimally freely on any regular fiber of μ . Thus G must act infinitesimally freely on P . Let us make this assumption.

Let us recall the definition of a connection one-form on P . Recall that for $X \in \mathfrak{g}$, we denote by X_P the vector field on P defined by

$$X_P(p) = \left. \frac{d}{dt} \right|_{t=0} (\exp(-tX).p).$$

The condition that G acts infinitesimally freely means that $X \mapsto X_P(p)$ is a bijection of \mathfrak{g} with a subspace of $T_p P$. This subspace is called the vertical subspace and is denoted by $V_p P$.

Definition 36. *Let P be a manifold with an infinitesimally free action of the Lie group G . A connection one-form is an element $\theta \in (\mathcal{A}^1(P) \otimes \mathfrak{g})^G$ such that $\theta(X_P) = X$ for every $X \in \mathfrak{g}$.*

Let $H_p = \ker \theta_p \subset T_p P$. Then $(H_p)_{p \in P}$ is a smooth sub-bundle of TP with the following properties:

$$T_p P = H_p \oplus V_p P,$$

$$H_{g.p} = g.H_p.$$

H_p is called the horizontal subspace. Conversely, if these two conditions are satisfied by a sub-bundle $(H_p)_{p \in P}$, then the projection $T_p P \rightarrow V_p P \simeq \mathfrak{g}$ parallel to H_p is a connection.

Lemma 37. *Let G be a compact group acting infinitesimally freely on P . Then there exists a connection one-form θ .*

Proof. As G is compact, there exists a G -invariant metric on P . We define H_p to be the orthogonal of V_pP . \square

Given a connection one-form on P , we define a pre-Hamiltonian structure on $P \times \mathfrak{g}^*$ in the following way.

Lemma 38. *Let $\gamma = \langle \xi, \theta \rangle$ be the one-form on $P \times \mathfrak{g}^*$ defined by*

$$\gamma_{(p,\xi)}(X, \xi') = \langle \xi, \theta(X) \rangle$$

for $X \in T_pP$ and $\xi, \xi' \in \mathfrak{g}^$. Then $\Omega = -d\gamma$ is a G -invariant two-form on $P \times \mathfrak{g}^*$ such that the projection $\mu : P \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is a moment map.*

Proof. The one-form γ is G -invariant because of the G -invariance of the connection θ . We have seen that in this case, the map defined by $\mu_\gamma(X) = \gamma(X_M)$ for $X \in \mathfrak{g}$ is a moment map for $\Omega = -d\gamma$. We have immediately $\gamma_{(p,\xi)}(X_M) = \langle \xi, X \rangle$, ie μ_γ is indeed the projection $(p, \xi) \mapsto \xi$. \square

We can now describe all the G -invariant closed two-forms on $P \times \mathfrak{g}^*$ for which $P \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is a moment map.

Proposition 39 (A particular case of the Normal Form Theorem). *Let G be a Lie group acting infinitesimally freely on a manifold P . Let $M = P \times \mathfrak{g}^*$, with G acting on \mathfrak{g}^* by the coadjoint action. Let μ be the projection $P \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ and let p be the projection on the first factor $P \times \mathfrak{g}^* \rightarrow P$. Let $\theta \in (\mathcal{A}^1(P) \otimes \mathfrak{g})^G$ be a connection one-form on P . Let Ω_0 be a closed basic two-form on P and let α be a basic one-form on $P \times \mathfrak{g}^*$, such that its restriction $j^*\alpha$ to P vanishes. Then*

$$\Omega = p^*\Omega_0 - d\langle \xi, \theta \rangle + d\alpha$$

is a G -invariant closed two-form on $P \times \mathfrak{g}^$ for which $P \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is a moment map and every such form Ω can be written in this manner, with $\Omega_0 = j^*\Omega$.*

Proof. The moment condition requires that $\iota_{X_M} \cdot (p^*\Omega_0 + d\alpha) = 0$ for every $X \in \mathfrak{g}$. We have $\iota_{X_M} \cdot p^*\Omega_0 = 0$ because $p^*\Omega_0$ is horizontal and $\iota_{X_M} \cdot d\alpha = 0$ because α is basic.

Conversely, let Ω be a G -invariant closed two-form on $P \times \mathfrak{g}^*$ for which $P \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is a moment map. Then $\Omega + d\langle \xi, \theta \rangle$ is a closed basic two-form on $P \times \mathfrak{g}^*$. We observe that $j^*d\langle \xi, \theta \rangle = 0$. Thus we can apply the Poincaré Lemma with parameters to the form $\Omega + d\langle \xi, \theta \rangle$. \square

Remark 40. *The result still holds if we replace \mathfrak{g}^* by a G -invariant star-shaped open subset $U \subseteq \mathfrak{g}^*$.*

4.3 Push-forward of the Liouville measure.

From now on, we will assume that G is a compact Lie group. Let (M, Ω, μ) be a pre-Hamiltonian G -manifold. We assume that M is oriented and has even dimension 2ℓ . Then Ω^ℓ has top dimension. If ϕ is a compactly supported continuous function on M , the integral $\int_M \phi \Omega^\ell$ is defined. We recall how it is defined. Assume that the support of ϕ is contained in a coordinate open subset U with coordinates (x_1, \dots, x_n) . On U , we have $\Omega^\ell = v(x) dx_1 \wedge \dots \wedge dx_n$, where $v(x)$ is a C^∞ function. We choose x_1, \dots, x_n so that $dx_1 \wedge \dots \wedge dx_n$ is a positive orientation. Then (with the usual abuse of notations), the integral is defined by

$$\int_M \phi \Omega^\ell = \int_{\mathbb{R}^n} \phi(x) v(x) dx_1 \cdots dx_n.$$

Definition 41. *The Liouville measure β is defined by*

$$\int_M \phi(m) d\beta(m) = \frac{1}{(2\pi)^\ell} \int_M \phi \frac{\Omega^\ell}{\ell!}$$

Let us assume that the moment map is proper. Then the push-forward $\mu_*(\beta)$ of the Liouville measure is an absolutely continuous measure on \mathfrak{g}^* . The following result is quite remarkable. It was first discovered by H. Duistermaat and G. Heckman in the case of a Hamiltonian action of a torus, [7].

Theorem 42 (Duistermaat-Heckman). *Let G be a compact Lie group. Let M be a pre-Hamiltonian G -manifold with moment map μ . Assume that μ is proper. Assume that $0 \in \mathfrak{g}^*$ is a regular value of μ . Then, on the connected component of 0 in the set of regular values of μ , the density of the push-forward $\mu_*(\beta)$ of the Liouville measure is a polynomial function.*

Example 43. Let $M = \mathbb{T}^1 \times \mathbb{R}$, with coordinates $(\theta, \xi) \in [0, 2\pi] \times \mathbb{R}$. A T^1 -invariant two-form on M is of the form $\Omega = f(\xi)d\theta \wedge d\xi$. A T^1 -equivariant map $M \rightarrow \mathbb{R}$ is of the form $\mu(\theta, \xi) = F(\xi)$. The Hamiltonian condition with respect to Ω is

$$\frac{\partial \mu}{\partial \xi} = -\Omega\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \xi}\right) = -f(\xi),$$

hence $F'(\xi) = -f(\xi)$. Moreover we assume that $\xi \mapsto F(\xi)$ is proper. Then the push-forward of $\frac{1}{2\pi}\Omega$ is just $-d\xi$. The density is the constant polynomial equal to -1 . If we insist that the moment map is the projection $(\theta, \xi) \mapsto \xi$, then we must have $f(\xi) = -1$, ie $\Omega = -d\theta \wedge d\xi$.

On the other hand, if we do not assume that the projection $(\theta, \xi) \mapsto \xi$ is a moment map for $\Omega = f(\xi)d\theta \wedge d\xi$, then the push-forward of $\frac{1}{2\pi}\Omega$ by this projection is $f(\xi)d\xi$, ie any density.

Proof. Let us denote $P = \mu^{-1}(0)$. It is a submanifold of M . The condition μ proper implies that P is compact.

Let U be a starshaped open neighborhood of 0 contained in the set of regular values of μ . (Actually, the connected component of 0 in the set of regular values of μ is itself starshaped). There exists a G -invariant connection for the fibration $\mu^{-1}(U) \rightarrow U$. Then parallel transport along lines defines an isomorphism

$$\mu^{-1}(U) \rightarrow P \times U.$$

We are going to prove that $\mu_*(\beta)|_U$ has a polynomial density.

We apply Proposition 39. Thus

$$\Omega = \Omega_P - d\langle \xi, \theta \rangle + d\alpha$$

where

$$\Omega_P = p^*j^*\Omega$$

and α is a G -basic one-form on $P \times U$ such that $j^*\alpha = 0$. Let

$$\Omega_t = -d\langle \xi, \theta \rangle + \Omega_P + td\alpha,$$

so that $\Omega_1 = \Omega$ and $\Omega_0 = -d\langle \xi, \theta \rangle + \Omega_P$. The form Ω_t is closed, for every t . For $X \in \mathfrak{g}$, we have

$$\iota(X_M).\Omega_t = \iota(X_M).\Omega = \langle d\xi, X \rangle,$$

since $\iota(X_M).d\alpha = 0$, (α is G -basic).

Let us show that the push-forward $\mu_*(\Omega_t^\ell)$ does not depend on t . Let $f(\xi)$ be a compactly supported smooth function on U . Then

$$\begin{aligned} \frac{d}{dt} \int_U f(\xi) \mu_*(\Omega_t^\ell) &= \frac{d}{dt} \int_{P \times U} f(\xi) \Omega_t^\ell = \ell \int_{P \times U} f(\xi) \Omega_t^{\ell-1} \wedge d\alpha \\ &= -\ell \int_{P \times U} \Omega_t^{\ell-1} \wedge \alpha \wedge df. \end{aligned}$$

The last equality follows from Stokes Formula.

Let us show that

$$\Omega_t^{\ell-1} \wedge \alpha \wedge df = 0.$$

It is enough to prove this when f is a coordinate, say $f(\xi) = \xi_1$, associated to a basis (E_k) of \mathfrak{g} . It is equivalent to show that $\iota((E_1)_M) \cdot (\Omega_t^{\ell-1} \wedge \alpha \wedge d\xi_1) = 0$ as the vector field $(E_1)_M$ vanishes nowhere on $P \times U$, (this follows from the moment condition), hence $\iota((E_1)_M)$ is injective on the space of maximal degree differential forms supported in this open set. We have $\iota((E_1)_M) \cdot \alpha = 0$, and $\iota((E_1)_M) \cdot d\xi_1 = (E_1)_M \cdot \langle \xi, E_1 \rangle = \langle \xi, [E_1, E_1] \rangle = 0$. We have $\iota((E_1)_M) \cdot \Omega_t^{\ell-1} = (\ell-1)\Omega_t^{\ell-2} \wedge d\xi_1$, hence

$$\iota((E_1)_M) \cdot \Omega_t^{\ell-1} \wedge \alpha \wedge df = (\ell-1)\Omega_t^{\ell-2} \wedge d\xi_1 \wedge \alpha \wedge d\xi_1 = 0.$$

Since $\mu_*(\Omega_t^\ell)$ does not depend on t , it is enough to look at $t = 0$ and show that $\mu_*(\Omega_0^\ell)$ has a polynomial density on U .

Using a basis, we identify \mathfrak{g} and its dual with \mathbb{R}^n . Thus we write $\langle \xi, \theta \rangle = \sum_a \xi_a \theta_a$ and

$$\Omega_0 = (\Omega_P - \sum_a \xi_a d\theta_a) + \sum_a \theta_a \wedge d\xi_a.$$

Let us write $\ell = \ell_0 + n$. In the expansion of $((\Omega_P - \sum_a \xi_a d\theta_a) + \sum_a \theta_a \wedge d\xi_a)^\ell$ we must keep only the terms which contain $d\xi_1 \wedge \cdots \wedge d\xi_n$. So we obtain

$$\frac{\Omega_0^\ell}{\ell!} = \frac{(\Omega_P - \sum_a \xi_a d\theta_a)^{\ell_0}}{\ell_0!} \theta_1 \wedge d\xi_1 \cdots \wedge \theta_n \wedge d\xi_n.$$

Hence,

$$\frac{1}{(2\pi)^\ell} \mu_* \left(\frac{\Omega_0^\ell}{\ell!} \right) = v(\xi) d\xi_1 \wedge \cdots \wedge d\xi_n$$

with

$$v(\xi) = (\text{sign}) \frac{1}{(2\pi)^\ell} \int_P \frac{(\Omega_P - \sum_a \xi_a d\theta_a)^{\ell_0}}{\ell_0!} \theta_1 \wedge \cdots \wedge \theta_n. \quad (12)$$

It is clear that $v(\xi)$ depends polynomially on ξ . The sign depends on the orientations on M and P . In the case where Ω is symplectic, we choose the orientation on M which is determined by Ω^ℓ , so that the Liouville measure is positive. \square

Remark 44. *Formula(12) shows that near 0, the density $v(\xi)$ is a polynomial of degree at most ℓ_0 . Actually, the degree can be smaller.*

4.4 Push-forward of the Liouville measure and volume of the reduced space

We are going to relate the density $v(\xi)$ of the push-forward of the Liouville measure with volumes of the reduced spaces. We begin with the value $v(0)$. We have (ignoring the sign)

$$v(0) = \frac{1}{(2\pi)^\ell} \int_P \frac{\Omega_P^{\ell_0}}{\ell_0!} \theta_1 \wedge \cdots \wedge \theta_n.$$

The value $v(0)$ does not depend on the connection θ which we choosed for the fibration $P \rightarrow P/G$. It follows from the definition since $\mu_*(\beta)$ does not depend on θ . We can see it also on the right-hand-side. If θ' is another connection, then the forms $\theta_a - \theta'_a$ are horizontal. The form Ω_P is also horizontal. Being of top degree and horizontal, the form $\Omega_P^{\ell_0}(\theta_1 \wedge \cdots \wedge \theta_n - \theta'_1 \wedge \cdots \wedge \theta'_n)$ must be 0.

