TWO EXISTENCE RESULTS FOR THE VORTEX-WAVE SYSTEM

EVELYNE MIOT

Abstract. The vortex-wave system is a coupling of the two-dimensional Euler equations for the vorticity together with the point vortex system. It was introduced by C. Marchioro and M. Pulvirenti [7, 8] to modelize the evolution of a finite number of concentrated vortices moving in a bounded vorticity background. The purpose of this paper is to provide global existence of a solution in two cases where the background vorticity is not bounded. Part of this work is joint with M. C. Lopes Filho and H. J. Nussenzveig Lopes.

Keywords. Two-dimensional Euler equations, Incompressible flows, Global existence of weak solutions, Point vortices, Vortex sheets

Mathematics Subject Classification (2000): 35Q35, 76B03, 76B47.

1. Introduction

The purpose of this paper is to present the results of [4], joint with M. C. Lopes Filho and H. J. Nussenzveig Lopes, and some results of [9], which concern the two-dimensional incompressible Euler equations on the full plane

$$\partial_t \omega + u \cdot \nabla \omega = 0, \quad \operatorname{div} u = 0. \tag{1}$$

Here $u = u(t,x) : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^2$ is the transporting divergence-free velocity of the fluid, and $\omega = \omega(t,x) = \operatorname{curl} u : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ is the vorticity. In the case of the full plane, the velocity is determined by the vorticity by means of the Biot-Savart law

$$u(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(t,y)\, dy \equiv K * \omega(t,\cdot)(x), \tag{2}$$

where $(a,b)^\perp = (-b,a)$.

Our aim here is to investigate global existence of special measure-valued solutions to (1). In the setting of weak solutions to the Euler equations, a classical result by V. I. Yudovich [13] provides global existence and uniqueness of a solution with bounded vorticity $\omega \in L^\infty(L^1 \cap L^\infty)$. When the vorticity is bounded, the corresponding velocity field is almost-Lipschitz (see (12) below), so that one may define its flow $\phi = \phi(t,x)$:

$$\frac{\partial \phi}{\partial t}(t,x) = u(t, \phi(t,x)), \quad \phi(0,x) = x, \tag{3}$$

and, since (1) is a transport equation with field u, one has

$$\omega(t, \phi(t,x)) = \omega(0,x). \tag{4}$$

In addition, the divergence-free condition on u implies that for all t, $\phi(t,\cdot)$ preserves Lebesgue measure, so that all the norms $\|\omega(t)\|_{L^p}, 1 \le p \le +\infty$ are preserved. The formulation (3)–(4) is called Lagrangian point of view.

One can also deal with weaker solutions, solving (1) in the sense of distributions (Eulerian approach) but for which no Lagrangian description is available. For example, global existence in the space of bounded Radon measures $\mathcal{M}$ under sign and kinetic energy restrictions (i.e., $\omega \geq 0$ and $\omega \in L^\infty(\mathcal{M}\cap H^{-1})$) is due to J.-M. Delort [2] (see also [5, 3, 11]); but nothing is known about uniqueness in this class. These solutions are called *vortex sheets*.

Another kind of special solution, referred to as *point vortex dynamics*, is obtained assuming the vorticity is the superposition of Dirac masses centered at points called *point vortices*

$$\omega = \sum_{i=1}^{\ell} \alpha_i \delta_{z_i}, \quad \alpha_i \in \mathbb{R}. \tag{5}$$

Actually, point vortex dynamics is too singular to include in the usual weak formulations, see [11]. Indeed, according to the Biot-Savart law (2), the corresponding velocity field writes

$$u = \sum_{i=1}^{\ell} \alpha_i K\left(\cdot - z_i\right),$$

it becomes singular at the point vortices and is not, even locally, square integrable. One way to treat such solutions is to assume that each vortex moves with the speed induced by the *other* vortices. Then (1) reduces formally to an Hamiltonian system of ordinary differential equations for the vortex trajectories, known as *point vortex system* in the literature

$$\frac{dz_i}{dt} = \sum_{j\neq i} \alpha_j K(z_i - z_j), \quad i = 1, \ldots, \ell. \tag{6}$$

Finally, in order to handle weaker solutions (including, for instance, point vortex dynamics), F. Poupaud [10] proposed a generalized form of the Euler equations (see (10)), taking into account an additional defect measure due to nonlinearity defects, and obtained global existence of a solution belonging to $L^\infty(\mathcal{M})$ under sign restrictions.

Here we will be interested in special measure-valued solutions μ to (1) behaving like the superposition of a finite number of point vortices and a non atomic, compactly supported vorticity background

$$\mu = \omega + \sum_{i=1}^{\ell} \alpha_i \delta_{z_i}. \tag{7}$$

More precisely, we will study separately the following situations:

I) ω is a function belonging to $L^\infty(L^p)$, $p > 2$, *without* sign conditions, and $\alpha_i \geq 0$ for all i ;

II) ω is a *positive* measure belonging to $L^\infty(H^{-1})$, as in J.-M. Delort's theorem, and $\alpha_i \geq 0$ for all i.

For any vorticity μ_0 given by (7), the existence result by F. Poupaud [10] provides at least one global solution $\mu(t)$ to the generalized Euler equations such that $\mu(0) = \mu_0$. However, nothing more is known about the defect measure and about the structure of $\mu(t)$ at positive times. Here we will construct a global weak solution $\mu(t)$, without defect measure, and such that $\mu(t)$ satisfies (7) for $t \geq 0$.

