COLLISIONS OF VORTEX FILAMENT PAIRS

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ABSTRACT. We consider the problem of collisions of vortex filaments for a model introduced in [KMD95, Z88, Z99] to describe the interaction of almost parallel vortex filaments in three-dimensional fluids. Since the results in [C70] examples of collisions are searched as perturbations of antiparallel translating pairs of filaments, with initial perturbations related to the unstable mode of the linearized problem; most results are numerical calculations. In this article we first consider a related model for the evolution of pairs of filaments and we display another type of initial perturbation leading to collision in finite time. Moreover we give numerical evidence that it also leads to collision through the initial model. We finally study the self-similar solutions of the model.

CONTENTS

1. INTRODUCTION

In this article we investigate the possible collision scenarios for a system of equations modelling the dynamics of vortex filaments in three-dimensional incompressible fluids. Since the results in [C70], collisions in this model are searched as perturbations of an antiparallel pair of vortex filaments, with initial perturbations related to the unstable mode of the linearized problem; most results are numerical calculations.

Let us now present the model at stake and survey the known results on it. It was introduced by Klein, Majda and Damodaran [KMD95] and Zhakarov [Z88, Z99] in order to describe the evolution of N almost parallel vortex filaments in a three-dimensional fluid governed by the Navier-Stokes equations. According to this model, the vortex filaments are curves parametrized by

$$(\Re\psi_j(t,\sigma), \Im\psi_j(t,\sigma), \sigma), \quad \sigma \in \mathbb{R}, \quad 1 \le j \le N,$$

where $\Psi_j : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, and their dynamics is given by the following Hamiltonian system of one-dimensional Schrödinger equations with vortex type interaction

$$(1) \quad \begin{cases} i\partial_t \Psi_j + \alpha_j \Gamma_j \partial_\sigma^2 \Psi_j + \displaystyle\sum_{k \ne j} \Gamma_k \frac{\Psi_j - \Psi_k}{|\Psi_j - \Psi_k|^2} = 0, \quad 1 \le j \le N, \\ \Psi_j(0,\sigma) = \Psi_{j,0}(\sigma). \end{cases}$$

Here $\alpha_j \in \mathbb{R}$ is a parameter related to the core structure and Γ_j is the circulation of the jth filament. Particular solutions are the parallel filaments (X_j) with $X_j(t,\sigma) = (z_j(t), \sigma)$ for (z_j) solutions of the 2-D point vortex system

$$(2) \qquad\qquad i\dot{z}_j + \sum_{k \ne j} \Gamma_k \frac{z_j - z_k}{|z_j - z_k|^2} = 0, \quad 1 \le j \le N.$$

In this paper we will focus on pairs of filaments, namely $N = 2$.

The system (1) appears as the leading-order asymptotic equation under particular conditions relating the wavelength, the amplitude of the perturbations around (X_j), the core thicknesses of the filaments, and the distances between the filaments. The attention was focused in [KMD95, Z88, Z99] on perturbations of two parallel filaments (X_j) with the same core parameters, $\alpha_1 = \alpha_2$. In the case of circulations having the same sign, the perturbations around the corotating pair of parallel filaments (X_j) were proved to be linearly stable and global existence was conjectured, while for circulations of opposite signs instability was shown to appear for the linearized operator and numerical evidence of finite-time collapse was given. We send the reader to chapter 7 in [MB02] for a complete description of the model (1) and of these results.

Later Kenig, Ponce and Vega [KPV03] proved global existence for (1) with same core parameters and same circulations for perturbations around a corotating pair of parallel filaments (X_j) and for perturbations around (X_j) with (z_j) a rotating equilateral triangle.

Recently in [BM12, BM13] symmetric perturbations around (X_j), with (z_j) a rotating $N-$polygon, were proved to exist globally in time for (1) with same core parameters and circulations. Moreover, existence of travelling waves was displayed for (1), together with collision scenarios for at least three filaments.

Finally, let us mention two other related results. Lions and Majda [LM00] established global existence without uniqueness of weak solutions of (1) as

perturbations around any (X_j) with same core parameters and circulations. A work by Craig and Garcia [CG13] is in progress on quasi-periodic and periodic perturbations of corotating pairs of parallel filaments using KAM theory.

Summarizing, for pairs of filaments no collision for (1) has been displayed rigorously. Moreover, the only numerical simulations of collisions have been performed starting from a perturbation given by the unstable mode of the linearized system around the antiparallel translating pair of filaments (which is the solution of the 2D point vortex system (2) with $\Gamma_1 = -\Gamma_2$). In this paper we will construct exact examples of collisions of pairs of filaments in related models together with numerical simulations. More precisely, in Section §2 we consider the system obtained from (1) by setting $\alpha_1 = -\alpha_2$ and we display another type of explicit initial perturbation of the antiparallel translating pair of filaments leading to an exact collision for this system. Moreover we give numerical evidence that this initial perturbation also leads to collision in (1). Remarkably, this type of data is precisely the one used in [BM12] to construct an explicit collision in (1) for any number of filaments. Then in Section §3 we establish the existence of self-similar solutions of (1). Finally, the last Section §4 gathers all the numerical investigations.

2. Antiparallel vortex filament pairs collisions

We consider opposite circulations

$$\Gamma_1 = -\Gamma_2 = 1.$$

Then, for two nearly parallel vortex filaments, system (1) writes

(3)
$$\begin{cases} i\partial_t \Psi_1 + \alpha_1 \partial_\sigma^2 \Psi_1 - \dfrac{\Psi_1 - \Psi_2}{|\Psi_1 - \Psi_2|^2} = 0, \\ i\partial_t \Psi_2 - \alpha_2 \partial_\sigma^2 \Psi_2 - \dfrac{\Psi_1 - \Psi_2}{|\Psi_1 - \Psi_2|^2} = 0. \end{cases}$$

Moreover,

$$z_1(t) = -\frac{it}{2} + 1, \quad z_2(t) = -\frac{it}{2} - 1,$$

is an explicit solution of the vortex point system

$$\begin{cases} i\partial_t z_1 - \dfrac{z_1 - z_2}{|z_1 - z_2|^2} = 0, \\ i\partial_t z_2 - \dfrac{z_1 - z_2}{|z_1 - z_2|^2} = 0. \end{cases}$$

2.1. **Case of opposite core parameters.** We start by studying the system (3) with

$$\alpha_1 = -\alpha_2 = 1.$$

We consider the Gaussian

$$G(\sigma) = \frac{e^{-\frac{\sigma^2}{1-4i}}}{\sqrt{1-4i}}.$$

Below we construct a global solution of (3) such that the filaments $\psi_1(t,\sigma)$ and $\psi_2(t,\sigma)$ collapse at time $t = 1$ and height $\sigma = 0$.