The value $v(0)$ does depend on the Lebesgue measure $\lambda_{\mathfrak{g}^*} = d\xi_1 \wedge \cdots \wedge d\xi_n$ on \mathfrak{g}^* . So we look at the top degree form $v(0)\lambda_{\mathfrak{g}^*}$. This is an element of $\Lambda^{max} \mathfrak{g}^*$ which does not depend on any choice.

Let us assume that G acts freely on $P = \mu^{-1}(0)$. Then the quotient P/G is a manifold, the reduced space, which we denoted by M_{red} .

The differential form Ω_P is basic. It defines a differential form on P/G which we denote by Ω_{red} . The volume of M_{red} is defined by the normalized formula

$$\text{vol}(M_{red}) = \frac{1}{(2\pi)^{\ell_0}} \int_{M_{red}} \frac{\Omega_{red}^{\ell_0}}{\ell_0!}. \quad (13)$$

We choose $\lambda_{\mathfrak{g}} \in \Lambda^{max} \mathfrak{g}$. This determines a Haar measure on the group G . We denote by $\text{vol}(G, \lambda_{\mathfrak{g}})$ the volume of G with respect to this Haar measure.

Corollary 45. *We have*

$$\langle v(0)\lambda_{\mathfrak{g}^*}, \lambda_{\mathfrak{g}} \rangle = \text{vol}(G, \lambda_{\mathfrak{g}}) \text{vol}(M_{red}).$$

Remark 46. *This formula is very important. It allows to compute the volumes of reduced spaces.*

We apply the following lemma.

Lemma 47. *Let P be a manifold with a free action of G and a G -equivariant connection θ . (Here we do not need to assume that P is compact). Let Φ be a compactly supported top degree differential form on P/G . Let us denote the projection $P \mapsto P/G$ by π . Then*

$$\int_P \pi^*(\Phi) \wedge \theta_1 \wedge \cdots \wedge \theta_n = \text{vol}(G, \lambda_{\mathfrak{g}}) \int_{P/G} \Phi$$

Proof. The choice of a basis of \mathfrak{g} determines the top degree element $\lambda_{\mathfrak{g}}$. Let us observe that the product $\theta_1 \wedge \cdots \wedge \theta_n$ depends only on $\lambda_{\mathfrak{g}}$.

Using a partition of unity, we can assume that the fibration $P \mapsto P/G$ is trivialized in a neighborhood of the support of Φ . Thus we can assume that $P = U \times G$ where G acts by left translations on G and U is an open subset of \mathbb{R}^{ℓ_0} . Let λ_a be the basis of \mathfrak{g}^* dual to E_a which we have fixed. We denote also by λ_a the right-invariant one form on G associated to λ_a . Then

$$\theta_a = \lambda_a + \sum_k h_{a,k}(u, g) du_k.$$

Since Φ has top degree, we have

$$\pi^*(\Phi) \wedge \theta_1 \wedge \cdots \wedge \theta_n = \Phi \wedge \lambda_1 \wedge \cdots \wedge \lambda_n.$$

By definition, we have $\lambda_1 \wedge \cdots \wedge \lambda_n = \lambda_{\mathfrak{g}}$, hence the result. \square

4.5 What is particular about the value 0 ?

Actually, there is a trick to shift the general case to the case $\xi = 0$. Let $\mathcal{O}_{\xi_0} \subset \mathfrak{g}^*$ a coadjoint orbit. Let Ω_{ξ_0} be its Kirillov symplectic two-form. The moment map is the injection $j_{\xi_0} : \mathcal{O}_{\xi_0} \hookrightarrow \mathfrak{g}^*$. Thus, the product $\widetilde{M} = M \times \mathcal{O}_{\xi_0}$ is pre-Hamiltonian with respect to the two-form $\Omega - \Omega_{\xi_0}$ and the moment map $\tilde{\mu} = \mu - j_{\xi_0}$. Assume that ξ_0 is a regular value of the moment map μ . Then 0 is a regular value of $\tilde{\mu}$. It is easy to see that the reduced space $M_{red}(\xi_0) = \mu^{-1}(\xi_0)/G(\xi_0)$ at the value ξ_0 is isomorphic to

$$\widetilde{M}_{red} = \tilde{\mu}^{-1}(0)/G = \{(m, \eta) \in M \times \mathcal{O}_{\xi_0}; \mu(m) = \xi\}/G.$$

More precisely, the isomorphism is induced by the map $\mu^{-1}(\xi_0) \rightarrow M \times \mathcal{O}_{\xi_0}$ which sends m to (m, ξ_0) . It is clear that this isomorphism is compatible with the reduced two-forms. So they have the same volumes

$$\text{vol}(M_{red}(\xi_0)) = \text{vol}(\tilde{M}_{red}).$$

In a neighborhood of $\mathcal{O}_{\xi_0} \subset \mathfrak{g}^*$, the push-forward $\mu_*(\beta)$ is given by a G -invariant smooth density

$$\mu_*(\beta) = v(\xi)\lambda_{\mathfrak{g}^*}.$$

On the other hand, as we saw in the previous section, the push-forward $\tilde{\mu}_*(\tilde{\beta})$ is given in a neighborhood of 0 by a polynomial density

$$\tilde{\mu}_*(\tilde{\beta}) = \tilde{v}(\xi)\lambda_{\mathfrak{g}^*},$$

and the volume of \tilde{M}_{red} is given by $\text{vol}(G, \lambda_{\mathfrak{g}}) \text{vol}(\tilde{M}_{red}) = \tilde{v}(0)\langle \lambda_{\mathfrak{g}^*}, \lambda_{\mathfrak{g}} \rangle$. So there remains to compare the densities v and \tilde{v} of the two push-forward. Let ϕ be a test function on \mathfrak{g}^* supported in a small neighborhood of 0. We have

$$\int_{\mathfrak{g}^*} \phi(\xi)\tilde{\mu}_*(\tilde{\beta}) = \int_{M \times \mathcal{O}_{\xi_0}} \phi(\mu(m) - \xi)\beta(m)\beta_{\mathcal{O}_{\xi_0}}(\xi),$$

where $\beta_{\mathcal{O}_{\xi_0}}$ is the Liouville measure of the orbit. This implies, for small η ,

$$\tilde{v}(\eta) = \int_{\mathcal{O}_{\xi_0}} v(\eta + \xi)\beta_{\mathcal{O}_{\xi_0}}(\xi). \quad (14)$$

In particular, for $\eta = 0$ we obtain

$$\tilde{v}(0) = \text{vol}(\mathcal{O}_{\xi_0})v(\xi_0),$$

hence the formula relating the density and the volume of the reduced space

Proposition 48. *Let ξ be a regular value of the moment map. Then*

$$\text{vol}(G, \lambda_{\mathfrak{g}}) \text{vol}(M_{red}(\xi)) = \text{vol}(\mathcal{O}_{\xi})v(\xi)\langle \lambda_{\mathfrak{g}^*}, \lambda_{\mathfrak{g}} \rangle.$$

Remark 49. *If G is abelian, then the orbit \mathcal{O}_{ξ} is of course just a point. Thus, on any connected component $U \subset \mathfrak{g}^*$ of the set of regular values of the moment map, the manifold $(M_{red}(\xi))$ does not depend on ξ . However the reduced two-form $\Omega_{red}(\xi)$ depends on ξ . Its volume is given by a polynomial*

function on any connected component U . In other words, $\text{vol}(M_{\text{red}}(\xi))$ is a piecewise polynomial on \mathfrak{g}^* .

In their original article [7], Duistermaat-Heckman proved actually a stronger result: the cohomology class of $\Omega_{\text{red}}(\xi)$ is a polynomial function **of degree one** of ξ , for $\xi \in U$.

Remark 50. There is a nice formula which expresses the Duistermaat-Heckman measure for the group G in terms of the one for its maximal torus H . On this formula, one can see that the non abelian D-H measure is not always locally bounded.

5 Orbits and moment map for a linear action of a torus

5.1 Weights

Let V be a complex vector space of dimension n . Let $T_{\mathbb{C}} = (\mathbb{C}^*)^r$ be a complex torus with a linear action in V . Then the action of $T_{\mathbb{C}}$ is diagonalizable, with weights

$$(t_1, \dots, t_r) \mapsto t_1^{m_1} \cdots t_r^{m_r},$$

where the exponents m_k are **integers**.

We will study the $T_{\mathbb{C}}$ -orbits in V , in particular the closed orbits.

Example 51. Let $T_{\mathbb{C}} = (\mathbb{C}^*)^2$ act diagonally on $V = \mathbb{C}^3$ with weights t_1, t_2 and $(t_1 t_2)^{-1}$. Let $z = (z_1, z_2, z_3) \in V$ with $z_k \neq 0$ for $k = 1, 2, 3$. Then the orbit of z is closed. Indeed, this orbit is the set of $w \in \mathbb{C}^3$ such that $w_1 w_2 w_3 = z_1 z_2 z_3$.

Example 52. Let now $(\mathbb{C}^*)^2$ act diagonally on $V = \mathbb{C}^3$ with weights t_1, t_2 and $t_1 t_2$. Then the only closed orbit is $\{0\}$ and 0 belongs to the closure of every orbit.

Let $T = \mathbb{T}^r$. It is a maximal compact subgroup of $T_{\mathbb{C}}$. We denote the Lie algebra of T by \mathfrak{t} and its dual by \mathfrak{t}^* .

We fix a Hermitian scalar product (v, w) on V which is invariant under the action of T . The space V has an orthonormal basis $(e_k, 1 \leq k \leq n)$ of eigenvectors for the action of $T_{\mathbb{C}}$. Let $(\lambda_k \in \mathfrak{t}^*, 1 \leq k \leq n)$ be the (infinitesimal) weights of T in V , defined by $X.e_k = i\langle \lambda_k, X \rangle e_k$, $\exp X.e_k = e^{i\langle \lambda_k, X \rangle} e_k$, for $X \in \mathfrak{t}$.

Definition 53. Let $v = \sum_{k=1}^n z_k e_k$. We denote the set of indices k such that $z_k \neq 0$ by $\text{Supp}(v)$, and we denote the sequence of weights $(\lambda_k, k \in \text{Supp}(v))$ by Φ_v .

Up to reordering, Φ_v does not depend on the particular basis of eigenvectors e_k . Sometimes Φ_v itself is called the *support* of the vector v .

We will see that the closure of the orbit $T_{\mathbb{C}}.v$ can be described in terms of the polyhedral cone $\mathfrak{c}(\Phi_v) \subseteq \mathfrak{t}^*$ generated by the weights λ_k for $k \in \text{Supp}(v)$.

5.2 Polyhedral cones

We briefly recall some facts about polyhedral cones.

Let E be a real vector space. A polyhedral cone $\mathfrak{c} \subseteq E$ is a finite intersection of closed half-spaces bounded by (linear) hyperplanes of E . Thus \mathfrak{c} is defined by a finite set of linear inequalities $\mathfrak{c} = \{\lambda \in E; \langle f_k, \lambda \rangle \geq 0\}$. We denote by $\text{lin}(\mathfrak{c})$ the sub-vector space spanned by \mathfrak{c} . The relative interior of \mathfrak{c} in $\text{lin}(\mathfrak{c})$ is denoted by \mathfrak{c}^0 .

A hyperplane H is called a supporting hyperplane of \mathfrak{c} if \mathfrak{c} is contained in one of the half-spaces bounded by H , and $H \cap \mathfrak{c}$ has non empty relative interior in $H \cap \text{lin}(\mathfrak{c})$.

The intersection $H \cap \mathfrak{c}$ is called a facet (face of codimension 1) of \mathfrak{c} . It is a polyhedral cone in H .

A facet of a facet of \mathfrak{c} is called a face of codimension 2 of \mathfrak{c} etc... A face of dimension 1 is called an edge. The cone \mathfrak{c} itself is considered as a face, of dimension $\text{dim}(\text{lin}(\mathfrak{c}))$.

There is at most one face of dimension 0, namely $\{0\}$. Moreover, $\{0\}$ is a face if and only if there exist an hyperplane H of $\text{lin}(\mathfrak{c})$ such that $\mathfrak{c} \setminus \{0\}$ is contained in an open half-space bounded by H . When this is the case, the cone is called *salient* (or pointed).

The cone \mathfrak{c} is generated (as a cone) by its edges $\mathbb{R}_{\geq 0}\lambda_k$ where $\lambda_k \in E$.

Conversely, let $\Phi = (\lambda_1, \dots, \lambda_n)$ be a finite sequence of vectors in E . The cone generated by Φ is denoted by $\mathfrak{c}(\Phi)$.

$$\mathfrak{c}(\Phi) = \sum_{k=1}^n \mathbb{R}_{\geq 0}\lambda_k.$$

Then $\mathfrak{c}(\Phi)$ is a convex polyhedral cone contained in the subspace generated by Φ . Note that the elements of Φ need not be edges of $\mathfrak{c}(\Phi)$. Similarly, in the dual description by linear inequalities, all the hyperplanes corresponding to the linear inequalities need not be supporting hyperplanes. In general, the computation of the set of edges as well as the set of facets is difficult!