The situation I) was introduced and formulated by C. Marchioro and M. Pulvirenti [7, 8] in the early 90s. The resulting system, called *vortex-wave system*, was obtained by separating the evolution for the continuous component ω, evolved using the Euler equations, and the evolution for the atomic part on the other hand, evolved through the point vortex system, coupling these equations by means of the Biot-Savart law:

$$\begin{cases} \partial_t \omega + \left(v + \sum_{i=1}^{\ell} \alpha_i K(\cdot - z_i) \right) \cdot \nabla \omega = 0, \quad v = K * \omega, \\ \dfrac{dz_i}{dt} = v(t, z_i) + \sum_{j \neq i} \alpha_j K(z_i - z_j), \quad i = 1, \dots, \ell. \end{cases} \tag{8}$$

In particular one retrieves the point vortex system (6) whenever $\omega \equiv 0$.

In [7], C. Marchioro and M. Pulvirenti proved a global existence result for the vortex-wave system (8) for $p = +\infty$ and for single signed vortices. Uniqueness for (8) in the general case is still an open issue, but was achieved under additional assumptions on the behavior of ω_0 near the point vortices [12, 1].

The purpose of Section 3 below is to extend the global existence result of [7] to the case where $2 < p < +\infty$ (see Theorem 3.1 or Theorem 1 in [4]).

In the second situation II), the transport equation for ω in the vortex-wave system does not make sense anymore, because the velocity $v = K * \omega$ is too singular. Therefore we have to go back to the generalized formulation introduced in [11, 10] for the *whole* vorticity $\mu(t) = \omega(t) + \sum_i \alpha_i \delta_{z_i(t)}$. This formulation relies on the following basic observation: assume that ω is bounded. Then we have, using the Biot-Savart law and the symmetry properties of K,

$$\int (u \cdot \nabla \varphi)\, \omega \, dx = \frac{1}{2} \iint [\nabla \varphi(x) - \nabla \varphi(y)] \cdot \widehat{K}(x - y) \omega(x) \omega(y) \, dx \, dy \tag{9}$$

for all test function φ, where we have set

$$\widehat{K}(x) = K(x) \text{ for } x \neq 0 \quad \text{and } \widehat{K}(0) = 0.$$

Now, introducing

$$H_\varphi(x, y) = \frac{1}{2} [\nabla \varphi(x) - \nabla \varphi(y)] \cdot \widehat{K}(x - y),$$

we realize that H_φ is defined and bounded on $\mathbb{R}^2 \times \mathbb{R}^2$, vanishes at infinity, and is continuous outside the diagonal $\{(x, x), x \in \mathbb{R}^2\}$. In particular, the right-hand side of (9) is well-defined whenever $\omega \in \mathcal{M}(\mathbb{R}^2)$ is a bounded Radon measure. This motivates the following definition:

Let $\mu_0 \in \mathcal{M}(\mathbb{R}^2)$. We say that $\mu \in L^\infty_{\text{loc}}\left(\mathbb{R}_+, \mathcal{M}(\mathbb{R}^2)\right)$ is a global solution of the Euler equations with initial condition μ_0 if for all function $\varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$, we have

$$\begin{aligned} \int \int \partial_t \varphi(t, x)\, \mu(t, x) \, dx \, dt + \int \iint H_\varphi(x, y)\, \mu(t, x) \mu(t, y) \, dx \, dy \, dt \\ = - \int \varphi(0, x) \mu_0(x) \, dx. \end{aligned} \tag{10}$$

Section 4 will be devoted to the proof of global existence of a solution to (10) behaving like (7) (see Theorem 4.1 or Théorème 4.1 in [9]).

Before stating Theorems 3.1 and 4.1 we will present in Section 2 some basic properties of the Euler equations for later use.

Notations. We will set

$$\ln^-(r) = \max(0, -\ln(r)), \quad \ln^+(r) = \max(0, \ln(r)), \quad \text{for } r > 0.$$

We will use the smooth, cut-off function $\chi_0 : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\chi_0 \equiv 1 \text{ on } B(0,1), \quad \chi_0 \equiv 0 \text{ on } B(0,2)^c, \quad 0 \leq \chi_0 \leq 1. \tag{11}$$

2. Some useful properties of the Euler equations

First we collect some useful properties of the Biot-Savart Kernel:

$$K(-x) = -K(x) \quad \text{(antisymmetry)}, \quad |K(x)| \leq \frac{1}{|x|},$$
$$|K(x) - K(y)| = \frac{1}{2\pi} \frac{|x-y|}{|x||y|} \quad \text{(regularity away from the origin).}$$

Next we present some preserved quantities by the flow of the Euler equations. Assume that $(\omega, u = K * \omega)$, with $\omega \in L^\infty(L^\infty_c)$, is a solution to (1). Then

$$\|\omega(t)\|_{L^p} = \|\omega(0)\|_{L^p} \quad \text{for all} \quad 1 \leq p \leq \infty, \qquad \int \omega(t,x)\, dx = \int \omega(0,x)\, dx.$$

Moreover we have the conservation of the momentum of inertia

$$\mathcal{I}(\omega)(t) = \int |x|^2 \omega(t,x)\, dx = \mathcal{I}(\omega)(0),$$

and of the pseudo-energy

$$\mathcal{H}(\omega)(t) = \iint \ln|x-y| \omega(t,x)\omega(t,y)\, dx\, dy = \mathcal{H}(\omega)(0).$$

In Section 3 and 4, where we assume that $\omega + \sum_i \alpha_i \delta_{z_i}$, with $\omega \in L^\infty(L^\infty_c)$, is a solution to the vortex-wave system (8), the previous properties translate into

$$\mathcal{I}(\omega, \{z_i\})(t) = \int |x|^2 \omega(t,x)\, dx + \sum_i \alpha_i |z_i(t)|^2 = \mathcal{I}(\omega, \{z_i\})(0),$$
$$\mathcal{H}(\omega, \{z_i\})(t) = \iint \ln|x-y| \omega(t,x)\omega(t,y)\, dx\, dy + \sum_{i \neq j} \alpha_i \alpha_j \ln|z_i(t) - z_j(t)|$$
$$+ 2 \sum_i \alpha_i \int \ln|x - z_i(t)| \omega(t,x)\, dx = \mathcal{H}(\omega, \{z_i\})(0).$$

3. The vortex-wave system (with M. C. Lopes Filho and H. J. Nussenzveig Lopes)

3.1. **Presentation and main result.** This section is devoted to the vortex-wave system (8). As already mentionned, there are two kinds of solutions to (1) or (8), namely the Lagrangian and the Eulerian solutions.