Proposition 2.1. *The perturbation $\Psi_j(t,\sigma)$ of $X_j(t,\sigma) = (z_j(t),\sigma)$ generated by the data*

$$\Psi_{1,0}(\sigma) = 1 - G(\sigma), \quad \Psi_{2,0}(\sigma) = -1 + G(\sigma)$$

evolves through a global solution of system (3) with $\alpha_1 = -\alpha_2 = 1$ in the following (mild) sense:

$$\Psi_1(t,\sigma) = -\frac{it}{2} + e^{it\partial_\sigma^2}(1-G)(\sigma) - \frac{1}{2}D(t,\sigma), \quad t \in \mathbb{R},$$

$$\Psi_2(t,\sigma) = -\frac{it}{2} - e^{it\partial_\sigma^2}(1-G)(\sigma) - \frac{1}{2}D(t,\sigma), \quad t \in \mathbb{R},$$

where

$$D \in C(\mathbb{R}, L^q(\mathbb{R})), \quad \forall q \in [2, +\infty)$$

and is given by

$$D(t,\sigma) = i \int_0^t e^{i(t-s)\partial_\sigma^2} \left(\frac{1}{1 - \overline{e^{is\partial_\sigma^2}G(\sigma)}} - 1 \right) ds, \quad t \in \mathbb{R}.$$

In particular the filaments $\psi_1(t,\sigma)$ and $\psi_2(t,\sigma)$ collide at time $t = 1$ and height $\sigma = 0$.

Remark 2.2. *We notice that the $\Psi_j(t,\sigma)$ are symmetric perturbations of $X_j(t,\sigma)$ plus a non-linear shift D that acts for all times t and heights σ, so the configuration does not keep any symmetry (see Figure 1 in Section 4).*

Remark 2.3. *Once we have an initial configuration $\psi_{j,0}$ leading to collision, any shifted perturbation $\psi_{j,0}(\sigma) + u_0(\sigma)$, with $u_0 \in L^2(\mathbb{R})$, gives rise to the solution $\Psi_j(t,\sigma) + e^{it\partial_\sigma^2}u_0(\sigma)$ and therefore also leads to collision at the same time and height.*

Proof. Adding and substracting the equations in (3) we get a system of equations for the functions $\Psi = \Psi_1 - \Psi_2$ and $\Phi = \Psi_1 + \Psi_2$:

$$(4) \qquad \begin{cases} i\partial_t\Psi + \partial_\sigma^2\Psi = 0 \\ i\partial_t\Phi + \partial_\sigma^2\Phi = 2\dfrac{\Psi}{|\Psi|^2}. \end{cases}$$

The solution of the first equation is

$$\Psi(t,\sigma) = e^{it\partial_\sigma^2}\Psi_0(\sigma), \quad (t,\sigma) \in \mathbb{R} \times \mathbb{R}.$$

We look then for a mild solution of the second equation, namely

$$\Phi(t,\sigma) = e^{it\partial_\sigma^2}\Phi_0(\sigma) - 2i \int_0^t e^{i(t-s)\partial_\sigma^2} \frac{\Psi}{|\Psi|^2}(s,\sigma)\, ds.$$

Since we have

$$\Psi_0(\sigma) = 2(1 - G(\sigma)),$$

we retrieve the explicit example of [BM12]:

$$\Psi(t,\sigma) = 2e^{it\partial_\sigma^2}(1-G)(\sigma) = 2\left(1 - \frac{e^{-\frac{\sigma^2}{1-4i(1-t)}}}{\sqrt{1-4i(1-t)}}\right), \quad (t,\sigma) \in \mathbb{R} \times \mathbb{R},$$

so that collision occurs if and only if $(t,\sigma) = (1,0)$,

$$\Psi(1,0) = e^{i1\partial_\sigma^2}\Psi_0(0) = 0.$$

Note that for simplicity we have also fixed the initial symmetry

$$\Phi_0(\sigma) = 0,$$

so

$$\Phi(t,\sigma) = -i \int_0^t e^{i(t-s)\partial_\sigma^2} \frac{1}{1 - \overline{e^{is\partial_\sigma^2}G(\sigma)}}\, ds.$$

We obtain then the explicit expressions of $\Psi_1(t,\sigma)$ and $\Psi_2(t,\sigma)$ in the statement of the Proposition.

Finally we justify that $D \in C(\mathbb{R}, L^q(\mathbb{R}))$, thus Φ is globally defined even beyond the collision time $t = 1$. First we notice that there exists $m > 0, C > 0$ such that

$$(5) \qquad \left|\frac{1}{1 - e^{is\partial_\sigma^2}G(\sigma)}\right| \leq C \max\left\{1, \frac{1}{\sigma^2 + |1-s|}\right\}, \quad \forall|\sigma| \leq m, \quad s \in \mathbb{R}.$$

Indeed, by a second order Taylor expansion in variables (s,σ) for the function

$$f(s,\sigma) = e^{is\partial_\sigma^2}G(\sigma) = \frac{e^{-\frac{\sigma^2}{1-4i(1-s)}}}{\sqrt{1-4i(1-s)}}$$

we obtain the existence of $m > 0$ and $0 < \delta_0 < 1$ such that

$$f(1,0) - f(s,\sigma) = -2i(1-s) - \sigma^2 - 6(1-s)^2 + r(s,\sigma),$$

with

$$|r(s,\sigma)| \leq \frac{\sigma^2 + (1-s)^2}{2} \leq \frac{\sigma^2 + |1-s|}{2}, \quad \forall|\sigma| \leq m, \quad |s-1| \leq \delta_0.$$

So if $|\sigma| \leq m$ and $|s-1| \leq \delta_0$,

$$\frac{1}{|f(1,0) - f(s,\sigma)|} \leq \frac{1}{|\sigma^2 + 6(1-s)^2 + 2i(1-s)| - |r(s,\sigma)|} \leq \frac{2}{\sigma^2 + |1-s|}.$$

On the other hand for $|s-1| \geq \delta_0$ we have $|f(s,\sigma)| \leq (1 + 16\delta_0^2)^{-1/4}$, so we get the existence of $C > 0$ such that

$$\frac{1}{|f(1,0) - f(s,\sigma)|} \leq C.$$

This establishes (5). We infer that there exists $C > 0$ such that

(6)

$$\left|\frac{e^{is\partial_\sigma^2}G(\sigma)}{1 - e^{is\partial_\sigma^2}G(\sigma)}\right| \leq C\frac{e^{-\frac{\sigma^2}{1+16(1-s)^2}}}{(1+16(1-s)^2)^{\frac{1}{4}}} \max\left\{1, \frac{1}{\sigma^2 + |1-s|}\right\}, \quad \forall(\sigma,s) \in \mathbb{R}\times\mathbb{R}.$$