The relative interior of $\mathfrak{c}(\Phi)$ is $\mathfrak{c}^0(\Phi) = \sum_{\lambda \in \Phi} \mathbb{R}_{> 0}\lambda$. Note that the relative interior of $\{0\}$ is $\{0\}$ itself.

We state two easy and useful lemmas. The second one is proven by induction on the codimension of the face.

Lemma 54. *The origin 0 belongs to $\mathfrak{c}^0(\Phi)$ if and only if $\mathfrak{c}(\Phi) = \mathfrak{c}^0(\Phi) = \sum_{\lambda \in \Phi} \mathbb{R}\lambda$.*

Lemma 55. *Let \mathfrak{f} be a face of \mathfrak{c} . There exists a linear form X on E such that X vanishes on \mathfrak{f} and $\langle \lambda, X \rangle > 0$ for any $\lambda \in \mathfrak{c} \setminus \mathfrak{f}$.*

By means of the moment map $\mu : V \rightarrow \mathfrak{t}^*$, we are going to describe the orbit $T_{\mathbb{C}}.v$ of an element $v \in V$ in terms of the open cone $\mathfrak{c}^0(\Phi_v) \subseteq \mathfrak{t}^*$. Then we will describe the orbit closure $T_{\mathbb{C}}.v$ in terms of the closed cone $\mathfrak{c}(\Phi_v)$ and its faces.

5.3 Image of a $T_{\mathbb{C}}$ -orbit under the moment map

The moment map $\mu : V \rightarrow \mathfrak{t}^*$, for the action of the compact torus T on V , is given by

$$\langle \mu(v), X \rangle = -\frac{i}{2}(X.v, v), \quad \text{for } X \in \mathfrak{t} \text{ and } v \in V.$$

If v is decomposed in the above orthonormal basis of eigenvectors as $v = \sum_{k=1}^d z_k e_k$, with $z_k \in \mathbb{C}$, we have

$$\mu(v) = \frac{1}{2} \sum_{k=1}^d |z_k|^2 \lambda_k.$$

In the theory of Hamiltonian manifolds, a fundamental problem is to compute the image of the moment map. The following theorem gives the answer in the simple and beautiful case where the manifold is an orbit of a complex torus acting on a Hermitian vector space.

Theorem 56 (Kac-Peterson [11]). *Let $v \in V$. The image of the orbit $T_{\mathbb{C}}.v$ under the moment map $\mu : V \rightarrow \mathfrak{t}^*$ is the cone $\mathfrak{c}^0(\Phi_v)$. Furthermore, the moment map induces a homeomorphism $T_{\mathbb{C}}.v/T \simeq \mathfrak{c}^0(\Phi_v)$.*

Proof. Let $u = \exp(X + iY).v \in T_{\mathbb{C}}.v$ with X and $Y \in \mathfrak{t}$. Then

$$u = \sum_{k=1}^d e^{i\langle \lambda_k, X \rangle} e^{-\langle \lambda_k, Y \rangle} z_k e_k.$$

We have $\text{Supp}(u) = \text{Supp}(v)$. Thus, to simplify the notations in the proof, we may assume that $v = \sum_{k=1}^d z_k e_k$ with all $z_k \neq 0$, and that Φ generates \mathfrak{t}^* . We have

$$\mu(u) = \frac{1}{2} \sum_{k=1}^d e^{-2\langle \lambda_k, Y \rangle} |z_k|^2 \lambda_k.$$

We see that $\mu(u) \in \mathfrak{c}^0(\Phi_v)$. Because of our assumption, the map $\mathfrak{t} \rightarrow T_{\mathbb{C}.v} : Y \mapsto \exp(iY).v$ induces a diffeomorphism from \mathfrak{t} onto the quotient $T_{\mathbb{C}.v}/T$, hence the theorem follows immediately from the next lemma.

Lemma 57. *Let $\Phi = (\lambda_1, \dots, \lambda_s)$ be a finite sequence of linear forms on a real vector space L such that Φ generates L^* . Let $c_k > 0$ for $1 \leq k \leq s$. Then the map $m : L \rightarrow L^*$ given by*

$$m(Y) = \sum_{k=1}^s c_k e^{\langle \lambda_k, Y \rangle} \lambda_k$$

is a diffeomorphism of L onto $\mathfrak{c}^0(\Phi)$.

Proof. We compute the differential of m at the point $Y \in L$ and show that it is injective. We have

$$dm_Y(Z) = \sum_{k=1}^s c_k \langle \lambda_k, Z \rangle e^{\langle \lambda_k, Y \rangle} \lambda_k.$$

If $Z \neq 0$, the numbers $\langle \lambda_k, Z \rangle$ are not all 0, as the sequence λ_k generates L^* . Hence,

$$\langle dm_Y(Z), Z \rangle = \sum_{k=1}^s c_k |\langle \lambda_k, Z \rangle|^2 e^{\langle \lambda_k, Y \rangle} \neq 0.$$

Let us show that m is one to one. Let $Y_1 \neq Y_2 \in L$. Consider the function

$$f(t) = \langle m(Y_1 + t(Y_2 - Y_1)), Y_2 - Y_1 \rangle.$$

Then $f'(t) = \sum_{k=1}^s c_k |\langle \lambda_k, Y_2 - Y_1 \rangle|^2 e^{\langle \lambda_k, Y_1 + t(Y_2 - Y_1) \rangle}$. So $f'(t) > 0$ for every $t \in [0, 1]$, hence $m(Y_1) = f(0) \neq f(1) = m(Y_2)$.

So m is a diffeomorphism of L onto its image. There remains to prove that this image is the cone $\mathfrak{c}^0(\Phi)$. Let $\lambda = \sum_{k=1}^s a_k \lambda_k$, with $a_k > 0$. We want $Y \in L$ such that $m(Y) = \lambda$. We consider the function $F : L \rightarrow \mathbb{R}$ given by

$$F(Y) = \sum_{k=1}^s c_k e^{\langle \lambda_k, Y \rangle} - \langle \lambda, Y \rangle.$$

Then $dF_Y = \sum_{k=1}^s c_k e^{\langle \lambda_k, Y \rangle} \lambda_k - \lambda$. Hence, the equation $m(Y) = \lambda$ means that Y is a critical point of F . We are going to show that $F(Y)$ is bounded

from below and reaches its minimum at a point Y_0 , which has to be a critical point.

We write $F(Y) = \sum_{k=1}^s (c_k e^{\langle \lambda_k, Y \rangle} - a_k \langle \lambda_k, Y \rangle)$. For $c > 0$ and $a > 0$, the function of one variable $y \mapsto ce^y - ay$ is bounded from below and tends to $+\infty$ when $y \rightarrow \pm\infty$. It follows that $F(Y)$ is bounded from below and tends to $+\infty$ when $\|Y\|$ does. As F is continuous and the space L is finite-dimensional, this implies that $F(Y)$ reaches its minimum at a point Y_0 . \square

\square

5.4 Image of a $T_{\mathbb{C}}$ -orbit closure under the moment map

We study now the image of the orbit closure $\overline{T_{\mathbb{C}} \cdot v}$ under the moment map. First, we state a corollary of Theorem 56.

Corollary 58. *Let $u \in V$ be an element of the orbit closure $\overline{T_{\mathbb{C}} \cdot v}$. Assume that $\mu(u) \in \mathfrak{c}^0(\Phi_v)$. Then $u \in T_{\mathbb{C}} \cdot v$.*

Proof. Let $\mathfrak{t}(v)$ be the infinitesimal stabilizer of v in \mathfrak{t} and let \mathfrak{h} be a supplementary subspace of $\mathfrak{t}(v)$ in \mathfrak{t} , so that $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \oplus i\mathfrak{t}(v) \oplus i\mathfrak{h}$. We can write $u = \lim_{n \rightarrow \infty} \exp(iX_n)h_nv$, with $h_n \in T$ and $X_n \in \mathfrak{h}$. As T is compact, we can assume that h_n has a limit $h \in T$. By replacing u with $h^{-1}u$ we can assume that $u = \lim_{n \rightarrow \infty} \exp(iX_n)v$. By Theorem 56, the map $X \mapsto \mu(\exp(iX)v)$ is a diffeomorphism of \mathfrak{h} onto $\mathfrak{c}^0(\Phi_v)$. As $\mu(u) \in \mathfrak{c}^0(\Phi_v)$, it follows that X_n has a limit $X \in \mathfrak{h}$, hence $u = \exp(iX)v$ belongs to the orbit. \square

Next, we will show that the orbits which are contained in a given orbit closure are in one to one correspondance with the faces of the moment cone $\mathfrak{c}(\Phi_v)$. By applying Lemma 55 to this cone, we obtain immediately the following.

Lemma 59. *If \mathfrak{f} is a face of $\mathfrak{c}(\Phi_v)$, there exists an $X \in \mathfrak{t}$ such that $\langle \lambda_k, X \rangle > 0$ if $\lambda_k \notin \mathfrak{f}$ and $\langle \lambda_k, X \rangle = 0$ if $\lambda_k \in \mathfrak{f}$. (If \mathfrak{f} is the whole cone, then $X = 0$, otherwise $X \neq 0$.)*

Theorem 60. *Let $v = \sum_{k=1}^d z_k e_k$. If \mathfrak{f} is a face of $\mathfrak{c}(\Phi_v)$, let*

$$v_{\mathfrak{f}} = \sum_{\{k, \lambda_k \in \mathfrak{f}\}} z_k e_k.$$

(i) *The image of $T_{\mathbb{C}} \cdot v_{\mathfrak{f}}$ under the moment map is the relative interior \mathfrak{f}^0 .*

(ii) The orbit closure $\overline{T_{\mathbb{C}}.v}$ is the union of the various orbits $T_{\mathbb{C}}.v_{\mathfrak{f}}$, as \mathfrak{f} runs over the set of faces of the cone $\mathfrak{c}(\Phi_v)$. In particular, $0 \in \overline{T_{\mathbb{C}}.v}$ if and only if the cone $\mathfrak{c}(\Phi_v)$ is salient.

Proof. (i) is Theorem 56 applied to $v_{\mathfrak{f}}$.

Let $X \in \mathfrak{t}$ satisfy the condition of Lemma 59. Then $\exp(itX).v = \sum_k e^{-t\langle \lambda_k, X \rangle} z_k e_k$ tends to $v_{\mathfrak{f}}$ when $t \rightarrow \infty$. In particular, $v_{\mathfrak{f}}$ belongs to $\overline{T_{\mathbb{C}}.v}$.

Conversely, let u be a point in the boundary of $\overline{T_{\mathbb{C}}.v}$. As $\mu(u) \in \mathfrak{c}(\Phi_v)$, there exists a face $\mathfrak{f} \subsetneq \mathfrak{c}(\Phi_v)$ such that $\mu(u) \in \mathfrak{f}^0$. Let us prove that $u \in T_{\mathbb{C}}.v_{\mathfrak{f}}$. By Corollary 58, it is enough to prove that $u \in \overline{T_{\mathbb{C}}.v_{\mathfrak{f}}}$.

We have $u = \lim_{n \rightarrow \infty} \exp(iX_n)h_n v$ with $h_n \in T$ and $X_n \in \mathfrak{t}$. By taking a converging subsequence $h_n \rightarrow h$ and replacing v by $h^{-1}.v$, we can assume that $u = \lim_{n \rightarrow \infty} \exp(iX_n)v$.

We write $v = v_{\mathfrak{f}} + (v - v_{\mathfrak{f}})$, thus $v - v_{\mathfrak{f}} = \sum_{k; \lambda_k \notin \mathfrak{f}} z_k e_k$. Let $X \in \mathfrak{t}$ Then

$$\exp(iX_n).(v - v_{\mathfrak{f}}) = \sum_{k; \lambda_k \notin \mathfrak{f}} e^{-\langle \lambda_k, X_n \rangle} z_k e_k.$$

Let $X \in \mathfrak{t}$ be, as above, such that $\langle \lambda_k, X \rangle > 0$ if $\lambda_k \notin \mathfrak{f}$ and $\langle \lambda_k, X \rangle = 0$ if $\lambda_k \in \mathfrak{f}$. Then

$$\langle \mu(\exp(iX_n).v_{\mathfrak{f}}), X \rangle = \frac{1}{2} \sum_{\{k, \lambda_k \in \mathfrak{f}\}} e^{-2\langle \lambda_k, X_n \rangle} |z_k|^2 \langle \lambda_k, X \rangle = 0,$$

$$\langle \mu(\exp(iX_n).(v - v_{\mathfrak{f}})), X \rangle = \frac{1}{2} \sum_{\{k, \lambda_k \notin \mathfrak{f}\}} e^{-2\langle \lambda_k, X_n \rangle} |z_k|^2 \langle \lambda_k, X \rangle.$$

By assumption, $\lim_{n \rightarrow \infty} \mu(\exp(iX_n).v) = \mu(u)$. Since $\mu(u) \in \mathfrak{f}$, we have $\langle \mu(u), X \rangle = 0$. Therefore each term in $\langle \mu(\exp(iX_n).(v - v_{\mathfrak{f}})), X \rangle$ tends to 0, hence $\lim_{n \rightarrow \infty} e^{-\langle \lambda_k, X_n \rangle} = 0$ if $\lambda_k \notin \mathfrak{f}$. Finally we obtain, as desired

$$u = \lim_{n \rightarrow \infty} \exp(iX_n).v_{\mathfrak{f}}.$$

In the particular case where $u = 0$, the face \mathfrak{f} such that $0 \in T_{\mathbb{C}}.v_{\mathfrak{f}}$ must be $\mathfrak{f} = \{0\}$, so the cone is pointed. Thus we have completed the proof of (ii). \square

Corollary 61 (Hilbert-Mumford criterion). *Let $u \in \overline{T_{\mathbb{C}}.v}$. Then there exists $g \in T_{\mathbb{C}}$ and $X \in \mathfrak{t}$ such that*

$$u = \lim_{t \rightarrow +\infty} \exp(itX)g.v.$$

In particular, if $0 \in \overline{T_{\mathbb{C}}.v}$, then there exists $X \in \mathfrak{t}$ such that

$$\lim_{t \rightarrow +\infty} \exp(itX)v = 0.$$

Proof. There exists a face \mathfrak{f} such that $u = g.v_{\mathfrak{f}}$. Take X as in Lemma 59. Then $v_{\mathfrak{f}} = \lim_{t \rightarrow \infty} \exp(itX).v$, hence $u = \lim_{t \rightarrow \infty} \exp(itX)g.v$. \square

5.5 Closed orbits

Finally, we obtain the characterization of closed orbits.