Let $p \geq 1$ and $\omega_0 \in L^p_c$. We say that $(\omega, \{z_i\})$ is a Lagrangian solution to (8) on $[0,T]$ with initial condition $(\omega_0, \{z_{i0}\})$ if

$$\omega \in L^\infty\left([0,T], L^p_c(\mathbb{R}^2)\right), \quad z_i \in C^1([0,T]), \quad \phi(\cdot, x) \in C^1([0,T]) \quad \forall x \neq z_{i0},$$

where for $t \in [0,T]$

$$\begin{cases} \omega(t,\phi(t,x)) = \omega_0(x), \quad v = K * \omega, \\ \dfrac{dz_i}{dt} = v(t,z_i) + \displaystyle\sum_{j\neq i} \alpha_j K\left(z_i - z_j\right), \quad z_i(0) = z_{i0}, \\ \dfrac{\partial \phi}{\partial t}(t,x) = v(t,\phi(t,x)) + \displaystyle\sum_{j=1}^{\ell} \alpha_j K\left(\phi(t,x) - z_j(t)\right), \\ \phi_0(x) = x, \ x \neq z_{i0}. \end{cases} \tag{ODE}$$

Additionally, for all t, $\phi_t(\cdot) = \phi(t,\cdot)$ is a homeomorphism from $\mathbb{R}^2 \setminus \{z_{10},\dots,z_{\ell 0}\}$ into $\mathbb{R}^2 \setminus \{z_1(t),\dots,z_\ell(t)\}$ preserving Lebesgue's measure. Of course, due to the divergence of K at the origin, the ordinary differential equations make sense only if the fluid particles $\phi(t,x)$ do not intersect the vortex trajectories $z_i(t)$ and if there is no collapse among the vortex trajectories on $[0,T]$.

In order to allow the singular fields to become infinite, one can also define another notion of solution without involving the flow ϕ: solutions to the PDE in the sense of distributions. More precisely, we say that $(\omega,\{z_i\})$ is an Eulerian solution if $v = K*\omega$ and $\dot{z}_i = v(t,z_i) + \sum_{j\neq i} \alpha_j K\left(z_i - z_j\right)$; moreover for all $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^2)$

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^2} \omega \left(\partial_t \varphi + (v + \sum_{i=1}^{\ell} \alpha_i K(\cdot - z_i)) \cdot \nabla \varphi\right) dx\, dt \\ = -\int_{\mathbb{R}^2} \omega_0(x)\varphi(0,x)\, dx, \end{aligned} \tag{PDE}$$

Such a formulation requires to give sense to the products ωv and $\omega K(\cdot - z_i)$. Since K belongs to L^q_{loc} for all $q<2$, then $\omega K(\cdot - z_i)$ belongs to L^1_{loc} provided $p>2$. On the other hand, the velocity $v = K*\omega$ is uniformly bounded for all $p>2$ (see (14) below); hence ωv belongs to L^1_{loc}. It is therefore natural to focus on the case where $p>2$.

In [7], C. Marchioro and M. Pulvirenti obtained a global existence result for the vortex-wave system (8), in Lagrangian formulation, for $p=+\infty$ and for single signed vortices. The single sign assumption on the α_i implies that the the vortices cannot collide in finite time. Since ω is bounded, the velocity $v = K*\omega$ is almost-Lipschitz [7, 8]: for all $x,y \in \mathbb{R}^2$,

$$|v(t,x) - v(t,y)| \leq C(\|\omega(t)\|_{L^1}, \|\omega(t)\|_{L^\infty})\, |x-y|(1 + \ln^- |x-y|). \tag{12}$$

In particular, for fixed v all ordinary differential equations involved in (ODE) are locally well-posed. Using again the almost-Lipschitz regularity for the velocity enables to establish a priori positive lower bounds on the distances

$$|\phi(t,x) - z_i(t)| \geq \mathcal{F}(|x - z_i(0)|), \quad i = 1,\dots,\ell, \tag{13}$$

for a positive function $\mathcal{F}$ vanishing only at the origin. It follows that there is no collapse among the fluid trajectories and the vortex trajectories, so that the solution is global in time.

Finally, for $p=+\infty$ it has been established in [1] that both Lagrangian and Eulerian formulations are equivalent.