Now we consider $2 \leq q < \infty$. For $t > 0$, using dispersion estimate we get

$$\|D(t)\|_{L^q} \leq C \int_0^t \left\| \frac{e^{is\partial_\sigma^2} G(\sigma)}{1 - e^{is\partial_\sigma^2} G(\sigma)} \right\|_{L^{\bar{q}}} \frac{ds}{|t-s|^{\frac{1}{2} - \frac{1}{q}}},$$

where $\bar{q}$ denotes the conjugate exponent of q. We split the integral in σ into the regions $|\sigma| \leq 1, 1 \leq |\sigma| \leq \max\{1, |1-s|^2\}, \max\{1, |1-s|^2\} \leq |\sigma|$ and use (6) to get

$$\left\| \frac{e^{is\partial_\sigma^2} G(\sigma)}{1 - e^{is\partial_\sigma^2} G(\sigma)} \right\|_{L^{\bar{q}}} \leq \left\| 1 + \frac{1}{\sigma^2 + |1-s|} \right\|_{L^{\bar{q}}(|\sigma| \leq 1)} + \left\| \frac{e^{-\frac{\sigma^2}{17|1-s|^2}}}{|1-s|^{\frac{1}{2}}} \right\|_{L^{\bar{q}}} + \left\| e^{-\frac{\sigma}{17}} \right\|_{L^{\bar{q}}}.$$

We perform in the integrals above changes of variables and we obtain

$$\|D(t)\|_{L^q} \leq C \int_0^t \left(1 + |1-s|^{-\frac{1}{2} - \frac{1}{2q}} + |1-s|^{\frac{1}{2} - \frac{1}{q}} \right) \frac{ds}{|t-s|^{\frac{1}{2} - \frac{1}{q}}} \leq C(1+t).$$

When $t < 0$ we have

$$\|D(t)\|_{L^q} \leq C \int_{-|t|}^0 \left\| \frac{e^{is\partial_\sigma^2} G(\sigma)}{1 - e^{is\partial_\sigma^2} G(\sigma)} \right\|_{L^{\bar{q}}} \frac{ds}{|t-s|^{\frac{1}{2} - \frac{1}{q}}}$$

$$\leq \int_{-|t|}^0 \left\| e^{-\frac{\sigma^2}{17|1-s|^2}} \right\|_{L^{\bar{q}}} \frac{ds}{|t-s|^{\frac{1}{2} - \frac{1}{q}}} \leq C(1+|t|),$$

where we have used that $|s-1| \geq 1$ for $s \in [-|t|, 0]$. This proves that $D \in L^\infty_{\text{loc}}(\mathbb{R}, L^q(\mathbb{R}))$. The time continuity can be established by the same arguments. $\square$

2.2. Case of positive core parameters.
In the previous subsection we have obtained an initial configuration that leads to collisions in the case when the numbers α_1, α_2 have opposite signs. However, in the derivation of (1) these numbers are vortex core parameters supposed to be positive. We consider again opposite circulations but same sign core parameters

$$\Gamma_1 = -\Gamma_2 = 1, \quad \alpha_1 = \alpha_2 = 1.$$

So system (3) is

$$(7) \qquad \begin{cases} i\partial_t \Psi_1 + \partial_\sigma^2 \Psi_1 - \dfrac{\Psi_1 - \Psi_2}{|\Psi_1 - \Psi_2|^2} = 0, \\ i\partial_t \Psi_2 - \partial_\sigma^2 \Psi_2 - \dfrac{\Psi_1 - \Psi_2}{|\Psi_1 - \Psi_2|^2} = 0. \end{cases}$$

As we will see in the section devoted to numerical simulations, it is worth mentioning that the same initial data from the previous subsection $\Psi_1(0) = 1 - G$, $\Psi_2(0) = -1 + G$ yieldS a very nice collision behavior, see Figure 2 in Section 4.

Remark 2.4. *To establish the occurence of collision we need a solution of (7) such that $(\Psi_1 - \Psi_2)(t,\sigma)$ values $2(1-G(\sigma))$ at time $t=0$, stays close to $2(1-e^{it\partial_\sigma^2}G)$ and vanishes at $(t,\sigma)=(1,0)$. The difference $\Psi_1-\Psi_2$ solves the equation*

$$(8) \qquad \partial_t^2 f + \partial_\sigma^4 f + \frac{2\partial_\sigma^2\overline{f}}{\overline{f}^2} - \frac{4(\partial_\sigma\overline{f})^2}{\overline{f}^3} = 0.$$

It is not obvious to perform a perturbative argument for constructing such solutions - note for instance that the first nonlinear term valued on the ansatz $2(1-e^{it\partial_\sigma^2}G)$ is not in $L^1([0,1],L^2)$.

Remark 2.5. *One can write the difference $(\Psi_1-\Psi_2)(t,\sigma)$ using an integral equation,*

$$(9) \qquad \int \left(\frac{e^{i\frac{(\sigma-y)^2}{4t}}}{\sqrt{4\pi it}}\Psi_{1,0}(y) - \frac{e^{-i\frac{(\sigma+y)^2}{4t}}}{\sqrt{4\pi it}}\Psi_{2,0}(y) \right) dy$$
$$+ 2\int_0^t \int \frac{\sin\frac{\sigma^2+y^2}{4(t-\tau)}e^{-i\frac{\sigma y}{2(t-\tau)}}}{\sqrt{4\pi i(t-\tau)}} \frac{1}{\overline{\Psi_1-\Psi_2}(\tau,y)}dyd\tau,$$

but it is not obvious to control pointwise the second term in order to get cancellation for some $t>0$ at some point σ.

Next we consider as in [Z88, Z99, MB02] an initial configuration of two filaments satisfying

$$\Psi_1(0) = -\overline{\Psi}_2(0).$$

Then, as long as the solution (Ψ_1, Ψ_2) to (3) exists, it satisfies

$$\Psi_1(t) = -\overline{\Psi}_2(t),$$

because $(-\overline{\Psi}_2, -\overline{\Psi}_1)$ is also solution to (1) with same initial datum. In particular $\Psi_1 - \Psi_2 = 2\Re(\Psi_1)$ is real and the equation for Ψ_1 in (1) reduces to

$$(10) \qquad i\partial_t\Psi_1 + \partial_\sigma^2\Psi_1 - \frac{1}{2\Re(\Psi_1)} = 0.$$

Collision occurs when the real part of ψ_1 vanishes. With the same initial data $\psi_1(0) = 1 - G$ as before, numerical simulations provide a symmetric[1] collision, see Figure 3 in section 4.