Theorem 62. (i) *The orbit $T_{\mathbb{C}}.v$ is closed if and only if $0 \in \mathfrak{c}^0(\Phi_v)$*
(ii) *The orbit $T_{\mathbb{C}}.v$ is closed if and only if it intersects $\mu^{-1}(0)$. Moreover if two points of $\mu^{-1}(0)$ are in the same $T_{\mathbb{C}}$ -orbit, then they are already in the same T -orbit.*

Proof. It follows from Theorem 60 that $0 \in \mu(\overline{T_{\mathbb{C}}.v})$. Assume that $T_{\mathbb{C}}.v$ is closed. Then 0 belongs to $\mu(T_{\mathbb{C}}.v)$ which is equal to $\mathfrak{c}^0(\Phi_v)$, by Theorem 56. Conversely, assume that $0 \in \mathfrak{c}^0(\Phi_v)$. Then it follows from Lemma 54 that $\mathfrak{c}(\Phi_v)$ has no strict face, hence $T_{\mathbb{C}}.v$ is closed by Theorem 60.

Since $\mu(T_{\mathbb{C}}.v) = \mathfrak{c}^0(\Phi_v)$, the first part of (ii) follows from (i). Regarding the second part of (ii), we already proved in Theorem 56 that the moment map induces an injection on $T_{\mathbb{C}}.v/T$. \square

5.6 T -invariant irreducible subvarieties of V

An irreducible subvariety of V is a subset defined by polynomial equations $M = \{z \in V, p_j(z) = 0\}$ M which is connected for the Zariski topology. Then M is T -invariant if and only if it is $T_{\mathbb{C}}$ -invariant. We denote

$$\Phi_M = \bigcup_{v \in M} \Phi_v.$$

Theorem 63. *Let $M \subseteq V$ be a T -invariant irreducible subvariety. Then the image of M under the moment map $\mu : V \mapsto \mathfrak{t}^*$ is the closed polyhedral cone $\mathfrak{c}(\Phi_M)$ generated by Φ_M .*

Proof. Let us show that there exists $v_0 \in M$ such that $\Phi_M = \Phi_{v_0}$. For $v \in M$, let $U_v = \{u \in M, \Phi_v \subseteq \Phi_u\}$. Then U_v is a Zariski open subset of M which is not empty since v belongs to it, There are only finitely many

distinct such open sets U_v , since there are only finitely many subsets of Φ . Therefore, as M is irreducible, the intersection $\bigcap_{v \in M} U_v$ is not empty. If v_0 belongs to $\bigcap_{v \in M} U_v$, then $\Phi_M = \Phi_{v_0}$. Hence

$$\mu(M) = \bigcup_{v \in M} \mu(\overline{T_C \cdot v}) = \bigcup_{v \in M} \mathbf{c}(\Phi_v) = \mathbf{c}(\Phi_{v_0}) = \mathbf{c}(\Phi_M).$$

□

6 Torus action on projective varieties. Image of the moment map

Let V be a complex vector space with a Hermitian scalar product (u, v) . We denote the unitary group by $U(V)$ and its Lie algebra by $\mathfrak{u}(V)$. Recall that the projective space $\mathbb{P}(V)$ has a natural symplectic form ω for which the action of $U(V)$ is Hamiltonian, with moment map

$$\langle \mu([v]), X \rangle = -\frac{i}{2} \frac{(X.v, v)}{\|v\|^2}. \quad (15)$$

6.1 The moment map and symplectic coordinates on the open orbit in $\mathbb{P}(V)$

We fix an orthonormal basis $(e_k, 1 \leq k \leq n+1)$ of V . Thus $V = \mathbb{C}^{n+1}$, $U(V) = U(n+1)$, $\mathbb{P}(V) = \mathbb{P}_n(\mathbb{C})$. Let H be the n -dimensional compact torus which consists of diagonal matrices h with last entry equal to 1,

$$h = \begin{pmatrix} e^{i\theta_1} & & & \\ & \cdot & & \\ & & e^{i\theta_n} & \\ & & & 1 \end{pmatrix}.$$

We denote the Lie algebra of H by \mathfrak{h} . Let $(J_k, 1 \leq k \leq n)$ be the basis of \mathfrak{h} , where J_k is the diagonal matrix with (k, k) -entry equal to $i = \sqrt{-1}$ and other entries all 0. We denote the dual basis of \mathfrak{h}^* by $(\eta_k, 1 \leq k \leq n)$. Then the moment map for the action of H on $\mathbb{P}_n(\mathbb{C})$ is

$$\mu([z]) = \frac{1}{2\|z\|^2} \sum_{k=1}^n |z_k|^2 \eta_k, \text{ for } z = (z_1, \dots, z_{n+1}).$$

Its image is the simplex $\frac{1}{2}E_n$ where

$$E_n = \left\{ \sum_{k=1}^n t_k \eta_k; t_k \geq 0, \sum_{k=1}^n t_k \leq 1 \right\}.$$

The group $H_{\mathbb{C}}$ has an open orbit in $\mathbb{P}_n(\mathbb{C})$, namely the open set \mathcal{O} of points $[z_1, \dots, z_{n+1}]$ such that $z_k \neq 0$ for all k . Moreover $H_{\mathbb{C}}$ acts freely on \mathcal{O} . We

observe that the image of the open orbit \mathcal{O} under the moment map is exactly the interior of the simplex $\frac{1}{2}E_n$.

Let us construct an isomorphism of H -Hamiltonian spaces between \mathcal{O} and a neighborhood of the zero section in the cotangent bundle $T^*H \simeq H$. The \mathfrak{h}^* part is read out of the moment map, which for the action of H on $T^*H \simeq H \times \mathfrak{h}^*$ is the projection $H \times \mathfrak{h}^* \rightarrow \mathfrak{h}^*$.

We will also denote by E_n the standard simplex in \mathbb{R}^n , ie $E_n = \{t = (t_k) ; t_k \geq 0, \sum_{k=1}^n t_k \leq 1\}$. Its interior is denoted by E_n^0 .

We define an H -equivariant map $\mathcal{O} \rightarrow H \times \mathfrak{h}^*$ by

$$\exp(iX)h.m \mapsto (h, \mu(\exp(iX)m)).$$

The reciprocal is the map $H \times \frac{1}{2}E_n^0 \rightarrow \mathcal{O}$ given by

$$(e^{i\theta_1}, \dots, e^{i\theta_n}, t_1, \dots, t_n) \mapsto [z_1, \dots, z_{n+1}],$$

where

$$z_1 = \sqrt{2t_1}e^{i\theta_1}, \dots, z_n = \sqrt{2t_n}e^{i\theta_n}, z_{n+1} = \sqrt{1 - 2(t_1 + \dots + t_n)}.$$

We check easily that this map is indeed a diffeomorphism $H \times \frac{1}{2}E_n^0 \rightarrow \mathcal{O}$. Let us show that it is a *symplectomorphism*. We compute ω in the coordinates (θ_k, t_k) . Recall that the pullback of ω to the sphere is given by $q^*(\omega) = \Omega = \frac{i}{2} \sum_{k=1}^{n+1} dz_k \wedge d\bar{z}_k$. Let us denote $f(\theta, t) = (z_1, \dots, z_{n+1})$ with z_k given by the above formulas. Then we have $f^*(dz_{n+1} \wedge d\bar{z}_{n+1}) = 0$ since $z_{n+1} = \sqrt{1 - 2(t_1 + \dots + t_n)}$ is real. In order to compute the other terms, we observe that if $z = e^{i\theta}\sqrt{2t}$, then $dz \wedge d\bar{z} = dt \wedge d\theta$. Thus

$$\omega = \sum_{k=1}^n dt_k \wedge d\theta_k.$$

Thus, as expected, the pull back of ω is the canonical symplectic form of T^*H and the coordinates (θ_k, t_k) are Darboux coordinates on the open subset $\mathcal{O} \subset \mathbb{P}_n(\mathbb{C})$.

We have here a particular case of *action-angle coordinates*, (see for instance[2]): if there is a Hamiltonian action of a compact torus H of dimension n on a manifold M of dimension $2n$, with a free orbit $H.m$, then there exists a H -equivariant symplectomorphism of a neighborhood of $H.m$ onto a neighborhood of the zero section in $T^*H \simeq H \times \mathfrak{h}^*$. The \mathfrak{h}^* -component of the isomorphism (the action coordinates) is read out of the moment map, but the H -component (the angle coordinates) is not so obvious in the general case.

6.2 Convexity of the image of the moment map

6.2.1 Image of orbits and orbit closures under the moment map

Let T be a compact torus with a unitary action on V . Let $T_{\mathbb{C}}$ be the complexified torus. We study the induced action of $T_{\mathbb{C}}$ on the projective space $\mathbb{P}(V)$.

If Φ is a subset of \mathfrak{t}^* , we denote the (closed) convex hull of Φ by $E(\Phi)$ and its relative interior by $E^0(\Phi)$. Thus $E(\Phi)$ is a compact convex polyhedron (*polytope*). The definitions of supporting hyperplanes, faces, relative interior etc. which we recalled for polyhedral cones extend to convex polyhedrons.

Let $v \in V$. We write $v = \sum_{k=1}^{n+1} z_k e_k$ in the Hermitian basis of V which diagonalizes the action of T . Let λ_k be the corresponding weights. Recall that Φ_v is the set of λ_k such that $z_k \neq 0$, (see Section 5).

Theorem 64. *Let $v \in V \setminus 0$ and let $q(v)$ be the corresponding point in $\mathbb{P}(V)$.*

(i) *The image of the orbit $T_{\mathbb{C}}q(v)$ under the moment map μ is the relative interior $\frac{1}{2}E^0(\Phi_v)$ of the polytope $\frac{1}{2}E(\Phi_v) \subset \mathfrak{t}^*$.*

(ii) *The image of the orbit closure $\overline{T_{\mathbb{C}}q(v)}$ is the polytope $\frac{1}{2}E(\Phi_v)$.*

(iii) *Two points in $\overline{T_{\mathbb{C}}q(v)}$ have the same image under μ if and only if they are conjugate under T .*

(iv) *Furthermore, the moment map sets up a one to one correspondence between the $T_{\mathbb{C}}$ -orbits contained in $\overline{T_{\mathbb{C}}q(v)}$ and the faces of the polytope $E(\Phi_v)$.*

Proof. We can assume that Φ_v generates \mathfrak{t}^* .

For $X \in \mathfrak{t}$, we have $\exp(iX).v = \sum_k z_k e^{-\langle \lambda_k, X \rangle} e_k$. Thus

$$2\mu(q(\exp(iX).v)) = \frac{\sum_k |z_k|^2 e^{-2\langle \lambda_k, X \rangle} \lambda_k}{\sum_k |z_k|^2 e^{-2\langle \lambda_k, X \rangle}}$$

belongs to the convex hull $E(\Phi_v)$.

All we need is the following projective analogue of Lemma 57.

Lemma 65. *Let $\Phi = (\lambda_1, \dots, \lambda_s)$ be a finite sequence of linear forms on a real vector space L such that Φ generates L^* . Let $c_k > 0$ for $1 \leq k \leq s$. Then the map $m : L \rightarrow L^*$ given by*

$$m(Y) = \frac{\sum_{k=1}^s c_k e^{\langle \lambda_k, Y \rangle} \lambda_k}{\sum_{k=1}^s c_k e^{\langle \lambda_k, Y \rangle}}$$

is a diffeomorphism of L onto the interior of the convex hull $E(\Phi)$.

Proof. We introduce the vector space $L \oplus \mathbb{R}$ and $\tilde{\lambda}_k = (\lambda_k, 1) \in (L \oplus \mathbb{R})^*$. We consider the map $L \oplus \mathbb{R} \rightarrow L^* \oplus \mathbb{R}$ given by

$$(Y, x) \mapsto \left(\sum_k c_k e^{\langle \lambda_k, Y \rangle + x} \lambda_k, \sum_k c_k e^{\langle \lambda_k, Y \rangle + x} \right).$$

By Lemma 57, this map is a diffeomorphism of $L \oplus \mathbb{R}$ onto the open cone generated by the vectors $\tilde{\lambda}_k$. Let $u_k > 0$ such that $\sum_k u_k = 1$. Thus there exists a unique $Y \in L$ and $x \in \mathbb{R}$ such that

$$\sum_k c_k e^{\langle \lambda_k, Y \rangle + x} \lambda_k = \sum_k u_k \lambda_k$$

and

$$\sum_k c_k e^{\langle \lambda_k, Y \rangle + x} = 1.$$

Hence m is one to one from L onto the interior of the convex hull $E(\Phi)$. Similarly, the fact that m is a diffeomorphism follows from Lemma 57. \square

The other statements of the theorem are deduced from the corresponding statements of Theorem 60 in a similar way. \square

6.2.2 Image of a projective variety under the moment map

Let $\tilde{M} \subseteq V$ be an irreducible complex algebraic cone, defined by homogeneous polynomial equations. We assume that \tilde{M} is stable under the action of T , hence also stable under $T_{\mathbb{C}}$. Let $M = q(\tilde{M})$ be the corresponding projective variety. Then M is stable under the action of the compact torus T . Let $\Phi_M = \cup_{v \in \tilde{M}} \Phi_v$. Similarly to Theorem 63, one proves the following theorem.