Assume now that $2 < p < \infty$ and $\omega \in L^\infty(L^p_c)$. Then the corresponding velocity $v = K * \omega$ is bounded

$$\|v(t)\|_{L^\infty} \leq C(p)\|\omega(t)\|_{L^1}^{1-p'/2}\|\omega(t)\|_{L^p}^{p'/2}, \tag{14}$$

where p' denotes the conjugate exponent of p. Furthermore, v is Hölder continuous:

$$|v(t,x) - v(t,y)| \leq C(\|\omega(t)\|_{L^1}, \|\omega(t)\|_{L^p})\, |x-y|^{1-2/p},\ \forall x, y \in \mathbb{R}^2.$$

Thus for fixed $\omega \in L^\infty(L^p_c)$, there exist local in time solutions $(z_i(t), \phi_t(x))$ to the ordinary differential equations in (ODE). However, in the present case the lower bound (13) translates into

$$|\phi(t,x) - z_i(t)|^{1-2/p} \geq |x - z_i(0)|^{1-2/p} - Ct.$$

Hence one cannot exclude possible collapse in finite time between the fluid trajectories and the vortices and we do not hope to establish global existence of a Lagrangian solution to (8). Nevertheless, as already mentionned all terms involved in the weak formulation (PDE) are well-defined. The main result of this section is the following

Theorem 3.1. *Let $p > 2$ and let $\omega_0 \in L^p(\mathbb{R}^2)$ have compact support. Let $\{z_{i0}\}, i = 1, \ldots, \ell$ be ℓ distinct points in $\mathbb{R}^2$, and let $\alpha_i, i = 1, \ldots, \ell$, be positive numbers. Then there exists a global weak solution of the vortex-wave system with this initial data.*

Remark 3.2. Without the sign restriction on the intensities α_i we cannot hope for a global existence theorem, since, even in the absence of the continuous vorticity, collisions in finite time are known to exist (see [8]). However, for $\alpha_i \in \mathbb{R}$ one can prove *local* existence of a solution to the vortex-wave system [4].

3.2. **Some elements for the proof of Theorem 3.1.** We sketch now the proof of Theorem 3.1. First, we regularize the initial vorticity by introducing $\omega_0^\delta = \rho_\delta * \omega_0 \in L^\infty_c(\mathbb{R}^2)$, where $\{\rho_\delta\}_{0<\delta<1}$ is a standard mollifier, and we consider a resulting global solution $(\omega^\delta, \{z_i^\delta\}, \phi^\delta)$ of the vortex-wave system provided by C. Marchioro and M. Pulvirenti's result. We then establish uniform estimates with respect to δ to obtain compactness and a weak limit $(\omega, \{z_i\})$, and finally we show that $(\omega, \{z_i\})$ is a weak solution to (8).

We set

$$v^\delta = K * \omega^\delta, \quad \mathcal{K}^\delta(t,x) = \sum_{i=1}^{\ell} \alpha_i K\left(x - z_i^\delta(t)\right) = \sum_{i=1}^{\ell} \alpha_i K_i^\delta(t,x).$$

Uniform estimates and compactness. By Section 2 and (14), $\{\omega^\delta\}$ is uniformly bounded in $L^\infty(L^1 \cap L^p)$ and $\{v^\delta\}$ is uniformly bounded in L^∞. Moreover, the following bounds hold

Proposition 3.3. *We have*

$$\max_i |z_i^\delta(t)| \leq C(1+t), \quad |\phi_t^\delta(x)| \leq C(1+t), \quad \forall x \in \operatorname{supp}(\omega_0), \quad \forall t \geq 0.$$

Proof. We consider the momentum of inertia

$$I^\delta(t) = \sum_{i=1}^{\ell} \alpha_i |z_i^\delta(t)|^2.$$

For the point vortex system (6) it is constant in time. For the vortex-wave system this quantity is no longer conserved, but we can obtain some control of its growth.

Indeed, symmetry properties of the kernel K and the sign assumption on the α_i yield

$$\frac{dI^\delta}{dt}(t) = 2\sum_{i=1}^{\ell} \alpha_i z_i^\delta(t) \cdot v^\delta(t, z_i^\delta(t)) \leq C \sum_{i=1}^{\ell} \alpha_i |z_i^\delta(t)| \leq C\sqrt{I^\delta(t)},$$

whence the control on the vortex trajectories. To control the support of ω^δ, we consider $x \neq z_{i0} \in \mathrm{supp}\,(\omega_0)$. Then, if the flow $\phi_t^\delta(x)$ is close to one of the point vortices, it lies in $B(C(1+t))$ in view of the previous bounds. Otherwise it is far from all of them; but then its total velocity $v^\delta + \mathcal{K}^\delta$ is bounded. One can then conclude. □

Proposition 3.4. *There exists a positive and continuous function $t \mapsto d(t) > 0$ such that*

$$\min_{i \neq j} |z_i^\delta(t) - z_j^\delta(t)| \geq d(t), \qquad \forall t \geq 0.$$

Proof. Set

$$H^\delta(t) = \sum_{i \neq j} \alpha_i \alpha_j \ln^- |z_i^\delta(t) - z_j^\delta(t)|,$$

so that

$$H^\delta(t) \leq -\mathcal{H}(\omega^\delta, \{z_i^\delta\})(0) + \left| \iint \ln|x-y| \omega^\delta(t,x)\omega^\delta(t,y)\,dx\,dy \right|$$
$$+ 2\sum_{i=1}^{\ell} \alpha_i \left| \int \ln|x - z_i^\delta(t)| \omega^\delta(t,x)\,dx \right| + \sum_{i\neq j} \alpha_i\alpha_j \ln^+ |z_i^\delta(t) - z_j^\delta(t)|.$$

Using various uniform bounds for $\{\omega^\delta\}$ and the estimates of Proposition 3.3 we infer that

$$H^\delta(t) \leq C\,(1 + \ln(1+t))$$

and the conclusion follows, since all the α_i are positive.