Remark 2.6. *Note that the source term*

$$\int_0^t e^{i(t-s)\partial_\sigma^2} \frac{1}{2\Re(\Psi_1)(s)} \, ds$$

valued on the ansatz $\Psi_1(s,\sigma) = 2(1-e^{it\partial_\sigma^2}G)$ does not belong to $L^1([0,1]L^2)$, since the behavior of $\Re(1-e^{it\partial_\sigma^2}G)$ near $(t,\sigma)=(1,0)$ is $\sigma^2+6(1-t)^2$. This is in contrast with the non-symmetric data case for which the term to be

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[1] In the sense that $\Psi_1(t) = -\overline{\Psi}_2(t)$.

evaluated is $|1 - e^{it\partial_\sigma^2}G|^{-1} \le C(\sigma^2 + |1 - t|)^{-1}$. This is an obstacle in performing a perturbative argument to construct solutions of (10) *around $1 - e^{it\partial_\sigma^2}G$ like we did in Proposition 2.1.*

We conclude this section with a discussion on another approach in order to exhibit collision scenarios. Namely, one could try the method used by Merle and Zaag [MZ97] for the problem of vortex reconnection with the boundary in a superconductor under the planar approximation: for Ψ_1 solution of (10) we denote $u(t, \sigma) = \frac{1}{\Psi_1(t,\sigma) + \frac{it}{2}}$ that solves

$$(11) \qquad i\partial_t u + \partial_\sigma^2 u - \frac{u^2}{2}\left(1 - \frac{|u|^2}{\Re(u)}\right) - 2\frac{(\partial_\sigma u)^2}{u} = 0,$$

with boundary condition 1 at infinity. Now we note that

$$\Re(\Psi_1) = \frac{\Re(u)}{|u|^2},$$

so in order to have a cancellation for $\Re(\Psi_1)$ it is enough to obtain a solution $u(t, \sigma)$ blowing-up in L^∞ in finite time at one point. In [MZ97] instead of (11) a heat equation is obtained with power nonlinearity and a square gradient term. A pointwise blowing-up solution is constructed starting from the explicit profile of pointwise blowing-up solution of the heat equation with power nonlinearity. In here (11) is a nonlinear Schrödinger type equation and we are looking for what is called a dispersive blow-up solution. This kind of problem has been considered recently by Bona, Ponce, Saut and Sparber [BPSS13]. Introducing $u = v + 1$, the equation on v is of Gross-Pitaevskii type

$$(12) \qquad i\partial_t v + \partial_\sigma^2 v + \frac{1}{2}\Re(v) + F(v) = 0,$$

with

$$F(v) = \Re v\left(v + \frac{(\Re v)^2}{2} + i\Re v\Im v\right) + \frac{(\Im v)^2}{2} + i(\Im v)^3 - \frac{(\Im v)^4}{1 + \Re(v)} - 2\frac{(\partial_\sigma v)^2}{1 + v},$$

and [BPSS13] provides an explicit initial data that leads to dispersive blow-up for the Gross-Pitaevskii equation:

$$u_0^m(\sigma) = 1 + \frac{e^{ix^2}}{(1 + x^2)^m} + B_0\left(\frac{\cos(x^2)}{(1 + x^2)^m}\right), \quad \frac{1}{4} < m \le \frac{1}{2},$$

where B_0 has the symbol $-\frac{1}{(1+\xi^2)(\sqrt{\xi^2(1+\xi^2)^{-1}+1})}$. Numerics again exhibits a collision in finite time by taking $\Psi_1(0, \sigma) = \frac{1}{u_0^m(\sigma)}$.

The article [BPSS13] gives also a recipe to conclude the dispersive blow-up for equations of type (11), from the same initial data. It consists in controlling the Duhamel term, once the existence of a local nonlinear solution in some appropriate spaces is known. So up to multiplying the initial data with a small constant, a local Cauchy theory for small data regular as $u_0^m - 1$

for (12) is sufficient for our purposes. The regularity of the initial data is proved in [BPSS13] to be

$$u_0^m - 1 \in \mathcal{C}^\infty \cap L^\infty \cap H^s(\mathbb{R}), \forall s < 2m - \frac{1}{2}.$$

In our case, for proving local in time wellposedness of (12), the first terms in $F(v)$ can be treated at the L^2 level in $L^\infty([0,T[,L^2) \cap L^4([0,T[,L^\infty)$. Then for treating $\frac{(\Im v)^4}{1+\Re v}$ one needs to control through the fixed point argument a positive lower bound for $1 + \Re v$, which is not obvious. Finally, inglobing in the fixed point the term involving $(\partial_\sigma v)^2$ is not obvious neither, since the initial data is of regularity at most $\mathcal{C}^\infty \cap L^\infty \cap H^{\frac{1}{2}^-}(\mathbb{R})$.

3. Self-similar vortex filament pairs collisions

In this section we investigate the existence of a particular class of solutions to (1) in the case

$$\Gamma_1 = -\Gamma_2 = 1, \quad \alpha_1 = \alpha_2 = 1.$$

More precisely, as suggested in [Z88, Z99, KMD95] we look for a self-similar solution of (10):

$$\psi_1(t,\sigma) = \sqrt{t}\, u\left(\frac{\sigma}{\sqrt{t}}\right).$$

Setting $x = \sigma/\sqrt{t}$, note that if the profile $u(x)$ behaves like $\alpha|x|$ when $|x| \to +\infty$ then the corresponding two filaments are localized at $+\infty$ near the lines passing through the origin and generated by $(1, \Re\alpha, \Im\alpha)$ and $(1, -\Re\alpha, \Im\alpha)$ respectively, and at $-\infty$ near the lines passing through the origin and generated by $(1, -\Re\alpha, -\Im\alpha)$ and $(1, \Re\alpha, -\Im\alpha)$ respectively. The filaments are not parallel, so a priori this kind of configuration does not enter the setting of the modelisation (1) proposed by [KMD95]. As [Z88, Z99] mention, this is actually a case for getting information on possible reconnection scenarios.

We obtain as an equation for the profile $u(x)$

(13) $$i(u - xu') + 2u'' - \frac{1}{\Re(u)} = 0.$$

Up to a change of scale we may rewrite Equation (13) as

(14) $$\begin{cases} v' = ixv - \dfrac{x}{\Re(u)} \\ u - xu' = v. \end{cases}$$

We look for a solution such that

$$v \text{ is even,} \quad v(0) = 1.$$

The condition $v(0) = 1$ corresponds to starting at time $t = 1$ with two filaments at distance 2 at the origin level $\sigma = 0$. For the function u we impose the following conditions at infinity:

(15) $$u(x) \sim \alpha|x|, \quad |x| \to +\infty$$

where $\alpha \in \mathbb{C}$ is such that $\Re(\alpha) > 0$.

Theorem 3.1. *Let $\alpha \in \mathbb{C}$. Let*

$$E = \left\{ w \in C^1(\mathbb{R}),\ w(0) = w'(0) = 0,\ w\ even\ ,\ \|w\|_{L^\infty} + \||x|^{-1}w'\|_{L^\infty} \leq \frac{\alpha}{4} \right\}.$$

There exists a numerical constant $K_0 > 1$ such that if $\Re(\alpha) > K_0$, there exists a unique $v \in 1 + E$ such that the couple (u, v), with u defined by

$$(16) \qquad u(x) = 1 + |x| \left(\alpha + \int_{|x|}^{+\infty} \frac{v(z) - 1}{z^2}\, dz \right), \forall x \neq 0, \quad u(0) = 1,$$

is a solution of the system (14) satisfying the condition (15). Moreover,

$$(17) \qquad\qquad\qquad \Re(u(x)) > 0,\ \forall x \in \mathbb{R},$$

and u is a Lipschitz function on $\mathbb{R}$ with $u' \in C(\mathbb{R}^*)$.