Theorem 66. *The image of M under the moment map μ is the convex hull $E(\Phi_M)$ of Φ_M .*

6.2.3 Fixed points and vertices of the moment polytope

A point $q(v)$ is fixed under T if and only if v is a weight vector for the action of T . Let $V = \oplus_k V_{\lambda}$ be the decomposition of V in weight spaces for the action of T . The connected components of the variety of fixed points $\mathbb{P}(V)^T$ are the projective subspaces $\mathbb{P}(V_{\lambda})$. The image $2\mu(\mathbb{P}(V)^T)$ is the finite set Φ_V of weights of T in V .

Now let M be a T -invariant subset of $\mathbb{P}(V)$. Then $\mu(M^T)$ is a subset of Φ_V which is clearly contained in Φ_M . However, in general $\mu(M^T)$ is strictly contained in Φ_M .

Example 67. Let $\widetilde{M} = \{z \in \mathbb{C}^3 ; z_1 z_2 - z_3^2 = 0\}$ and $M = \mathbb{P}(\widetilde{M})$, and let T be the two dimensional torus acting on \mathbb{C}^3 by $(t_1^2 z_1, t_2^2 z_2, t_1 t_2 z_3)$. Thus the set of weights of \mathfrak{t} in V is $(2\eta_1, 2\eta_2, \eta_1 + \eta_2)$. The set Φ_M is also $(2\eta_1, 2\eta_2, \eta_1 + \eta_2)$.

The fixed points in $\mathbb{P}(V)$ are $[1, 0, 0], [0, 1, 0], [0, 0, 1]$. The first two points belong to M but the third one $[0, 0, 1]$ does not. Thus $\mu(M^T) = (2\eta_1, 2\eta_2)$.

On the other hand we have

Lemma 68. *If the image of $m \in M$ is a vertex of $\frac{1}{2}E\Phi_M$, then $m \in M^T$.*

Proof. We know that $\mu(m)$ is a relatively interior point of $\overline{\mu(T_{\mathbb{C}}).m} \subseteq \mu(M)$. So if $\mu(m)$ is an extremal point of $\mu(M) = E\Phi_M$, then $\overline{\mu(T_{\mathbb{C}}).m}$ must consist of just one point, hence Φ_m itself must consist of just one point. \square

Remark 69. *The points in $\mu(M^T)$ need not be extremal points of $\mu(M)$, as shown by the previous example where we take now the whole projective space.*

In conclusion, the convexity theorem for a projective variety M can be stated independently of the realization of M .

Theorem 70. *Let M be a closed irreducible subvariety of $\mathbb{P}(V)$ which is stable under the action of a compact torus T . Then the image of M^T under the moment map is a finite subset of \mathfrak{t}^* and the image of M is its convex hull.*

6.3 The convexity theorem for a Hamiltonian torus action on a compact manifold

In the case of a smooth compact manifold with a Hamiltonian action of a compact torus T , there is a similar convexity theorem, proved (independently) by M. Atiyah [1] and V. Guillemin and S. Sternberg [8]. The two situations have an intersection: the smooth projective varieties, but neither is contained in the other. Indeed, in the discussion above, we did not assume that the variety is smooth.

So, let M be a smooth compact manifold with a Hamiltonian action of a compact torus T . The fixed point set M^T is a closed submanifold

with finitely many connected components M_a^T . We saw that the moment map μ is constant on each M_a^T . Thus the image of M^T is the finite subset $\{\lambda_a = \mu(M_a^T)\}$ of \mathfrak{t}^* .

Theorem 71. *The image of M under the moment map is the convex hull of the points λ_a .*

We will not give the proof here. Good references are [2] and [3].

Remark 72. *For the convexity theorem, it is important that the symplectic two-form be non-degenerate. An example is $V = \mathbb{C}^2$ with a modified two-form. Let θ be the canonical 1-form,*

$$\theta = \frac{1}{2} \sum_{k=1,2} x_k dy_k - y_k dx_k.$$

Let $\Omega = d((\|z\|^2 - 1)\theta)$. Then Ω_z is non-degenerate except for $\|z\|^2 = 1/2$. Indeed, by invariance, it is enough to compute Ω_w for $w = (u, 0)$ with $u = \|z\|$. Then we have $\Omega_w = (2u^2 - 1)dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, which is non-degenerate except for $2u^2 - 1 = 0$. The natural action of \mathbb{T}^2 on V admits the moment map

$$\mu(z) = (\|z\|^2 - 1)(|z_1|^2 \eta_1 + |z_2|^2 \eta_2).$$

The image of the sphere $\|z\|^2 = r$ is the segment $(r - 1)(t_1 \eta_1 + t_2 \eta_2)$, with $t_1 > 0, t_2 > 0, t_1 + t_2 = r$. Thus the image of V is the union of these segments, the "butterfly" with one infinite wing defined by $xy \geq 0, x + y \geq -1/4$, See Fig. ??.

This example can be modified to become compact, with a butterfly $\{xy \geq 0, |x + y| \leq K\}$ as image of the moment map.

7 Orbits and moment map for a linear action of a complex reductive group

There are several equivalent definitions of a complex reductive group, see [6]. A complex reductive group is in particular a complex Lie group. We will much use the following property: let K be a maximal compact subgroup of G , then G is the complexification of K . We write $G = K_{\mathbb{C}}$.

Thus some properties of an action of G can be obtained by restricting the action to K and constructing K -invariants by integration over K . This method was called the "unitarian trick" by Herman Weyl. In particular, any linear action of G on a finite dimensional complex vector space is *completely reducible*, hence the word reductive.

In this section, V will be a finite dimensional complex vector space with a linear action of G .

7.1 Some properties of G orbits in V

We recall some properties, see [4].

- Orbit closures $\overline{G.v}$ for the usual topology coincide with orbit closures for the Zariski topology.
- A G -orbit is Zariski open in its closure.
- The algebra of invariants $\mathbb{C}[V]^G$ is finitely generated. (Examples ...).

It follows that for any $u \in \overline{G.v} \setminus G.v$, the dimension of $G.u$ is strictly smaller than the dimension of $G.v$.

Another consequence is the

Proposition 73. (i) *G -invariant polynomials separate Zariski-closed G -invariant subsets of V , in particular closed orbits: if F_1 and F_2 are Zariski-closed, G -invariant and distinct, there exists $p \in \mathbb{C}[V]^G$ such that $p(v_1) \neq p(v_2)$.*
(ii) *The closure of an orbit contains exactly one closed orbit.*

Proof. If F_1 and F_2 are Zariski closed and disjoint, there exists a polynomial $p \in \mathbb{C}[V]$ which takes the value 0 on F_1 and the value 1 on F_2 . Indeed, let I_i be the ideal of polynomials which vanish on F_i , for $i = 1, 2$. Then $I_1 + I_2 = \mathbb{C}[V]$, by the nullstellensatz. Assume moreover that F_1 and F_2 are

G -invariant. Let dk be a Haar measure on K . Then $\tilde{p}(v) = \int_K p(k.v)dk$ is a K -invariant polynomial which still takes the value 0 on F_1 and the value 1 on F_2 . Let us show that \tilde{p} is actually G -invariant. (This is an instance of the unitarian trick). For $v \in V$ fixed, the function $g \mapsto \tilde{p}(g.v) - \tilde{p}(v)$ is holomorphic on G and vanishes on K , hence it is identically 0.

Any invariant polynomial is constant on the closure $\overline{G.v}$, therefore, by (i), there can be at most one closed orbit contained in $\overline{G.v}$. Let $G.u \subset \overline{G.v}$ be an orbit of minimal dimension. Then $G.u$ is closed. \square

7.2 Stable and semi-stable points

Definition 74. Let $v \neq 0$ in V .

v is called *unstable* if $0 \in \overline{G.v}$.

v is called *semi-stable* if $0 \notin \overline{G.v}$.

v is called *stable* if $G.v$ is closed.

The set of stable points is denoted by V_s . The set of semi-stable points is denoted by V_{ss} .

Let N be the set of unstable points. The *nilcone* $N \cup \{0\}$ is the set of common zeroes of the ideal $\mathbb{C}[V]_+^G \subset \mathbb{C}[V]^G$ of invariant polynomials without constant term. Let q_1, \dots, q_r be a set of generators of $\mathbb{C}[V]^G$, homogeneous of degree > 0 . Then $N \cup \{0\}$ is also the set of common zeroes of q_1, \dots, q_r . The set V_{ss} of semi-stable points is Zariski open. A point v is semi-stable if and only if there exists an invariant polynomial without constant term $p \in \mathbb{C}[V]_+^G$ such that $p(v) \neq 0$.

The affine variety with ring of regular functions $\mathbb{C}[V]^G$ is denoted by $V//G$. Its points are the closed orbits of G in V . It can be seen as the quotient of V by an equivalence relation which is not the G -action, but an enlarged equivalence relation: v is equivalent to v' if there exists a curve $g(t) \in G$ such that $\lim_{t \rightarrow \infty} g(t).v = \lim_{t \rightarrow \infty} g(t).v'$. We will see later that the moment map provides a realization of $V//G$ as a quotient of a submanifold of V under the compact group K .

We fix a K -invariant hermitian scalar product $h(u, v)$ on V .

Let $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$ be the Cartan decomposition of \mathfrak{g} . Thus $i\mathfrak{k}$ acts on V by hermitian operators with real eigenvalues. Let $v \in V$ and $X \in \mathfrak{k}$. Then $\lim_{t \rightarrow +\infty} \exp(itX).v = 0$ if and only if v is a sum of eigenvectors with eigenvalues < 0 , with respect to iX . This remark is the basis of Hilbert-Mumford instability criterion.

Theorem 75 (Hilbert-Mumford criterion). *A point $v \in V$ is unstable if and only if there exists $X \in \mathfrak{k}$ such that $\lim_{t \rightarrow +\infty} \exp(itX).v = 0$.*

More generally, let $G.u$ be the unique closed orbit contained in $\overline{G.v}$, then there exists $X \in \mathfrak{k}$ such that the limit $\lim_{t \rightarrow +\infty} \exp(itX).v$ exists and belongs to $G.u$.

Remark 76. *When G is a torus, any orbit $G.w \subseteq \overline{G.v}$ can be reached by the action of a one-parameter subgroup, as we saw in Corollary 61. This is no longer true when G is not commutative. The criterion holds for the closed orbit contained in $\overline{G.v}$.*

Example Let $G = \mathrm{SL}(2, \mathbb{C})$ and let $V = S^3(\mathbb{C}^2)$ be the space of homogeneous polynomials in two variables $p(x, y)$. We consider $v = xy^2$.

If $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $(g.p)(x, y) = p(ax + by, cx + dy)$.

For $X = \begin{pmatrix} it & 0 \\ 0 & -it \end{pmatrix}$, we have $\exp(itX).v = e^{-t}xy^2$, hence $\lim_{t \rightarrow +\infty} \exp(itX).v =$

0. Let $g_n = \begin{pmatrix} n & n^2 \\ 0 & \frac{1}{n} \end{pmatrix}$. Then $g_n^{-1}.v = \frac{1}{n}xy^2 + y^3$, hence $w = y^3 \in \overline{G.v}$.

On the other hand, let us show that there is no one-parameter subgroup $\exp(tX)$ such that $\lim_{t \rightarrow +\infty} \exp(tX).v \in G.y^3$.

Indeed, if $\lim_{t \rightarrow +\infty} \exp(tX).v = g.y^3$, then $\exp(tX)g.y^3 = g.y^3$ for every t . Therefore (replacing X with $g^{-1}Xg$), we have $X.y^3 = 0$. Here we denote by $X.v$ the infinitesimal action of the Lie algebra. If E, F, H is the usual basis of $\mathfrak{sl}(2, \mathbb{C})$, we have $E.v(x, y) = -y\frac{\partial v}{\partial x}$, $F.v(x, y) = x\frac{\partial v}{\partial y}$, $H.v(x, y) = -x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}$. Therefore we must have $X \in \mathbb{C}E$. But $\exp(tzE).xy^2 = xy^2 - tzy^3$ does not have a limit when $t \rightarrow \infty$.

Proof of Theorem 75, after H.Kraft [17]. Let $T \subseteq K$ be a maximal compact torus and let \mathfrak{t} be its Lie algebra. The proof uses the Cartan decomposition $G = KAK$ where $A = \exp(it)$. See for instance [13] or [9]. By Theorem ??, it is enough to prove that there exists $k \in K$ such that $\overline{T_{\mathbb{C}}(k.v)}$ intersects $G.u$. Indeed, if $\overline{T_{\mathbb{C}}(k.v)}$ intersects $G.u$, there exists $X \in \mathfrak{t}$ such that $\lim_{t \rightarrow \infty} \exp(itX)k.v$ exists and belongs to $G.u$. Writing $\exp(itX)kv = k \exp(itk^{-1}.X).v$, we obtain the result, as $k^{-1}.X \in \mathfrak{k}$.