□

Passing to the limit[1]**.** Invoking the previous estimates and standard compactness arguments, we are in position to find $(\omega, \{z_i\})$ such that, up to a subsequence still denoted by δ, we have

1) $\omega^\delta(t) \rightharpoonup \omega(t)$ weakly in L^p for all $t \geq 0$.
2) $v^\delta = K * \omega^\delta \to v = K * \omega$ locally uniformly on $\mathbb{R}_+ \times \mathbb{R}^2$.
3) $z_i^\delta \to z_i$ locally uniformly on $\mathbb{R}_+$, $\forall i$, therefore
4) $K_i^\delta \to K_i = K(\cdot - z_i)$ locally uniformly away from z_i, $\forall i$.

It remains to show that $(\omega, \{z_i\})$ is a weak solution to the vortex-wave system. In view of 2), 3) and 4), and using Proposition 3.4 it is straightforward to establish the ordinary differential equations satisfied by the $\{z_i\}$, since $\{v^\delta(z_i^\delta) + \sum_{j\neq i} \alpha_j K_j^\delta(z_i^\delta)\}$ converge locally uniformly. Next we want to prove that

$$\partial_t \omega + \mathrm{div}\,((v + \mathcal{K})\omega) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^2),$$

where $\mathcal{K} = \sum_i \alpha_i K_i$. In fact, given 1) and 2) we only have to worry about the convergence of the non linear term $\omega^\delta \mathcal{K}^\delta$, given in the following

Proposition 3.5. *We have $\omega^\delta \mathcal{K}^\delta \to \omega\mathcal{K}$ in $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^2)$.*

---
[1] [4] presents a slightly different proof, based on the symmetry properties of K.

Proof. As a matter of fact, by 1) it suffices to show that

$$\omega^\delta(\mathcal{K}^\delta - \mathcal{K}) \to 0 \quad \text{in } \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^2). \tag{15}$$

By virtue of 1), 3) and 4) we already know that (15) holds away from the vortices. Let φ be a test function such that $\operatorname{supp}(\varphi) \subset [0,T] \times \mathbb{R}^2$. We introduce a parameter $0 < \varepsilon < 1$ and we set

$$\chi_\varepsilon^i(t,z) = \chi_0\left(\frac{z - z_i(t)}{\varepsilon}\right), \quad \chi_\varepsilon = \sum_{i=1}^{\ell} \chi_\varepsilon^i,$$

where χ_0 is defined in (11). By Proposition 3.4, one can choose ε small enough so that $\chi_\varepsilon^i(t,\cdot)$ and $\chi_\varepsilon^j(t,\cdot)$ have disjoint supports on $[0,T]$ for $i \neq j$ and

$$\chi_\varepsilon(t,\cdot) \equiv 1 \quad \text{on } \bigcup_{i=1}^{\ell} B\left(z_i(t), \frac{\varepsilon}{2}\right).$$

Then, we set

$$\varphi = \varphi(1 - \chi_\varepsilon) + \sum_{i=1}^{\ell} \varphi \chi_\varepsilon^i.$$

Fixing ε and letting $\delta \to 0$, we readily obtain by definition of χ_ε and 1), 4)

$$\iint \omega^\delta \left[(\mathcal{K}^\delta - \mathcal{K}) \cdot \varphi(1 - \chi_\varepsilon) + \sum_{i=1}^{\ell} \sum_{j \neq i} \left(K_j^\delta - K_j\right) \cdot \varphi \chi_\varepsilon^i\right] dx\, dt \to 0. \tag{16}$$

On the other hand, Cauchy-Schwarz inequality yields for all i

$$\left|\iint \omega^\delta (K_i^\delta - K_i) \cdot \varphi \chi_\varepsilon^i \, dx\, dt\right|$$

$$\leq C \|\omega_0\|_{L^p} \|\varphi\|_{L^\infty} \sup_{t \in [0,T]} \left(\left\| \frac{1}{|x - z_i^\delta(t)|} \right\|_{L^{p'}(B(z_i(t), 2\varepsilon))} + \left\| \frac{1}{|x - z_i(t)|} \right\|_{L^{p'}(B(z_i(t), 2\varepsilon))} \right).$$

One may chose δ sufficiently small so that $\sup_{t \in [0,T]} |z_i(t) - z_i^\delta(t)| \leq \varepsilon$. Therefore

$$\left|\iint \omega^\delta (K_i^\delta - K_i) \cdot \varphi \chi_\varepsilon^i \, dx\, dt\right| \leq C \| |x|^{-1} \|_{L^{p'}(B(0,3\varepsilon))} \leq C \varepsilon^{2/p' - 1}. \tag{17}$$

In conclusion, (16) and (17) yield

$$\limsup_{\delta \to 0} \left| \iint \omega^\delta (\mathcal{K}^\delta - \mathcal{K}) \cdot \varphi \, dx\, dt \right| \leq C \varepsilon^{2/p' - 1},$$

and (15) follows by letting finally $\varepsilon \to 0$.

□

4. Vortex sheets and point vortices

4.1. **Presentation and main result.** The main result of this section is the following

Theorem 4.1. *Let ω_0 be a* positive, *compactly supported Radon measure belonging to $H^{-1}(\mathbb{R}^2)$. Let $z_0 \in \mathbb{R}^2$ not belonging to the support of ω_0. Let $\alpha \geq 0$ and set $\mu_0 = \omega_0 + \alpha \delta_{z_0}$. There exists a global solution of the Euler equations μ, in the* *sense of Definition* (10)*, such that* $\mu \geq 0$*,* $t \mapsto \int \varphi(t,x)\mu(t,x)\,dx$ *is continuous* $\forall \varphi \in C_0^0(\mathbb{R}^2)$*, and* $\mu(0) = \mu_0$*. Moreover, we have*

$$\mu(t) = \omega(t) + \alpha\delta_{z(t)}, \quad \forall t \geq 0,$$

where $z \in C^{1/2}(\mathbb{R}_+, \mathbb{R}^2)$ *and* $\omega \in L^\infty\left(\mathbb{R}_+, H^{-1}(\mathbb{R}^2)\right)$*.*

Remark 4.2. Using suitable test functions in the formulation (10), one can show that if $\mu = \omega + \alpha\delta_z$ is a solution of (10) and if moreover $\omega \in L^\infty(L^p_c)$ for some $p > 2$, then $z \in W^{1,\infty}$ and (ω, z) is a solution of the vortex-wave system (8).