Remark 3.2. *We could replace $v - 1$ by $v - v(0)$ with $v(0)$ any complex number satisfying $\Re(v(0)) > 0$ (so that the filaments are separated at $\sigma = 0$ and $t = 1$).*

Remark 3.3. *We will see below that if $v \in 1 + E$ the integral $\int_0^{+\infty} |v - 1|/z^2$ is finite so that u' is continuous on $\mathbb{R}^*$ and has a jump discontinuity at $x = 0$ with $u'(0)^+ = -u'(0)^- = \alpha + \int_0^{+\infty}(v - 1)/z^2$. This means that the filaments exhibit a corner at $\sigma = 0$.*

Proof. Note that since the system (14) is invariant by the transformation $(u, v) \mapsto (u - i\Im(\alpha)|x|, v)$ we may assume that

$$\alpha \in \mathbb{R}, \quad \alpha > 0.$$

We will show by a fixed point argument that there exists a unique solution as in Theorem 3.1 such that $v - 1$ belongs to the Banach space E endowed with the metric

$$d(w_1, w_2) = \|w_1 - w_2\|_\infty + \||x|^{-1}(w_1 - w_2)\|_\infty.$$

Before proving this, let us derive first its consequences. Let (u, v) be such a solution of (14) with $v - 1 \in E$. Integrating the second equation of (14) on the intervals $[x, +\infty)$ when $x > 0$ or $(-\infty, x]$ if $x < 0$ we find

$$u(x) = x\alpha + x \int_x^{+\infty} \frac{v(z)}{z^2} dz \quad \text{for } x > 0$$

and

$$u(x) = -x\alpha - x \int_{-\infty}^x \frac{v(z)}{z^2} dz \quad \text{for } x < 0.$$

In particular the first part of (16) is satisfied. Since v is even we infer that u is even as well. Note that u' has a jump discontinuity at $x = 0$ but $u - xu' = v \in C^1(\mathbb{R})$. From now on we only consider $x > 0$.

We now check that $\Re(u) > 0$ on $\mathbb{R}$. We set

$$w = v - v(0) = v - 1.$$

In order to consider converging integrals we rewrite u as

$$(18) \qquad u(x) = 1 + x\alpha + x \int_x^{+\infty} \frac{w(z)}{z^2}\, dz = 1 + x\left(\alpha + \int_x^{+\infty} \frac{w(z)}{z^2}\, dz\right).$$

Note that the integral is well-defined since w is bounded. We first observe that $w \in E$ implies

$$(19) \qquad \sup_{x \in \mathbb{R}_+} \left|\int_x^{+\infty} \frac{w}{z^2}\, dz\right| < \frac{\alpha}{2}.$$

Indeed, for $w \in E$,

$$(20) \qquad \left|\int_x^{+\infty} \frac{w}{z^2}\, dz\right| \leq \int_0^1 \frac{|w|}{z^2}\, dz + \int_1^{+\infty} \frac{|w|}{z^2}\, dz \leq \frac{\||x|^{-1}w'\|_\infty}{2} + \|w\|_\infty,$$

where we have used that $w(0) = 0$ and

$$|w(z)| \leq \int_0^z |w'(t)|\, dt \leq \frac{z^2 \||x|^{-1}w'\|_\infty}{2},$$

so the inequality (19) follows.

In particular (18) and (19) yield

$$(21) \qquad \Re(u(x)) \geq 1 + \frac{x\alpha}{2} > 0, \quad \forall x \geq 0,$$

together with $u(0) = 1$, so all the claims of the Theorem are verified.

We are left with the proof of the existence of solutions in E. In view of the first equation in (14) we define the following operator:

$$P(w)(x) = e^{\frac{ix^2}{2}} - 1 - e^{\frac{ix^2}{2}} \int_0^x \frac{ye^{-\frac{iy^2}{2}}}{\Re(u(y))}\, dy, \quad x \in \mathbb{R},$$

where $u(y)$ is defined by (18).

We next show that P has a unique fixed point $w \in E$ if $\alpha > K_0$ with K_0 sufficiently large.

When $w \in E$ the function u is even therefore $P(w)$ is also even. Moreover, integrating by parts we rewrite $P(w)$ as

$$P(w) = \begin{cases} e^{\frac{ix^2}{2}} - 1 - e^{\frac{ix^2}{2}} \int_0^x \frac{ye^{-\frac{iy^2}{2}}}{\Re(u(y))}\, dy, \quad x \in [0,1], \\[2mm] e^{\frac{ix^2}{2}}\left(1 - \int_0^1 \frac{ye^{-\frac{iy^2}{2}}}{\Re(u(y))}\, dy\right) - 1 \\[2mm] \quad -i\left(\frac{1}{\Re(u(x))} - \frac{e^{\frac{i(x^2-1)}{2}}}{\Re(u(1))}\right) - ie^{\frac{ix^2}{2}} \int_1^x \frac{e^{-\frac{iy^2}{2}}\Re(u'(y))}{\Re(u(y))^2}\, dy, \quad x \in [1,+\infty). \end{cases}$$

We first show that $P(E) \subset E$. We set for $y \geq 0$

$$f(y) = \Re\left(\alpha + \int_y^{+\infty} \frac{w(z)}{z^2}\, dz\right)$$

so that
$$\Re(u(y)) = 1 + yf(y).$$

Then $f \in C^1(\mathbb{R}_+)$ and by (19) it satisfies

(22) $$|f'(y)| \le \frac{\alpha}{4y^2}, \quad |f(y)| \le \frac{3\alpha}{2}, \quad f(y) \ge \frac{\alpha}{2} > 0.$$

On the one hand, this yields $\Re(u(y)) \ge 1$ so $P(w)(0) = P(w)'(0) = 0$ and

$$\int_0^1 \frac{y}{\Re(u(y))} \, dy \le 1,$$

therefore
$$\|P(w)\|_{L^\infty([0,1])} \le 3.$$

On the other hand, using (22) we get

$$\int_1^{+\infty} \frac{|\Re(u')|}{\Re(u)^2} \, dy \le \int_1^{+\infty} \frac{y|f'| + f}{(1 + yf)^2} \, dy \le \int_1^{+\infty} \left(\frac{y|f'|}{2yf} + \frac{f}{y^2 f^2} \right) dy$$
$$\le \int_1^{+\infty} \frac{|f'|}{2f} \, dy + \frac{2}{\alpha} \int_1^{+\infty} \frac{dy}{y^2}$$
$$\le \frac{\alpha + 8}{4\alpha}.$$

It follows that
$$\|P(w)\|_{L^\infty([1,+\infty))} \le 5 + \frac{\alpha + 8}{4\alpha}.$$

Next, we have
$$P'(w) = ix(1 + P(w)) - \frac{x}{\Re(u)}$$

so that combining the previous estimates
$$\||x|^{-1} P'(w)\|_{L^\infty} \le 7 + \frac{\alpha + 8}{4\alpha}.$$

In conclusion, we obtain
$$\|P(w)\|_{L^\infty} + \||x|^{-1} P'(w)\|_{L^\infty} \le \frac{\alpha}{4}$$

provided that

(23) $$12 + \frac{\alpha + 8}{2\alpha} \le \frac{\alpha}{4},$$

which holds for $\alpha > K_0$ sufficiently large.