Assume that $\overline{T_{\mathbb{C}}.w}$ does not meet $G.u$ for any $w \in K.v$. Then for every $w \in K.v$, there exists a $T_{\mathbb{C}}$ -invariant polynomial f_w on V which takes the value 0 on $G.u$ and the value 1 on $\overline{T_{\mathbb{C}}.w}$. Let

$$U_w = \{z \in V, f_w(z) \neq 0\}.$$

Then the compact set $K.v$ is contained in $\bigcup_{w \in K.v} U_w$, therefore it is contained in a finite union

$$K.v \subseteq U_{w_1} \cup \dots \cup U_{w_n}.$$

Consider the function $f(z) = \sum_{k=1}^n |f_{w_k}(z)|$. Then $f(z)$ is $T_{\mathbb{C}}$ -invariant, continuous, positive, > 0 on $K.v$ and equal to 0 on $G.u$. Therefore there exists $a > 0$ such that $f(z) \geq a$ for $z \in K.v$. Then $f(z) \geq a$ for $z \in \overline{T_{\mathbb{C}}K.v}$ as well, since f is $T_{\mathbb{C}}$ -invariant. Hence, $\overline{T_{\mathbb{C}}K.v}$ does not intersect $G.u$, hence $K\overline{T_{\mathbb{C}}K.v}$ does not intersect $G.u$ either.

Now, by the Cartan decomposition, we have $K\overline{T_{\mathbb{C}}K.v} = \overline{G.v}$, which contains $G.u$, thus a contradiction. \square

Corollary 77. *If $T_{\mathbb{C}}.v$ is closed for every maximal compact torus $T \subset K$, then $G.v$ is closed.*

Proof. Fix a compact torus $T \in K$. Then every other compact torus in K is conjugate to T , so $T' = k^{-1}Tk$, for $k \in K$. Thus $T'_{\mathbb{C}}.v$ is closed if and only if $T_{\mathbb{C}}kv$ is closed. If every $T_{\mathbb{C}}kv$ is closed, then $G.v$ is closed by Theorem 75. \square

Remark 78. *The converse is false, as shown by the following example. Let $G = SL(3, \mathbb{C})$ acting on its Lie algebra by the adjoint representation. Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then A is diagonalizable, therefore its G -orbit is closed (see Example 82 below). On the other hand, the orbit under the group of diagonal matrices is not closed.*

7.3 Closed orbits and the moment map

A moment map $V \rightarrow \mathfrak{k}^*$ for the K action on V is given in terms of the K -invariant Hermitian scalar product. For $X \in \mathfrak{k}$ and $v \in V$,

$$\langle \mu(v), X \rangle = -\frac{i}{2}h(X.v, v).$$

The moment map is the differential of the norm $\|v\|^2$ along the G -orbit of v , up to a constant factor. More precisely,

Lemma 79. *Let $X \in \mathfrak{k}$ and $v \in V$. Let $f(t) = \|\exp(itX).v\|^2$. Then $f(t)$ is convex, with derivative at $t = 0$*

$$\frac{d}{dt}\Big|_{t=0} \|\exp(itX).v\|^2 = 2ih(X.v, v) = -4\langle \mu(v), X \rangle. \quad (16)$$

Proof. Let e_k be an orthonormal basis of eigenvectors for the action of iX , with (real) eigenvalues λ_k . Let $v = \sum_k z_k e_k$. Let $f(t) = \|\exp(itX).v\|^2 = \sum_k e^{2\lambda_k t} |z_k|^2$. Then $f''(t) = 4 \sum_k e^{2\lambda_k t} \lambda_k^2 |z_k|^2 \geq 0$, thus $f(t)$ is convex. \square

Proposition 80. *Let \mathcal{O} be a closed G -orbit in V and let \mathcal{O}_{min} be the set of points of \mathcal{O} with minimum norm. Then $\mathcal{O}_{min} = \mu^{-1}(0) \cap \mathcal{O}$. Moreover \mathcal{O}_{min} is a K -orbit.*

Proof. The set \mathcal{O}_{min} (called the orbit core) is not empty if \mathcal{O} is closed. If $v \in \mathcal{O}_{min}$, then $t = 0$ is a critical point for the function $\|\exp(itX).v\|^2$ hence $\mu(v) = 0$ by (16).

Conversely, let $v \in \mu^{-1}(0) \cap \mathcal{O}$. We want to show that $\|g.v\| \geq \|v\|$ for every $g \in G$. Any $g \in G$ can be written $g = \exp(iX)k$ with $k \in K$ and $X \in i\mathfrak{k}$. As $\|k.v\| = \|v\|$, we can assume that $g = \exp(iX)$.

Let $f(t) = \|\exp(itX).v\|^2$. We have $f'(0) = 0$, hence $\|g.v\|^2 = f(1) \geq f(0) = \|v\|^2$. Thus $v \in \mathcal{O}_{min}$.

Assume that $\exp(iX).v$ is also in \mathcal{O}_{min} . Then $f(1) = f(0)$ therefore $f(t)$ must be constant for $t \in [0, 1]$. This implies $\lambda_k = 0$ for all k such that $z_k \neq 0$, hence $\exp(iX).v = v$. Thus \mathcal{O}_{min} is a K -orbit. \square

Theorem 81. *A G orbit \mathcal{O} is closed if and only if it intersects $\mu^{-1}(0)$. Moreover two points of $\mu^{-1}(0)$ which are conjugate under G are conjugate under K .*

Proof. Assume that $\mu(v) = 0$. Then for every compact torus T , the moment map relative to T vanishes at v . As we have seen in Chapter ??, then the $T_{\mathbb{C}}$ orbit $T_{\mathbb{C}}.v$ is closed. Hence the G -orbit $G.v$ is also closed, by Corollary 77. The second statement has already been proved in Proposition 80. \square

In other words, the space of closed orbits $V//G$ is in one to one correspondence with the topological space $\mu^{-1}(0)/K$. Therefore $\mu^{-1}(0)/K$ has a structure of affine variety, with affine algebra $\mathbb{C}[V]^G$.

Example 82. Let $V = \mathfrak{k}_{\mathbb{C}}$ with $G = K_{\mathbb{C}}$ acting by the adjoint action. We take the Hermitian product given by $h(X, Y) = Q(X, \bar{Y})$ where Q is the Killing form. We identify \mathfrak{k} with \mathfrak{k}^* by means of $-Q$. Then the moment map is

$$\mu(A + iB) = -[A, B], \quad \text{for } A, B \in \mathfrak{k}.$$

Indeed, we have

$$\langle \mu(A + iB), X \rangle = \frac{i}{2} Q([X, A + iB], A - iB) = Q(X, [A, B]).$$

A and B are semi-simple, (adA and adB are diagonalizable endomorphisms of $\mathfrak{k}_{\mathbb{C}}$). If $[A, B] = 0$, then $A + iB$ is also semi-simple. So we have shown that a $K_{\mathbb{C}}$ -orbit in $\mathfrak{k}_{\mathbb{C}}$ is closed if and only if it consists of semi-simple elements.

Proposition 83. Let $v \in \mu^{-1}(0)$. Then the stabilizer G_v of v in G is the complexification of its stabilizer K_v in K .

Proof. Let $g = \exp(iX)k$ be such that $g.v = v$. We consider again the convex function $f(t) = \|\exp(itX)k.v\|^2$. We have $\mu(kv) = 0$, hence $f'(0) = 0$. On the other hand, we have $f(0) = \|kv\|^2 = \|v\|^2 = f(1)$, hence $f(t)$ is constant on $[0, 1]$. Let e_k be an orthonormal basis of eigenvectors for the action of iX , with (real) eigenvalues λ_k . Let $k.v = \sum_k z_k e_k$. Computing $f''(t)$ as in the proof of Proposition 80, we obtain $\lambda_k = 0$ for all eigenvalues of iX such that $z_k \neq 0$. Thus $X.k.v = 0$, hence $g.v = k.v$, and finally $v = k.v$. \square

Remark 84. The converse statement is not true. A point v can have a reductive stabilizer although its orbit Gv is not closed, thus $v \notin \mu^{-1}(0)$. An example is $v = xy^2$ for the action of $\mathrm{SL}(2, \mathbb{C})$. The stabilizer is just $\{1\}$, but the orbit is not closed, for instance $ax(a^{-1}y)^2 = a^{-1}xy^2$ belongs to it for any $a \in \mathbb{C}$.

Proposition 85. Assume that 0 is a regular value for the restriction of μ to the unit sphere $\mu^{-1}(0)$. Then every semi-stable point $v \in V_{ss}$ has a closed orbit and a finite stabilizer.

Proof. Let $v \in V_{ss}$. Let $G.u \subseteq \overline{G.v}$ be the (unique) closed orbit contained in $\overline{G.v}$. We can assume that $\mu(u) = 0$. By Lemma 27, the point $\frac{u}{\|u\|}$ has a finite stabilizer in K . By proposition 83, u has also a finite stabilizer in G . It follows that the dimension of $G.u$ cannot be strictly smaller than the dimension of $G.v$, hence $G.u = G.v$. \square

8 Action of a complex reductive group on a projective variety: Kirwan-Mumford convexity theorem.

In this section, we will use implicitly some results about actions of algebraic groups on algebraic varieties. A good reference is the lecture notes by Michel Brion http://ccirm.cedram.org/cedram-bin/article/CCIRM_2010__1_1_1_0.pdf

8.1 Highest weights of the space of regular functions on a G -invariant algebraic cone

Let V be a complex vector space. We denote the projective space by $\mathbb{P}(V)$ and the map $V \setminus 0 \rightarrow \mathbb{P}(V)$ by $u \mapsto q(u)$ or simply $u \mapsto [u]$. The algebra of polynomial functions on V is identified with the symmetric algebra $S(V^*)$. We write $S(V^*) = \bigoplus_{n=0}^{\infty} S^n(V^*)$, where $S^n(V^*)$ is the space of homogeneous polynomial of degree n .

Let $G \subseteq \mathrm{GL}(V)$ be a connected complex reductive subgroup of $\mathrm{GL}(V)$.

We fix a maximal compact subgroup K of G and a K -invariant Hermitian scalar product $h(u, v)$ on V .

The Lie algebra of K is denoted by \mathfrak{k} and its dual by \mathfrak{k}^* . We recall that the moment map $\mu_V : V \rightarrow \mathfrak{k}^*$ is given by

$$\langle \mu_V(v), X \rangle = -\frac{i}{2} h(Xv, v),$$

and that the moment map $\mu : \mathbb{P}(V) \rightarrow \mathfrak{k}^*$ is given by

$$\mu([v]) = \frac{\mu_V(v)}{\|v\|^2}.$$

In this section, we consider a G -stable algebraic cone $C \subseteq V$.

By definition, a regular function on the cone C is the restriction to C of a polynomial function on V . We denote the space of regular functions on C by $R(C)$. We have

$$R(C) = \bigoplus_{n=0}^{\infty} R_n(C).$$

Let T be a maximal torus of K , with Lie algebra \mathfrak{t} . We fix a system of positive roots Δ^+ . For $\alpha \in \Delta^+$, we denote by $H_\alpha \in \mathfrak{t}$ the corresponding

co-root, defined by the equation $\langle \alpha, H \rangle = Q(H, H_\alpha)$ for every $H \in \mathfrak{t}$. The negative Weyl chamber is

$$\mathfrak{t}_-^* = \{\lambda \in \mathfrak{t}^*, i\langle \lambda, H_\alpha \rangle \leq 0 \text{ for every } \alpha \in \Delta^+\}. \quad (17)$$

Any K -invariant subset $E \subseteq \mathfrak{t}^*$ is determined by its intersection with \mathfrak{t}_-^* , since we have $E = K(E \cap \mathfrak{t}_-^*)$. Our goal is to describe in this manner the image $\mu(\mathbb{P}(C))$ under the moment map, by computing $\mu(\mathbb{P}(C)) \cap \mathfrak{t}_-^*$ in terms of the representation of K in $R(C)$.

Remark 86. *It is only to simplify some computations that we consider the negative Weyl chamber rather than the positive one \mathfrak{t}_+^* . A K -invariant subset $E \subseteq \mathfrak{t}^*$ is determined as well by its intersection with \mathfrak{t}_+^* . The two sets are related by*

$$E \cap \mathfrak{t}_-^* = -w_0(E \cap \mathfrak{t}_+^*)$$

where w_0 is the longest element of the Weyl group.

Let \hat{K} be the set of irreducible representations of K , identified with finite dimensional rational irreducible representations of G . For $\pi \in \hat{K}$, we denote the isotypic subspace of type π of $R(C)$ by $R(C)_\pi$. We have

$$R(C) = \bigoplus_{\pi \in \hat{K}} R(C)_\pi.$$

The set \hat{T} of characters of the torus T is identified with the weight lattice $P \subset i\mathfrak{t}^*$. For $\Lambda \in P$, the corresponding character is denoted by e^Λ , it is given by $e^\Lambda(\exp X) = e^{\langle \Lambda, X \rangle}$. The highest weight of $\pi \in \hat{K}$ is denoted by Λ_π , so that $\Lambda_\pi = i\lambda_\pi$, with $\lambda \in \mathfrak{t}_+^*$, the positive Weyl chamber.