Remark 4.3. Theorem 4.1 easily extends to the case of several point vortices having all positive intensities. Moreover, the same conclusion holds replacing the assumption $z_0 \notin \operatorname{supp}(\omega_0)$ by the assumption $\ln|z_0 - \cdot|\omega_0 \in \mathcal{M}(\mathbb{R}^2)$.

Without loss of generality, we will further assume that $\alpha = 1$.

4.2. **Sketch of the proof of Theorem 4.1.** In order to establish Theorem 4.1, we will adapt the result obtained by A. J. Majda [5] for vorticites without atomic part ($\alpha = 0$) to the present case ($\alpha = 1$). In particular, we will exploit as in [5] the notion of pseudo-energy, which has been already defined in Section 2 for bounded vorticities:

$$\mathcal{H}(\omega) = \iint \ln|x-y|\omega(x)\omega(y)\,dx\,dy.$$

In fact, this definition extends to positive, compactly supported measures belonging to H^{-1}; one has (see e.g. [5] or [6]):

$$|\mathcal{H}(\omega)| \leq C, \tag{18}$$

where C depends only on $\int \omega, \|\omega\|_{H^{-1}}$ and $\operatorname{supp}(\omega)$.

One has also the converse estimate

$$\|\omega\|_{H^{-1}} \leq C', \tag{19}$$

where C' depends only on $\int \omega, |\mathcal{H}(\omega)|$ and $\int |x|^2\omega$. It should be mentionned that, in contrary to estimate (18), estimate (19) does not involve the size of the support of ω.

We proceed as in the proof of Theorem 3.1, considering a sequence of global solutions $(\omega^\delta, z^\delta)$, with $\omega^\delta \in L^\infty$, to the vortex-wave system (8). Introducing the full vorticity $\mu^\delta = \omega^\delta + \delta_{z^\delta}$, we obtain a sequence of global solutions to (10). We then establish uniform estimates and pass to the limit in (10).

Uniform estimates and compactness. For $t \geq 0$ and $0 < \delta < 1$, $\omega^\delta(t)$ is positive, bounded and compactly supported. By assumption on μ_0 we get uniform bounds on the full momentum of inertia and on the full pseudo-energy

$$\mathcal{I}(\omega^\delta, z^\delta)(t) = \mathcal{I}(\omega^\delta, z^\delta)(0) \leq C \tag{20}$$

and

$$|\mathcal{H}(\omega^\delta, z^\delta)(t)| = |\mathcal{H}(\omega^\delta, z^\delta)(0)| \leq C. \tag{21}$$

We next establish some compactness for $\{\mu^\delta\}$. It is uniformly bounded in $L^\infty(\mathcal{M})$. Moreover, since μ^δ satisfies (10) we have for all $\varphi \in C_c^\infty$ and $s, t \geq 0$

$$\left|\int_{\mathbb{R}^2} \varphi(x)\mu^\delta(t,x)\,dx - \int_{\mathbb{R}^2} \varphi(x)\mu^\delta(s,x)\,dx\right| \leq C\|D^2\varphi\|_{L^\infty}|t-s|, \tag{22}$$

hence $t \mapsto \int \varphi(x)\mu^\delta(t,x)\,dx$ is uniformly bounded and equicontinuous. By a standard density argument and Ascoli's theorem we conclude that there exists $\mu \geq 0$ such that, up to a subsequence, $\mu^\delta(t)$ converges to $\mu(t)$ and $\mu^\delta(t)\otimes\mu^\delta(t)$ to $\mu(t)\otimes\mu(t)$ vaguely, locally uniformly with respect to $t \geq 0$.

The next step is to show that ω^δ does not concentrate in the limit $\delta \to 0$. For this the sign assumption on μ_0 plays a crucial role.

Lemma 4.4. *There exists a constant C depending only on μ_0 such that for $0 < r < 1/2$ we have*

$$\sup_{t\geq 0}\ \sup_{0<\delta<1}\ \sup_{x_0\in\mathbb{R}^2} \int_{B(x_0,r)} \omega^\delta(t,x)\,dx \leq C|\ln r|^{-1/2}.$$

Proof. Recall that $\omega^\delta \geq 0$; therefore, by virtue of a result of [5] (page 932), in order to prove Lemma 4.4 it suffices to obtain a uniform bound for the pseudo-energy $\mathcal{H}(\omega^\delta)(t)$ associated to ω^δ. We have by direct computations

$$\begin{aligned}\big|\mathcal{H}(\omega^\delta)(t)\big| &\leq |\mathcal{H}(\omega^\delta, z^\delta)(t)|\\ &+ C\left(\iint \ln^+|x-y|\omega^\delta(t,x)\omega^\delta(t,y)\,dx\,dy + \int \ln^+|z^\delta(t)-x|\omega^\delta(t,x)\,dx\right).\end{aligned}$$

Using the basic inequality $\ln^+|x-y| \leq |x|^2+|y|^2$ for $x,y\in\mathbb{R}^2$ and the estimates (20) and (21) we obtain a uniform bound on $|\mathcal{H}(\omega^\delta)(t)|$, as we wanted. □

Lemma 4.5. *There exists $z \in C^{1/2}(\mathbb{R}_+,\mathbb{R}^2)$ such that, up to a subsequence, z^δ converges to z uniformly on compact sets of $\mathbb{R}_+$.*