We then show that P is a contraction on E. For $w_1, w_2 \in E$ we set $f_i(y) = \Re\left(\alpha + \int_y^{+\infty} w_i/z^2 \, dz\right)$, $i = 1, 2$. We have

$$P(w_1)(x) - P(w_2)(x) = -e^{\frac{ix^2}{2}} \int_0^x y e^{-\frac{iy^2}{2}} g(y) \, dy,$$

where
$$g(y) = \frac{y(f_2(y) - f_1(y))}{(1 + yf_1(y))(1 + yf_2(y))},$$

so that
$$g'(y) = \frac{y(f_2' - f_1') + f_2 - f_1}{(1 + yf_1)(1 + yf_2)}$$
$$- y(f_2 - f_1)\frac{(1 + yf_1)(yf_2' + f_2) + (1 + yf_1)(yf_2' + f_2)}{(1 + yf_1)^2(1 + yf_2)^2}.$$

Since
$$f_1(y) - f_2(y) = \Re\left(\int_y^{+\infty} \frac{w_1 - w_2}{z^2}\, dz\right)$$

we obtain by (20) for $y > 0$

$$(24) \qquad |f_1(y) - f_2(y)| \le d(w_1, w_2), \quad |f_1'(y) - f_2'(y)| \le \frac{d(w_1, w_2)}{y^2},$$

while

$$(25) \qquad \frac{\alpha}{2} \le f_i(y) \le \frac{3\alpha}{2}, \quad i = 1, 2.$$

Using the inequality $1 + yf_i(y) \ge yf_i(y) \ge y\alpha/2$, we obtain by (24) and (25)

$$(26) \qquad |g(y)| \le \frac{y|f_2(y) - f_1(y)|}{yf_2(y)} \le \frac{2}{\alpha}d(w_1, w_2),$$

and there exists some $K_1 > 1$ such that

$$(27) \qquad |g'(y)| \le \frac{K_1}{\alpha^2 y^2}d(w_1, w_2), \quad y \ge 1.$$

Hence
$$\|P(w_1) - P(w_2)\|_{L^\infty([0,1])} \le \frac{1}{\alpha}d(w_1, w_2).$$

And for $x \ge 1$ an integration by parts yields

$$P(w_1)(x) - P(w_2)(x) = -e^{\frac{ix^2}{2}}\int_0^1 ye^{-\frac{iy^2}{2}}hg(y)\, dy - ig(x) + ie^{\frac{i(x^2-1)}{2}}g(1)$$
$$+ ie^{\frac{ix^2}{2}}\int_1^x e^{-\frac{iy^2}{2}}g'(y)\, dy$$

hence in view of (26)-(27)

$$\|P(w_1) - P(w_2)\|_{L^\infty([1,+\infty))} \le \left(\frac{5}{\alpha} + \frac{K_1}{\alpha^2}\right)d(w_1, w_2).$$

On the other hand,
$$P'(w_1)(x) - P'(w_2)(x) = ix\left(P(w_1)(x) - P(w_2)(x)\right) - xg(x)$$

therefore using again (24)-(25) we get

$$\|x^{-1}(P'(w_1) - P'(w_2))\|_{L^\infty} \le \left(\frac{7}{\alpha} + \frac{K_1}{\alpha^2}\right)d(w_1, w_2).$$

Finally,

$$d(P(w_1), P(w_2)) \leq \left(\frac{12}{\alpha} + \frac{2K_1}{\alpha^2} \right) d(w_1, w_2),$$

which establishes that P is a contraction provided that α satisfies

$$(28) \qquad \frac{12}{\alpha} + \frac{2K_1}{\alpha^2} < 1,$$

and increasing possibly K_0 we are ensured that both conditions (23) and (28) are satisfied. This concludes the proof of Theorem 3.1.

$\square$

4. NUMERICAL SIMULATIONS

To compute an approximation of the system (3), we use a splitting algorithm between the nonlinear and linear parts. Indeed, the solution of the linear part

$$(29) \qquad \begin{cases} i\partial_t \Psi_1 + \alpha_1 \partial_\sigma^2 \Psi_1 = 0, \\ i\partial_t \Psi_2 - \alpha_2 \partial_\sigma^2 \Psi_2 = 0, \end{cases}$$

is given explicitly in Fourier, and can be very easily computed using the Fast Fourier Transform algorithm. On the other hand, the solution of the nonlinear part

$$\begin{cases} i\partial_t \Psi_1 - \dfrac{\Psi_1 - \Psi_2}{|\Psi_1 - \Psi_2|^2} = 0, \\ i\partial_t \Psi_2 - \dfrac{\Psi_1 - \Psi_2}{|\Psi_1 - \Psi_2|^2} = 0, \end{cases}$$

is given explicitly by the formula

$$\Psi_k(t, x) = \Psi_k(0, x) - it \frac{\Psi_1(0, x) - \Psi_2(0, x)}{|\Psi_1(0, x) - \Psi_2(0, x)|^2}.$$

Denoting by $\varphi_k^t(\Psi_1(0), \Psi_2(0))$ this application, the algorithm used to compute the solutions are thus based on the classical Lie approximation

$$\Psi_k(t) \simeq e^{it\alpha_k \partial_\sigma^2} \circ \varphi_k^t(\Psi_1(0), \Psi_2(0)),$$

as well as the symmetric version (Strang splitting)

$$\Psi_k(t) \simeq \varphi_k^{t/2} \circ e^{it\alpha_k \partial_\sigma^2} \circ \varphi_k^{t/2}(\Psi_1(0), \Psi_2(0)),$$

To discretize in space these formulae, we take a large periodic box $L[-\pi, \pi]$ using equidistant grid points $x_k = Lk\pi/K$, for $k \in B_K := \{K/2 - 1, K/2\}$. The number K is the number of nodes. Then we can evaluate the flow φ^t at the fix nodes x_k, while the approximation of the free equation $e^{it\alpha_k}$ is made using the discrete Fourier transformation:

$$\forall\, k \in B_K, \quad v_k = (\mathcal{F}_K)_k := \frac{1}{K} \sum_{\ell \in B_K} u_\ell e^{-ikx_\ell}$$

and the calculation $v_k(t) = e^{-itk^2L^{-2}}v_k(0)$ as solution of (29) in Fourier variables. We then go back to the x-variable using the inverse of $\mathcal{F}_K$ that, together with $\mathcal{F}_K$, can be computed in $K\log K$ operations.