Definition 87. *We denote by $P_n(C)$ the set of highest weights of [irreducible sub-representations of] $R_n(C)$.*

Let $i\mathfrak{t}_\mathbb{Q}^* \subset i\mathfrak{t}^*$ be the rational vector space generated by P over \mathbb{Q} . We consider the following subset of $i\mathfrak{t}_\mathbb{Q}^*$

$$\mathfrak{p}(C) = \bigcup_{n=0}^{\infty} \frac{P_n(C)}{n}. \quad (18)$$

Proposition 88. *The set $\mathfrak{p}(C)$ is convex. Moreover it is the convex hull (over \mathbb{Q}) of a finite set.*

Proof. Let $i\lambda$ be a highest weight in $R_m(C)$ and $i\mu$ be a highest weight in $R_n(C)$. Then there is a polynomial $f_\lambda \in R_m(C)$ which is a highest weight vector of weight $i\lambda$ and a polynomial $f_\mu \in R_n(C)$ which is a highest weight vector of weight $i\mu$. The product $f_\lambda f_\mu \in R_{m+n}(C)$ is non zero, hence it is a highest weight vector of weight $i(\lambda + \mu)$. Let $a \in \mathbb{Q}$, $0 < a < 1$. Let $i\nu = a\frac{i\lambda}{m} + (1-a)\frac{i\mu}{n}$. Let us show that $i\nu \in \mathfrak{p}(C)$. Let $a = \frac{p}{q}$ with p, q integers. Then $qmn i\nu = pn i\lambda + (q-p)m i\mu$ is the highest weight of a K -type of $R_{qmn}(C) = R_{pmn+(q-p)mn}(C)$. So we have proved that $\mathfrak{p}(C)$ is convex.

Next we show that $\mathfrak{p}(C)$ is the convex hull of a finite set. We deduce this result from the fact that the algebra $R(C)^{n^+}$ is finitely generated, which will be proven below (Theorem 90). Let $(f_k, k \in K)$ be a finite set of generators of $R(C)^{n^+}$. We can assume that f_k is homogeneous of degree n_k and is a weight vector of weight $i\lambda_k$. Let $f \in R_n(C)$ be a highest weight vector with weight $i\lambda$. By Theorem 90, we can write f as a polynomial in the generators f_k . Therefore there exist non-negative integers p_k such that $n = \sum_{k \in K} p_k n_k$ and $\lambda = \sum_{k \in K} p_k \lambda_k$. Hence $\frac{\lambda}{n}$ belongs to the convex hull of the points $\frac{\lambda_k}{n_k}$. \square

We will now prove that $R(C)^{n^+}$ is finitely generated.

Remark 89. *If a reductive group G acts on an affine algebraic variety A , then the algebra of invariants $R(A)^G$ is finitely generated (see for instance [6]). In contrast, if N is a unipotent group with a linear action on a vector space L , then $S(L^*)^N$ need not be finitely generated, as shown by M.Nagata in his famous counterexample to Hilbert's fourteen's problem, for an action of \mathbb{C}^{13} on \mathbb{C}^{32} [21]. More recently, an example where \mathbb{C}^6 acts linearly on \mathbb{C}^{18} has been given by R.Steinberg [23].*

Theorem 90. *Let L be complex vector space with a linear action of a complex reductive group G . Let $A \subseteq L$ be a G -stable algebraic subset of L . Then $R(A)^{n^+}$ is finitely generated.*

Proof. Let $R(G)$ be the ring of regular functions on G . First we map $R(A)$ into $R(G \times A) = R(G) \otimes R(A)$ by the map

$$\Phi(f)(g, x) = f(g.x).$$

Example. Let $G = \text{GL}(2, \mathbb{C})$ with the standard representation on $L = \mathbb{C}^2$. Take $f(x) = x_1 x_2 \in S^2(L^*)$. For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have $f(gx) = acx_1^2 + (bc + ad)x_1 x_2 + bdx_2^2$.

Let G act on $G \times A$ by $(gg_0^{-1}, g_0.x)$. Then Φ is a linear bijection of $R(A)$ onto $R(G \times A)^G$, with inverse map given by $f(x) = F(1, x)$. Furthermore Φ commutes with the actions of N , where N acts on $G \times A$ by acting only on the first component G on the left, that is (g_0g, x) . Thus it is enough to prove that $R(G)^N$ is finitely generated.

Any function $h \in R(G)^N$ can be written as a coefficient of a finite-dimensional representation W , in the form $h(g) = \langle g^{-1}v, w \rangle$, where $v \in W^{n^+}$. We decompose W as a sum of irreducible representations V_λ with highest weight λ . We decompose each such λ as a sum of fundamental weights $\lambda = k_1\omega_1 + \dots + k_r\omega_r$. Thus V_λ is a quotient of the tensor product $V_{\omega_1}^{\otimes k_1} \otimes \dots \otimes V_{\omega_r}^{\otimes k_r}$, the highest weight vector v_λ being the image of the product $v_{\omega_1}^{\otimes k_1} \otimes \dots \otimes v_{\omega_r}^{\otimes k_r}$. In this way, we obtain h as a polynomial in the N -invariant coefficients of the fundamental representations $V_{\omega_1}, \dots, V_{\omega_r}$. \square

8.2 Kirwan-Mumford convexity theorem

Theorem 91 (Kirwan-Mumford). *Let $C \subseteq V$ be a G -stable algebraic cone. Then the intersection $\mu(\mathbb{P}(C)) \cap \mathfrak{t}_{\mathbb{Q},-}^*$ is equal to $-i\mathfrak{p}(C)$.*

Proof. Let w_0 be the longest element of the Weyl group W . The highest weight of π_Λ^* is $-w_0\Lambda$. Let $\tilde{P}_n(C)$ be the set of weights Λ such that π_Λ^* is a sub-representation of $R_n(C)$. Let

$$\tilde{\mathfrak{p}}(C) = \bigcup_{n \in \mathbb{N}} \frac{\tilde{P}_n(C)}{n}.$$

In terms of the positive Weyl chamber, the theorem can be rephrased as follows.

$$\mu(\mathbb{P}(C)) \cap \mathfrak{t}_{\mathbb{Q},+}^* = i\tilde{\mathfrak{p}}(C). \quad (19)$$

We first prove a key particular case of the theorem, in the form (19).

Example 92. *let $C = V$ and $G = \mathrm{GL}(V)$ with maximal compact subgroup $K = \mathrm{U}(V)$ and maximal torus the diagonal. For every $n \in \mathbb{N}$, $R_n(C) = S^n(V^*)$ is an irreducible representation of G , with highest weight $(n, 0, \dots, 0)$. The projective space $\mathbb{P}(V)$ is a single K -orbit and its image $\mu(\mathbb{P}(V))$ under the moment map is the K -orbit of the weight $(1, 0, \dots, 0)$.*

Example 93. *More generally, let $V = V_\Lambda$ be the space of the irreducible representation of K with highest weight $\Lambda = i\lambda$ and let $C_\Lambda = G(\mathbb{C}v_\Lambda)$ be*

the G -orbit of the line $\mathbb{C}v_\Lambda$, where v_Λ is a highest weight vector. In the next lemma, we prove that $\tilde{\mathfrak{p}}(C_\Lambda) = \{\Lambda\}$. This is a particular case of Theorem 91.

Lemma 94. (i) C_Λ is a closed algebraic cone, equal to $K(\mathbb{C}v_\Lambda)$.

(ii) We have $\mu([v_\Lambda]) = \lambda$. The moment map is an isomorphism between the projective variety $\mathbb{P}(C_\Lambda)$ and the orbit $K\lambda \subset \mathfrak{k}^*$.

(iii) For every $n \in \mathbb{N}$, $R_n(C_\Lambda)$ is irreducible and isomorphic to $V_{n\Lambda}^*$ as a G -module.

Proof. We have $\mathfrak{n}^+v_\Lambda = 0$. Let $\mathfrak{b} = \mathfrak{t}_\mathbb{C} \oplus \mathfrak{n}^+$. Let $B = \exp \mathfrak{b}$, the Borel subgroup of G associated to Δ_+ . Then $G = KB$ and $Bv_\Lambda = \mathbb{C}^*v_\Lambda$, hence $G(\mathbb{C}v_\Lambda) = K(\mathbb{C}v_\Lambda)$. Therefore $G(\mathbb{C}v_\Lambda)$ is a constructible set which is closed for the Euclidean topology, hence it is also closed for the Zariski topology.

Let us prove (ii). By K -invariance of the Hermitian product, we have $h(Zu, v) + h(u, \theta(Z)v) = 0$, for any $Z \in \mathfrak{g}$, where θ is the Cartan involution. Recall that any element of \mathfrak{k} can be written as a sum $X + Y - \theta(Y)$, with $X \in \mathfrak{t}$ and $Y \in \mathfrak{n}^+$. If $X \in \mathfrak{t}$ we have

$$\langle \mu([v_\Lambda]), X \rangle = -i \frac{h(Xv_\Lambda, v_\Lambda)}{\|v_\Lambda\|^2} = \langle \lambda, X \rangle.$$

If $Y \in \mathfrak{n}^+$, we have $Y.v_\Lambda = 0$, hence $h((Y - \theta(Y))v_\Lambda, v_\Lambda) = h(Yv_\Lambda, v_\Lambda) + h(v_\Lambda, Yv_\Lambda) = 0$. Hence $\mu([v_\Lambda]) = \lambda$ and the moment map μ induces a surjective map of $\mathbb{P}(C_\Lambda) = K.[v_\Lambda]$ onto $K\lambda$. It can be shown that the stabilizer of λ in K is always equal to the stabilize of $[v_\Lambda]$. If λ is regular, it is just T .

Let us prove (iii). Let $v_{-\Lambda}^* \in V_\Lambda^*$ be the lowest weight vector, with weight $-\Lambda$. As a function on V , it does not vanish on C_Λ since $\langle v_{-\Lambda}^*, v_\Lambda \rangle = 1$. Its power $(v_{-\Lambda}^*)^n$ is a non zero element of $R_n(C_\Lambda)$ and a lowest weight vector of weight $-n\Lambda$. Hence $V_{n\Lambda}^* \subseteq R_n(C_\Lambda)$ for every $n \in \mathbb{N}$.

Conversely, let $F \in R_n(C_\Lambda)$ be a lowest weight vector. Thus in particular, $F(gv_\Lambda) = F(v_\Lambda)$ for every $g \in N_-$. Therefore $F(v_\Lambda) \neq 0$, since by assumption $F(v)$ is not identically zero on C_Λ . Next, let us show that the weight of F is $-n\Lambda$. For $X \in \mathfrak{t}$ we have $(\exp XF)(v_\Lambda) = F(\exp(-X)v_\Lambda) = F(e^{-\langle \Lambda, X \rangle} v_\Lambda) = e^{-n\langle \Lambda, X \rangle} F(v_\Lambda)$. \square

Remark 95. *B.Kostant [?] has proved that the ideal which vanish on the algebraic cone C_Λ is generated by its homogeneous elements of degree 2.*

Let us now prove the theorem in general. First let us show that $-i\mathfrak{p}(C) \subseteq \mu(\mathbb{P}(C))$. Let $\Lambda = i\lambda$ be a highest weight of $R_n(C)$. We want to find $x \in C \setminus 0$ such that $\mu([x]) = -\frac{\lambda}{n}$.

Since V_Λ is a sub-representation of $R_n(C)$, there exists an injective intertwining operator $T \in \text{Hom}_G(V_\Lambda, R_n(C))$. We can consider T as an element of $\text{Hom}_G(V_\Lambda, S^n(V^*))$ with the property that if $v \in V_\Lambda, v \neq 0$, then $x \mapsto T(v)(x)$ is not identically 0 on C .

Let $W = V \oplus V_\Lambda$. For $w = (x, v) \in W$, let $Q(w) = T(v)(x)$. Then Q is a G -invariant polynomial function on W , homogeneous of degree 1 with respect to $v \in V_\Lambda$ and homogeneous of degree n with respect to $x \in V$. Let \mathbb{C}^* act on W by $t(x, v) = (tx, t^{-n}v)$. Then Q is invariant under the action of $G \times \mathbb{C}^*$. Let $\mu_W : W \rightarrow \mathfrak{k}^* \oplus \mathbb{R}$ be the moment map for the action of $K \times S^1$.

Let $x_0 \in C$ be such that $Q(x_0, v_\Lambda) = T(v_\Lambda)(x_0) \neq 0$. There exists a point (x, v) in the closure of the $(G \times \mathbb{C}^*)$ -orbit of (x_0, v_Λ) such that $\mu_W(x, v) = 0$. As Q is constant on orbit closures, we have $Q(x, v) = Q(x_0, v_\Lambda) \neq 0$. Therefore $x \neq 0$ and $v \neq 0$. As C is G -invariant, the point x is in $C \setminus 0$.

The vector v belongs to $G(\mathbb{C}v_\Lambda)$ which is equal to $K(\mathbb{C}v_\Lambda)$ as we saw in Lemma 94 (i). Up to K -action, we can assume that $v = tv_\Lambda$. The relation $\mu_W(x, v) = 0$ means that $\mu_V(x) + \|tv_\Lambda\|^2\lambda = 0$ and $\|x\|^2 - n\|tv_\Lambda\|^2 = 0$, hence $\mu([x]) = -\frac{\lambda}{n}$.