Proof. The sequence $\{z^\delta\}$ is uniformly bounded on $\mathbb{R}_+$ by (20). It is also uniformly equicontinuous. Indeed, let $\varphi \in C_c^\infty(\mathbb{R}^2)$. By (22) we have

$$\Big|\varphi(z^\delta(t)) - \varphi(z^\delta(s))\Big| \leq C\|D^2\varphi\|_\infty|t-s| + \int_{\mathbb{R}^2}\omega^\delta(t,x)|\varphi(x)|\,dx + \int_{\mathbb{R}^2}\omega^\delta(s,x)|\varphi(x)|\,dx.$$

Let $0<\eta<1$ and $K>1$ be two constants, depending only on μ_0, to be determined later. For t,s satisfying $|t-s|\leq\eta$, we assume by contradiction that

$$|z^\delta(t)-z^\delta(s)| > K|t-s|^{1/2}.$$

Set

$$r = \frac{K|t-s|^{1/2}}{2};$$

decreasing η if necessary, we may assume that $r<1/4$. Next, we choose

$$\varphi(x) = \chi_0\left(\frac{x-z^\delta(s)}{r}\right),$$

where χ_0 is the cut-off function defined by (11). Clearly we have $\varphi(z^\delta(t))=0$, while $\varphi(z^\delta(s))=1$. On the other hand, Lemma 4.4 implies that

$$\int_{\mathbb{R}^2}\omega^\delta(s,x)\varphi(x)\,dx \leq \int_{B(z^\delta(s),2r)}\omega^\delta(s,x)\,dx \leq C|\ln r|^{-1/2},$$

and the same estimate holds true for $\omega^\delta(t)$. Finally, using the fact that $\|D^2\varphi\|_\infty \leq Cr^{-2}$ we find

$$1 \leq C|\ln r|^{-1/2} + \frac{C}{r^2}|t-s| \leq C\big(|\ln r|^{-1/2}+K^{-2}\big), \tag{23}$$

where C depends only on μ_0. One may choose K large enough, then η sufficiently small, so that the right-hand side of (23) is smaller than 1. We are led to a contradiction, therefore $\{z^\delta\}$ is uniformly equicontinuous on $\mathbb{R}_+$. The existence of $z \in C^{1/2}$ such that z^δ converges to z (up to a subsequence) uniformly on compact sets follows from Ascoli's theorem. $\square$

Lemma 4.6. *There exists a positive $\omega \in L^\infty\left(\mathbb{R}_+, \mathcal{M} \cap H^{-1}(\mathbb{R}^2)\right)$ such that, up to a subsequence, $\omega^\delta(t)$ converges to $\omega(t)$ vaguely, locally uniformly with respect to $t \geq 0$. The estimate of Lemma 4.4 holds for ω.*

Proof. Setting $\omega(t) = \mu(t) - \delta_{z(t)}$, the only point to check is the fact that $\omega \in L^\infty\left(H^{-1}\right)$. We already know that $|\mathcal{H}(\omega^\delta)(t)|$ is uniformly bounded. Thanks to (19) and (20), we obtain that $\{\omega^\delta\}$ is uniformly bounded in $L^\infty\left(H^{-1}\right)$. The conclusion follows. $\square$

Passing to the limit. We finally show that μ satisfies the formulation (10). As a matter of fact, in view of the weak convergence of μ^δ to μ, we only have to prove that for all $T > 0$, for all $\psi \in C_c^\infty([0,T))$ and $\varphi \in C_c^\infty(\mathbb{R}^2)$ the non linear term

$$\int\iint \psi(t) H_\varphi(x,y)\mu^\delta(t,x)\mu^\delta(t,y)\,dx\,dy\,dt$$

passes to the limit as δ tends to zero. Observe that, since $H_\varphi(\cdot,\cdot)$ vanishes on the diagonal, we have

$$H_\varphi(x,y)\mu^\delta(x)\mu^\delta(y) = H_\varphi(x,y)\omega^\delta(x)\omega^\delta(y) + H_\varphi(x,z^\delta)\omega^\delta(x) + H_\varphi(z^\delta,y)\omega^\delta(y).$$

Hence it suffices to establish that

$$\iiint \psi H_\varphi(x,y)\omega^\delta(t,x)\omega^\delta(t,y)\,dx\,dy\,dt \to \iiint \psi H_\varphi(x,y)\omega(t,x)\omega(t,y)\,dx\,dy\,dt \tag{24}$$

and

$$\iint \psi H_\varphi(x,z^\delta(t))\omega^\delta(t,x)\,dx\,dt \to \iint \psi H_\varphi(x,z(t))\omega(t,x)\,dx\,dt. \tag{25}$$

In doing this, the major difficulty is due to the discontinuity of H_φ on the diagonal, therefore the weak convergences of ω^δ and $\omega^\delta \otimes \omega^\delta$ to ω and $\omega \otimes \omega$ do not allow to pass to the limit in (24) and (25). However, the crucial observation in [2, 5, 11] is the fact that, since ω^δ does not concentrate on the diagonal (see Lemma 4.4), the contribution of $H_\varphi(x,y)\omega^\delta \otimes \omega^\delta$ on $\{(x,y) : |x-y| \leq \varepsilon\}$ or $H_\varphi(x,z^\delta)\omega^\delta$ on $\{x : |x - z^\delta| \leq \varepsilon\}$, for a small parameter ε, is small *uniformly* with respect to δ.

First, (24) directly follows from the by now standard arguments of [2, 5, 11], since $\{\omega^\delta\}$ satisfies all required assumptions.