The algorithms obtained are then symplectic, and used with a time stepsize τ satisfying the CFL condition $\tau K^2L^{-2} < \pi$ which is known to be a necessary condition to ensure the existence of a modified energy as well as some stability results in simpler situations (cubic Schrödinger equation for instance, see [1, F12]).

4.1. **Opposite core parameters.** We consider now the equation (3), in the case where $\Gamma_1 = -\Gamma_2 = 1$ and $\alpha_1 = -\alpha_2 = 1$. We take $K = 1024$, and the stepsize $\tau = \pi K^{-2}$. We take $L = 10$ (so that the CFL number is of order 10^{-2}), and the initial data $\Psi_1(0) = 1 - G$ and $\Psi_2(0) = -1 + G$. The non-symmetric evolution of the corresponding filaments is depicted in Figure 1. We observe that the collision occurs at time $t = 1$ as predicted by Proposition 2.1.

4.2. **Positive core parameters.** We now consider exactly the same initial condition as before, but we just change the sign of α_2 setting $\alpha_1 = \alpha_2 = 1$, so the evolution is governed by (7). As mentioned above, the solution starting with the initial data $\Psi_1(0, \sigma) = 1 - G(\sigma)$ and $\Psi_2(0, \sigma) = -1 + G(\sigma)$ leads to a collision behavior in $\sigma = 0$ at a time of order $t \simeq 2.64$, see Figure 2. Note that the solution is more oscillatory compared to the situation where $\alpha_1 = -\alpha_2$, and the support slightly larger. Hence we have taken $L = 20$ instead of $L = 10$ in this simulation.

In a second simulation, we take as initial data $\Psi_1(0) = 1 - G$ and $\Psi_2(0) = -1 + \overline{G} = -\overline{\Psi}_1(0)$ so that $\Psi_2(t) = -\overline{\Psi}_1(t)$. With the same parameters as before (and with $L = 10$) we observe a symmetric collision at time $t \simeq 0.83$, see Figure 3.

To investigate the type of collision appearing in the symmetric situation $\Psi_1(0) = -\overline{\Psi}_2(0)$ we consider numerical approximations of Equation (10). We first consider again the case where $\Psi_1(0) = 1 - G$. In Figure 4 we plot the evolution of the real part of the solution of (10) using the same splitting method (which extends straightforwardly to the this situation) and using again 1024 grid points and the same CFL condition (and $L = 10$). We observe that the collision (which means that the real part vanishes) occurs around the time $t = 0.83$, which confirms the previous numerical experiment.

In the last figure 5, we perform a numerical simulation of (10) but with initial data $\Psi_1(0) = 0.6 - G$. In this situation we observe a collision at a time close to $t = 0.18$. Note that in this situation, the collision arises at two symmetric points near $\sigma = 0$. We mention that this kind of collision is also observed in [K93] where numerical simulations of the interaction of
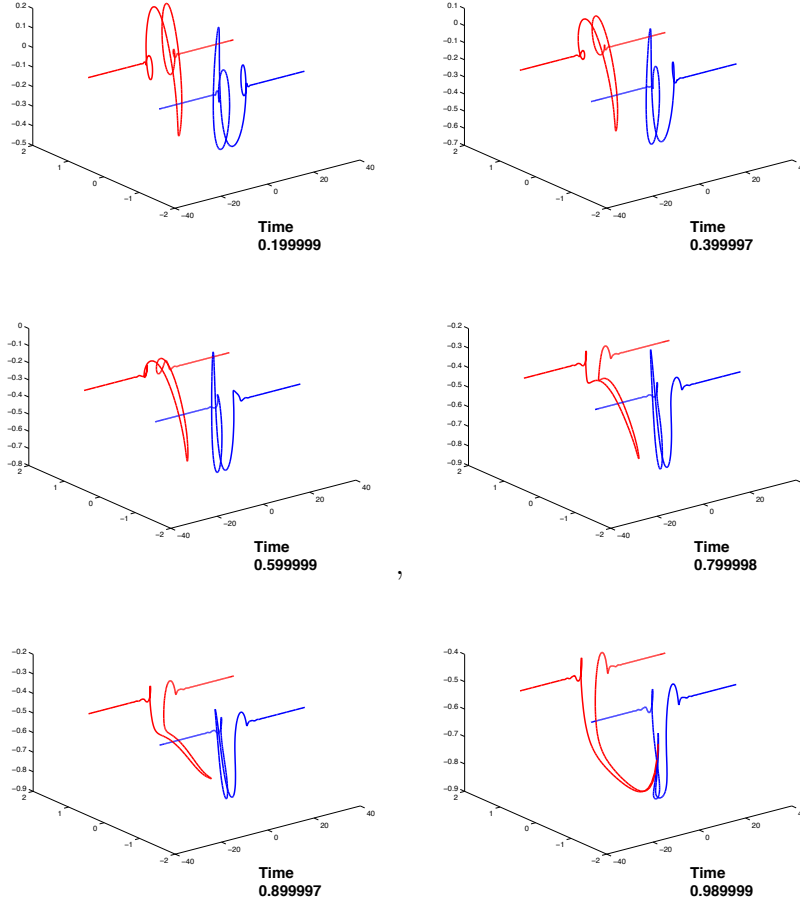
FIGURE 1. Collision in the case $\alpha_1 = -\alpha_2 = 1$, $\Psi_1(0) = 1 - G$, $\Psi_2(0) = -1 + G(\sigma)$, no symmetry kept by $(\Psi_1, \Psi_2)(t, \sigma)$.