Conversely, let $\gamma \in \mu(\mathbb{P}(C)) \cap \mathfrak{t}_{\mathbb{Q}, -}^*$. We want to find $N \in \mathbb{N}$ such that $-iN\gamma$ is a highest weight of $R_N(C)$. First we choose n so that $\Lambda = -in\gamma$ is a dominant weight. We consider as above $W = V \oplus V_\Lambda$ with the action of $G \times \mathbb{C}^*$ and moment map μ_W . Let $v_0 \in C$ such that $\|v_0\| = 1$ and $\mu_V(v_0) = \gamma$. Let $v_\Lambda \in V_\Lambda$ a highest weight vector with norm $\|v_\Lambda\| = 1$. Then $\mu_W(n^{1/2}v_0, v_\Lambda) = 0$. Hence the $(G \times \mathbb{C}^*)$ -orbit of $(n^{1/2}v_0, v_\Lambda)$ is closed and $\neq \{(0, 0)\}$. Therefore there exists a $(G \times \mathbb{C}^*)$ -invariant polynomial $Q(x, v)$ on W which takes the value 0 at $(0, 0)$ and the value 1 at $(n^{1/2}v_0, v_\Lambda)$. Then Q has a constant term equal to 0. Considering the \mathbb{C}^* -action, we have $S(W^*)^{\mathbb{C}^*} = \bigoplus_{k \in \mathbb{N}} S^{nk}(V^*) \otimes S^k(V_\Lambda^*)$. Therefore Q has at least one component $Q_k \in S^{nk}(V^*) \otimes S^k(V_\Lambda^*)$, for some $k > 0$, such that $Q_k((n^{1/2}v_0, v_\Lambda)) \neq 0$. Observe that Q_k is G -invariant.

Let $C_\Lambda = G(\mathbb{C}v_\Lambda) \subset V_\Lambda$ as in Lemma 94. The restriction of Q_k to the cone $C \oplus C_\Lambda$ is non zero. By Lemma 94, it gives a non zero element of $(R^{nk}(C) \otimes V_{k\Lambda}^*)^G$. Thus we have proved that $k\Lambda = -ink\gamma$ is a highest weight of $R_{nk}(C)$, as wanted. \square

8.3 An application of the convexity theorem

Proposition 96. *Let $\Lambda_1, \Lambda_2, \Lambda_3$ be dominant weights. Assume that $g_1\Lambda_1 + g_2\Lambda_2 + g_3\Lambda_3 = 0$, for some elements $g_i \in K$. Then there exists an integer n such that $(V_{n\Lambda_1} \otimes V_{n\Lambda_2} \otimes V_{n\Lambda_3})^G$ is non zero.*

Proof. The space $V = V_{\Lambda_1} \otimes V_{\Lambda_2} \otimes V_{\Lambda_3}$ is a representation of $G \times G \times G$ with highest weight $(\Lambda_1, \Lambda_3, \Lambda_3)$. As in Example 92, we consider the cone

$$C_{(\Lambda_1, \Lambda_3, \Lambda_3)} = (G \times G \times G)(\mathbb{C}v_{\Lambda_1} \otimes v_{\Lambda_2} \otimes v_{\Lambda_3}).$$

The graded ring of regular functions on $C_{(\Lambda_1, \Lambda_3, \Lambda_3)}$ is

$$R(C_{(\Lambda_1, \Lambda_3, \Lambda_3)}) = \bigoplus_{n \in \mathbb{N}} (V_{n\Lambda_1} \otimes V_{n\Lambda_2} \otimes V_{n\Lambda_3})^*.$$

The projective variety $\mathbb{P}(C_{(\Lambda_1, \Lambda_3, \Lambda_3)})$ is isomorphic via the moment map to $K\lambda_1 \times K\lambda_2 \times K\lambda_3$. Let us restrict the action to the diagonal of $G \times G \times G$. The moment map for the diagonal action of K on $K\lambda_1 \times K\lambda_2 \times K\lambda_3$ is $\mu_{diag}(\alpha_1, \alpha_2, \alpha_3) = \alpha_1 + \alpha_2 + \alpha_3$. By assumption, 0 belongs to the image of the moment map $\mu_{diag}(C_{(\Lambda_1, \Lambda_3, \Lambda_3)})$. By the convexity theorem, there exists $N \in \mathbb{N}$ such that 0 is a highest weight of the diagonal representation of G on $(V_{n\Lambda_1} \otimes V_{n\Lambda_2} \otimes V_{n\Lambda_3})^*$. In other words, the space $((V_{n\Lambda_1} \otimes V_{n\Lambda_2} \otimes V_{n\Lambda_3})^*)^G$ is non zero. Therefore $(V_{n\Lambda_1} \otimes V_{n\Lambda_2} \otimes V_{n\Lambda_3})^G$ is also non zero. \square

Remark 97. *If the group elements g_1, g_2, g_3 belong to the Weyl group, then already*

$$(V_{\Lambda_1} \otimes V_{\Lambda_2} \otimes V_{\Lambda_3})^G \neq \{0\}.$$

This is a hard result due independently to S.Kumar [18] and O.Mathieu [19]. They proved a conjecture of K.R.Parthasarathy, R.Rao and V.S.Varadarajan which gives a sufficient condition for a module V_Λ to appear in the tensor product $V_{\Lambda_1} \otimes V_{\Lambda_2}$. See also [14] about problems related to the PRV conjecture.

9 Kirwan-Mumford quotient

9.1 Mumford quotient

Let V be a complex vector space. Let $\mathcal{L} \mapsto \mathbb{P}(V)$ be the line bundle whose space of holomorphic sections is V^* . (As an associated fibre bundle, \mathcal{L} corresponds to the character t^{-1} of \mathbb{C}^\times). Let $G \subseteq \mathrm{GL}(V)$ be a complex reductive subgroup of $\mathrm{GL}(V)$. Let $C \subseteq V$ be a G -stable algebraic cone. The ring of polynomial functions on C is a graded ring $R(C) = \bigoplus_{n=0}^{\infty} R^n(C)$. We consider $M = \mathbb{P}(C)$ as a subvariety of $\mathbb{P}(V)$. The bundle \mathcal{L} restricts to a holomorphic bundle on M which we denote also by \mathcal{L} . The space $H^0(M, \mathcal{L}^n)$ of holomorphic sections of \mathcal{L}^n identifies with $R^n(C)$. We consider the graded ring of G -invariant polynomial functions

$$R(C)^G = \bigoplus_{n=0}^{\infty} R^n(C)^G.$$

The goal is to construct a variety $M//G$ with a holomorphic bundle $\mathcal{L}//G$ such that $R(C)^G$ is identified with

$$\bigoplus_{n=0}^{\infty} H^0(M//G, (\mathcal{L}//G)^n),$$

The variety $M//G$ is called the Mumford quotient of M with respect to the bundle \mathcal{L} .

A point $x \in C$ is called semi-stable if the closure of its G -orbit does not contain 0. Equivalently, there exists a non constant homogeneous G -invariant polynomial which does not vanish at x . The set $C_{ss} \subset C$ of semi-stable points is open in C .

Definition 98. *The open subset $M_{ss} \subset M$ of semi-stable points of M is the image of C_{ss} .*

This definition depends on the realization of M as a quotient $(C \setminus \{0\})/\mathbb{C}^*$. Here is a more intrinsic definition of M_{ss} . Let M be a projective variety with an action of a complex reductive group G . Let $\mathcal{L} \rightarrow M$ be a G -equivariant line bundle. A point $x \in M$ is called \mathcal{L} -semi-stable if there exists an integer $m > 0$ and a G -invariant section s of \mathcal{L}^m such that $s(x) \neq 0$.

Example 99. *Let \mathbb{C}^* act on $M = \mathbb{P}^1(\mathbb{C})$ by $t[z_1, z_2] = [tz_1, t^{-1}z_2]$. Then every point is semi-stable except the two poles. The qualifying section is z_1z_2 , a section of \mathcal{L}^2 .*

Definition 100. If $v \in V$ be a semi-stable point, let Gv_s be the unique closed orbit contained in the closure of Gv . Two semi-stable points v, v' are called projectively G -equivalent if $Gv_s = aGv'_s$ for some $a \in \mathbb{C}^*$. Denote this equivalence relation on C_{ss} by \mathcal{R}_G . The Mumford quotient $M//G$ of $M = \mathbb{P}^1(C)$ by the group G is the quotient C_{ss}/\mathcal{R}_G .

Example 101. Let \mathbb{C}^* act on \mathbb{C}^4 by $t(u_1, u_2, u_3, u_4) = (tu_1, t^{-1}u_2, u_3, u_4)$. Let C be the cone defined by the equation $u_1u_2 = u_3u_4$. The set of semi-stable points is $C_{ss} = \{u \in C; (u_3, u_4) \neq (0, 0)\}$. The map

$$u \mapsto [u_2, u_3] : C_{ss} \rightarrow \mathbb{P}^1(\mathbb{C})$$

defines a bijection of C_{ss}/\mathcal{R}_G onto $\mathbb{P}^1(\mathbb{C})$.

Thus, as a set, $M//G$ is just the set of closed orbits modulo homothety. The point is to define a structure of projective variety on $M//G$. This is done in the following way. Let P_1, \dots, P_r be a set of homogeneous generators of the ring $S(V^*)^G$, with degrees $n_1 > 0, \dots, n_r > 0$.

Lemma 102. The map $V \rightarrow \mathbb{C}^r$ given by $u \mapsto (P_1(u), \dots, P_r(u))$ maps the cone C on a Zariski closed subset Z of \mathbb{C}^r , which is invariant by the action of \mathbb{C}^* on \mathbb{C}^r given by $tq = (t^{n_1}q_1, \dots, t^{n_r}q_r)$. This map induces a bijection of $M//G$ on $(Z \setminus \{0\})/\mathbb{C}^*$.

Proof. By definition, the map $u \mapsto (P_1(u), \dots, P_r(u))$ defines a map from the set $M//G$ onto $(Z \setminus \{0\})/\mathbb{C}^*$. As invariant polynomials separate closed orbits, this map is one-to-one. \square

The quotient $(Z \setminus \{0\})/\mathbb{C}^*$ is compact for the Euclidean topology. It follows that it is a projective variety.

9.2 Mumford quotient and symplectic reduction

Let K be a maximal compact subgroup of G with Lie algebra \mathfrak{k} . We fix a K -invariant Hermitian product on V etc.

Proposition 103. Let $\mu : \mathbb{P}(V) \rightarrow \mathfrak{k}^*$ be the moment map. Define $M_{red} = (\mu^{-1}(0) \cap M)/K$. Then $M//G$ is isomorphic to M_{red} .

Proof. This follows from the fact that $(\mu^{-1}(0) \cap M)/K$ parameterizes the closed G -orbits in C which intersect the unit sphere of V . \square

Proposition 104. *Assume that the action of K on $\mu^{-1}(0) \cap M$ is infinitesimally free. Then every semi-stable G -orbit in C is closed, and the action of G on M_{ss} is infinitesimally free. Thus $M//G = M_{ss}/G$.*

Proof. Let $v \in V \setminus \{0\}$ and $X \in \mathfrak{g}$. Then $[\exp tX.v] = [v]$ if and only if $Xv = av$ with $a \in \mathbb{C}$. If v is semi-stable, then $a = 0$, otherwise 0 would be in the closure of $(\exp \mathbb{C}X)v = \mathbb{C}^*v$.

Assume now that there exists a semi-stable orbit $Gu \subset C$ which is not closed. Let Gv be the closed orbit contained in the closure of Gu . The dimension of Gv is strictly smaller than that of Gu , therefore there exists a non-zero $X \in \mathfrak{g}$ such that $Xv = 0$. Moreover we can assume that $\mu_V(v) = 0$, as every closed orbit meets $\mu_V^{-1}(0)$. We have seen that the G -stabilizer of v is the complexification of its K -stabilizer (Proposition 83). So there exists $Y \in \mathfrak{k}$ such that $Yv = 0$. This contradicts the hypothesis that the action of K on $\mu^{-1}(0) \cap M$ is infinitesimally free. \square

9.3 G -invariant rational sections

Theorem 105 (Mumford). *Let ϕ be a rational function, homogeneous of degree $k \geq 0$, defined on the open set C_{ss} . If ϕ is G -invariant, then ϕ is the restriction to C_{ss} of a homogeneous G -invariant polynomial.*

It follows that the space of G -invariant rational functions, homogeneous of degree $k \geq 0$, defined on the open set C_{ss} , is finite dimensional.

Remark 106. *The condition that ϕ is G -invariant is important. Let us look at Example 99 with $C = \mathbb{C}^2$. Then $C_{ss} = (\mathbb{C}^*)^2$. The function $\frac{z_1^{k+1}}{z_2}$ is homogeneous of degree k and defined on C_{ss} . However there is only one invariant function of degree $2k$ namely $z_1^k z_2^k$.*

Proof. Let $S \subset V$ be the unit sphere. We are going to show that ϕ is bounded from above on $S \cap C_{ss}$. This implies the theorem.

Let $F = \mu^{-1}(0) \cap S \cap C$. Then F is compact and ϕ is defined at every point of F . Let $A = \max_{v \in F} \phi(v)$. Let $u \in S \cap C_{ss}$. There exists $g \in G$ such that $v = gu \in \mu^{-1}(0)$. We have $\phi(u) = \phi(v)$ and $\|v\| \leq \|u\| = 1$. Thus $\phi(v) = \|v\|^k \phi(v/\|v\|) \leq A$. \square

If M and $M//G$ are smooth, the bundle \mathcal{L}^k restricted to M quotients out to a line bundle $\mathcal{L}^k//G$ on $M//G$. The theorem reads

$$H^0(M//G, \mathcal{L}^k//G) = H^0(M, \mathcal{L}^k)^G.$$

In other words "Quantization commutes with Reduction". This is a very important result. It has been proven in the symplectic setup by E.Meinrenken and R.Sjamaar [20]. In the algebraic setup, C.Teleman [24] and M.Braverman [5] have generalized the result in every degree in cohomology.

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