We next establish (25) by similar arguments. We introduce a small $\varepsilon > 0$ and the cut-off function $\chi_\varepsilon(t,x) = \chi_0((x - z(t))/\varepsilon)$, where χ_0 is defined in (11).

We will first show that for fixed ε we have as $\delta \to 0$

$$\int\int \psi(t)(1-\chi_\varepsilon)H_\varphi(x,z^\delta)\omega^\delta\,dx\,dt \to \int\int \psi(t)(1-\chi_\varepsilon)H_\varphi(x,z)\,\omega\,dx\,dt. \tag{26}$$

Indeed, in view of the definition of H_φ we have

$$\int\int \psi(t)(1-\chi_\varepsilon)H_\varphi(x,z^\delta)\omega^\delta\,dx\,dt = I^\delta + J^\delta + K^\delta,$$

where

$$
\begin{aligned}
I^\delta &= \int\int \psi(t)(1-\chi_\varepsilon)H_\varphi(x,z)\omega^\delta\,dx\,dt,\\
J^\delta &= \int\int \psi(t)(1-\chi_\varepsilon)\frac{1}{2}\Big[\nabla\varphi(z)-\nabla\varphi(z^\delta)\Big]\cdot K(x-z)\omega^\delta\,dx\,dt,\\
K^\delta &= \int\int \psi(t)(1-\chi_\varepsilon)\frac{1}{2}\Big[\nabla\varphi(x)-\nabla\varphi(z^\delta)\Big]\cdot\Big[K(x-z^\delta)-K(x-z)\Big]\omega^\delta\,dx\,dt.
\end{aligned}
$$

First, using the regularity of K away from zero, the fact that $1-\chi_\varepsilon$ vanishes in a neighborhood of $z(t)$ and Lemma 4.5 we obtain

$$
\limsup_{\delta\to 0}\Big(|J^\delta|+|K^\delta|\Big)\leq \limsup_{\delta\to 0}\left(C(\varepsilon,\varphi)\sup_{t\in[0,T]}|z(t)-z^\delta(t)|\right)=0.
$$

On the other hand, since $\psi(1-\chi_\varepsilon)H_\varphi(\cdot,z)$ belongs to $L^1\left(C_0^0\right)$ we have, using the convergence of ω^δ to ω,

$$
\lim_{\delta\to 0} I^\delta = \int\int \psi(t)(1-\chi_\varepsilon)H_\varphi(x,z)\,\omega\,dx\,dt.
$$

Hence we obtain (26).

Finally, invoking Lemma 4.4 we can estimate the remaining contribution to the integral as follows

$$
\sup_{0<\delta<1}\left|\int\int \psi(t)\chi_\varepsilon H_\varphi(x,z^\delta)\omega^\delta\,dx\,dt\right|\leq C|\ln\varepsilon|^{-1/2}, \tag{27}
$$

and the same estimate holds replacing ω^δ by ω.

Letting eventually ε go to zero, (26) and (27) lead to (25) and the proof of Theorem 4.1 is complete.

Acknowledgments. The author warmly thanks Didier Smets and Franck Sueur for helpful discussions.

References

[1] C. Lacave and E. Miot, *Uniqueness for the vortex-wave system when the vorticity is initially constant near the point vortex*, SIAM J. Math. Anal. **41** (2009), 1138–1163.

[2] J.-M. Delort, *Existence de nappes de tourbillon en dimension deux*, J. Amer. Math. Soc. **4** (1991), 553–586.

[3] M. C. Lopes Filho, H. J. Nussenzveig Lopes and Z. Xin, *Existence of vortex sheets with reflection symmetry in two space dimensions*, Arch. Rat. Mech. Anal. **158** (2001), 235–257.

[4] M. C. Lopes Filho, E. Miot and H. J. Nussenzveig Lopes, *Existence of a weak solution to the vortex-wave system*, preprint.

[5] A. J. Majda, *Remarks on weak solutions for vortex sheets with a distinguished sign*, Indiana Univ. Math. J. **42** (1993), 921–939.

[6] A. J. Majda and A. L. Bertozzi, *Vorticity and Incompressible flow*, Cambridge Texts in Applied Mathematics, 2002.

[7] C. Marchioro and M. Pulvirenti, *On the vortex-wave system*, in Mechanics, analysis, and geometry: 200 years after Lagrange, 79–95, M. Francaviglia (ed), Elsevier Science, Amsterdam 1991.

[8] C. Marchioro and M. Pulvirenti, *Mathematical Theory of Incompressible Nonviscous Fluids*, Springer-Verlag, New York 1994.

[9] E. Miot, *Quelques problèmes relatifs à la dynamique des points vortex dans les équations d'Euler et de Ginzburg-Landau complexe*, Ph. D. thesis, Université Pierre et Marie Curie, 2009.

[10] F. Poupaud, *Diagonal defect measures, adhesion dynamics and Euler equation*, Meth. Appl. Anal. **9** (2002), 533–561.
[11] S. Schochet, *The weak vorticity formulation of the 2-D Euler equations and concentration-cancellation*, Commun. Partial Diff. Eqs. **20** (1995), 1077–1104.
[12] V. Starovoitov, *Uniqueness of the solution to the problem of the motion of a point vortex*, Sibirsk. Mat. Zh. **35** (1994), 696–701.
[13] V. I. Yudovich, *Non-stationary flows of an ideal incompressible fluid*, Zh Vych Mat.i Mat. Fiz. **3** (1963), 1032–1066.

Evelyne Miot
Laboratoire de Mathématiques, Université Paris-Sud 11
Bât. 425, 91405 Orsay, France
e-mail: Evelyne.Miot@math.u-psud.fr