perturbed antiparallel vortex tubes are performed in the setting of three-dimensional incompressible Euler equations. We would like to conclude on this case by doing the following remarks. First, the collisions numerically observed are *locally* of a form very similar to the solutions constructed in the previous section §3: at the collision time, the solution is locally made of two straight lines forming an angle on a collisional corner. Second, the numerical solution remains very close to the solution of the linear flow. Hence, by combining the two arguments (pertubative approach and description of a corner collision) these observations give a hope to rigorously describe the collision and prove its existence mathematically.
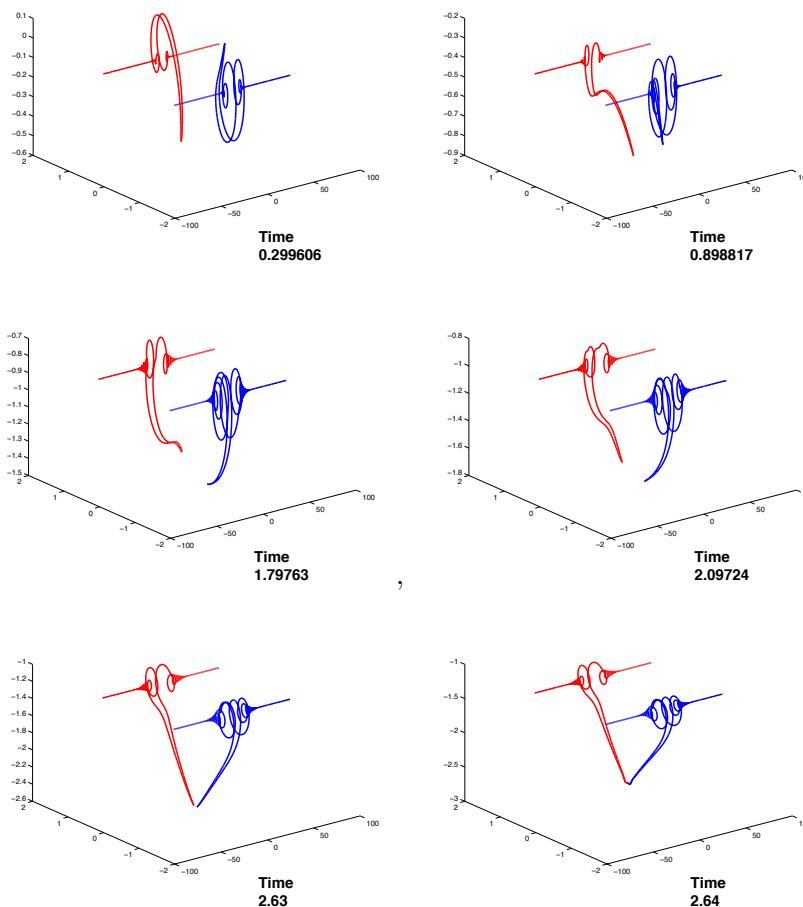
FIGURE 2. Collision in the case $\alpha_1 = \alpha_2 = 1$, $\Psi_1(0) = 1 - G$, $\Psi_2(0) = -1 + G$, no symmetry kept by $(\Psi_1, \Psi_2)(t)$.

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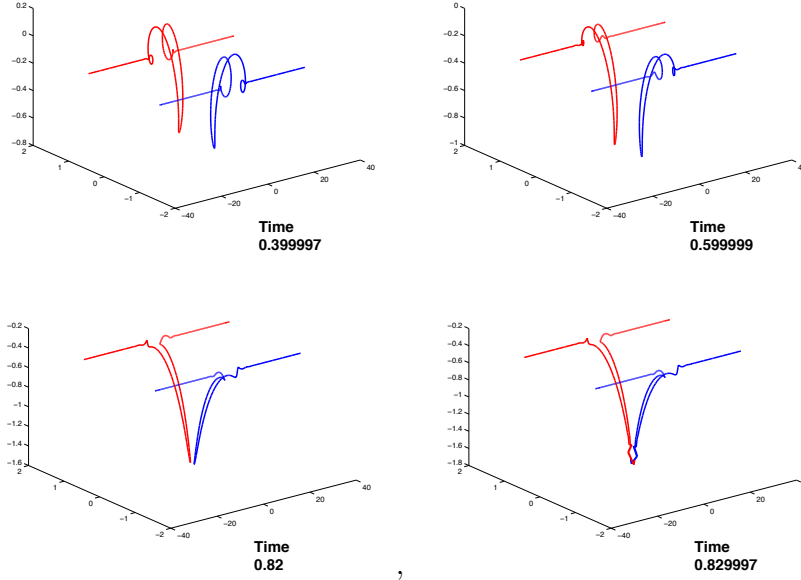
FIGURE 3. Collision in the case $\alpha_1 = \alpha_2 = 1$, $\Psi_1(0) = 1 - G$, $\Psi_2(0) = -1 + \overline{G}$, symmetry $\Psi_1(t) = -\overline{\Psi}_2(t)$.

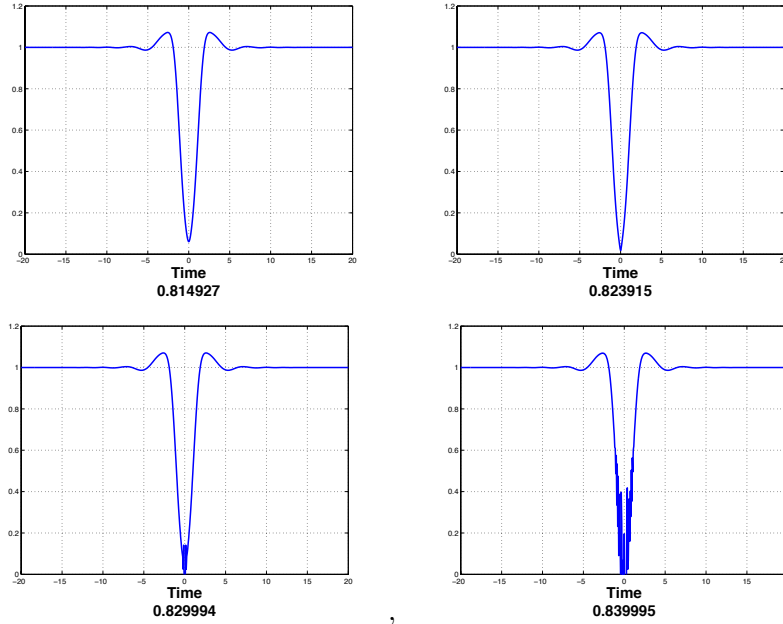

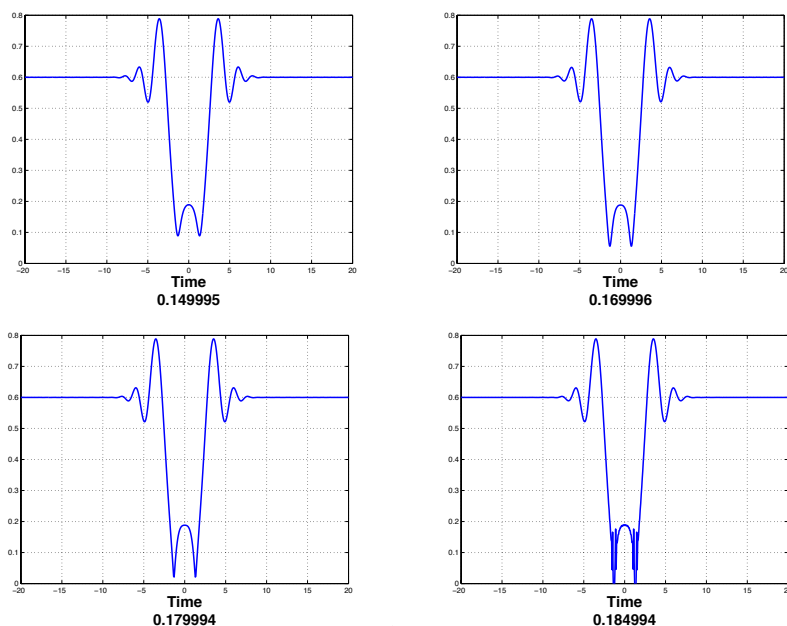FIGURE 4. Collision in the case for (10), $\Psi_1(0) = 1 - G$.

FIGURE 5. Collision in the case for (10), $\Psi_1(0) = 0.6 - G$.

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